Vortices and Two-Dimensional Fluid Motion

C. Eugene Wayne

The study of fluid motions is of obvious importance for a host of applications ranging in scale from the microscopic to the atmospheric. Since we live in a three-dimensional world, it may be less obvious why the understanding of two-dimensional fluid flows is of interest. However, in many applications, such as the atmosphere or the ocean, the fluid domain is much smaller in one direction than in the other two—and also smaller than the typical size of features of interest in the fluid. For example, in the case of the atmosphere, the thickness is a few tens of kilometers, while the lateral extent is tens of thousands of kilometers and the diameter of a feature such as a hurricane can be several hundreds of kilometers. Furthermore, in both the atmosphere and the ocean, the applicability of a two-dimensional approximation is enhanced by two additional effects: the stratification of the medium (which reduces the effective thickness of the domain) and the rotation of the earth, which tends to reduce variations in the vorticity field with height and means that in any cross-sectional plane, the flow is effectively two-dimensional. In such circumstances a two-dimensional approximation to the fluid motion can provide very accurate insights into the behavior of the physical system.

Even more interesting is the fact that two- and three-dimensional fluids behave in qualitatively different fashions. In three-dimensional flows energy typically flows from large-scale features to small ones until it is dissipated by the viscosity of the fluid. In two dimensions the phenomenon tends to reverse itself, and the energy concentrates itself in a few large vortex-like structures. This phenomenon, known as the "inverse cascade", manifests itself in a striking visual way through the coalescence of many small vortices into a smaller number of larger vortices.

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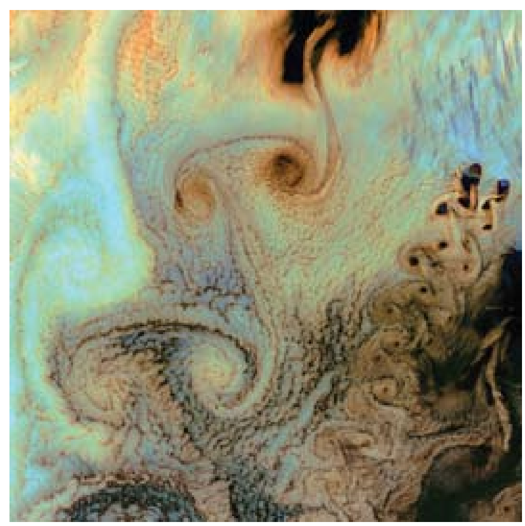

Figure 1. Atmospheric vortices formed by wind flowing past the Aleutian Islands, captured by Landsat 7 [17]**. Note that in this image, the vorticity field cannot be directly visualized. Instead, one views passive tracer particles (i.e., clouds!) that are carried along by the background flow and which are believed to accurately mimic the vorticity field.**

A beautiful visualization of this effect was created by Maarten Rutgers in turbulent soap films (see Figure 2). The patterns make visible differences in the vorticity of the fluid. The vorticity will be defined more precisely below but basically represents the rotational speed of the fluid either clockwise or counterclockwise. In this figure, the flow begins above the top of the picture and falls under the influence of gravity toward the bottom of the picture, so going down in the picture indicates a later stage in the evolution of the flow. The tendency of the vorticity to organize itself into larger and larger structures is clearly visible.

Figure 2. Two-dimensional turbulent flow visualized in a soap film by Maarten Rutgers—for an even more striking illustration of this phenomenon, see the video clip under the "Turbulence" section of http://maartenrutgers.org/ [20]**.**

The growth in the size of vortices and the reduction in their numbers is also visible in numerical experiments, such as those displayed in Figure 3, that are the result of research by the vortex dynamics group at the Technische Universiteit Eindhoven (TUE), Netherlands. One of the main goals in the study of two-dimensional turbulent flows has been to understand and explain this inverse cascade and in particular to explain the tendency of the vorticity to coalesce into a smaller and smaller number of larger and larger vortices. In this article I will explain how by exploiting ideas from the kinetic theory of gases one can show that almost all two-dimensional *viscous* flows eventually approach a single, large vortex.

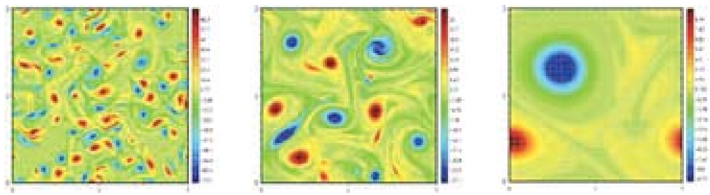

Figure 3. A numerical simulation of a two-dimensional turbulent flow. The figures display the vorticity field (with blue and red representing fluid swirling in opposite directions) at successively later and later times and clearly indicate the tendency of regions of vorticity of like sign to coalesce into a smaller and smaller number of larger vortices [21]**.**

The typical way of describing the motion of a fluid is through its velocity field, $\mathbf{v}(\mathbf{z},t)$; that is, through a vector field that at each point in space and time gives the velocity of the fluid at that point. However, more than 150 years ago Helmholtz realized that in addition to the velocity, the vorticity of the fluid carries important information about the nature of the flow. As mentioned above, the vorticity roughly measures the swirl in a fluid. More precisely, the vorticity is defined as the curl of the fluid velocity,

$$\boldsymbol{\omega}(\mathbf{z},t) = \nabla \times \mathbf{v}(\mathbf{z},t).$$

Note that we see already an important difference between two and three dimensions—in two dimensions, only one component of the vorticity is nonzero, and thus we can treat the vorticity essentially as a scalar field.

For an incompressible fluid, the fluid velocity satisfies the Navier-Stokes equations, the system of coupled nonlinear partial differential equations

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \nabla p \, , \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \, , \tag{2}$$

where ν is the kinematic viscosity of the fluid (which we will assume is constant), ρ is the fluid density (which is constant due to the incompressibility condition), and $p = p(\mathbf{z},t)$ is the pressure in the fluid. The first of these equations is just Newton's law for the fluid, with the left-hand side representing the acceleration of the fluid and the right-hand side the forces acting on it. We will assume that the only forces present are the internal viscous forces, modeled by the first term on the right-hand side, and the pressure forces, represented by the second. External forces acting on the fluid could be incorporated by adding additional terms to the right-hand side of the equation. We will also ignore the effects of boundaries on the fluid by assuming that the fluid occupies all of $\mathbb{R}^d$ with $d = 2, 3$.

In order to determine how the vorticity evolves, one can take the curl of both sides of the first of the Navier-Stokes equations. The dynamics represented by the two- and three-dimensional equations are strikingly different despite the close relationship between the equations. In three dimensions one has the system of equations

$$\partial_t \boldsymbol{\omega}(\mathbf{z}, t) - \boldsymbol{\omega} \cdot \nabla \mathbf{v}(\mathbf{z}, t) + \mathbf{v} \cdot \nabla \boldsymbol{\omega}(\mathbf{z}, t) = \nu \Delta \boldsymbol{\omega}(\mathbf{z}, t), \tag{3}$$

while in two dimensions one has only the single, scalar equation

$$\partial_t \omega(\mathbf{z}, t) + \mathbf{v} \cdot \nabla \omega(\mathbf{z}, t) = \nu \Delta \omega(\mathbf{z}, t). \tag{4}$$

(We will use a boldface $\boldsymbol{\omega}$ to denote the vorticity vector and an ω to denote the single, nonzero component of the vorticity in two dimensions.) The presence of the "vortex stretching term", $-\boldsymbol{\omega} \cdot \nabla \mathbf{v}(\mathbf{z}, t)$ in (3), is a critical physical difference between these two equations. For certain special fluid configurations, this term can lead to a sort of feedback mechanism in which the vorticity begins to grow. While it is not known if this growth can continue without bound, there is no obvious mechanism to stop its growth, and this is the physical source of the uncertainty as to whether or not smooth solutions of the Navier-Stokes equations exist for all time in three dimensions. The fact that it is still unknown whether or not the partial differential equations that are believed to describe such a basic system as fluid motion have unique, smooth solutions makes this obviously an extremely important question, and the successful resolution of this question (or the discovery of an example demonstrating the formation of a singularity in the solution in finite time) was chosen by the Clay Mathematics Institute as one of the one-million-dollar Millennium Prize Problems. The role of the vortex-stretching term and its relationship both to the possible formation of singularities and the analysis of the Navier-Stokes equation is discussed in detail in [5]. In contrast, in two dimensions the absence of this term allows one to construct global solutions of the two-dimensional vorticity equation, even for initial data with very little regularity [8].

One respect in which the vorticity formulation of the fluid equations is less convenient than the velocity-pressure formulation is that the velocity still appears in equations (3) and (4) for the evolution of the vorticity. However, if we remember that the vorticity is the curl of a divergence-free vector field (i.e., the velocity), then we can recover the vorticity field from the velocity field via the Biot-Savart law, which inverts the curl, and which in the two-dimensional case on which we will focus from now on takes the form

$$\mathbf{v}(\mathbf{z}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{z} - \tilde{\mathbf{z}})^\perp}{|\mathbf{z} - \tilde{\mathbf{z}}|^2} \cdot \omega(\tilde{\mathbf{z}}, t) d\tilde{\mathbf{z}}\ . \tag{5}$$

Here, if $\mathbf{z} = (x, y) \in \mathbb{R}^2$, then $\mathbf{z}^\perp = (-y, x)$. Thus the velocity can be regarded as a linear, but nonlocal, function of the vorticity. With this point of view the two-dimensional vorticity equation (4) can be regarded as a nonlinear heat equation in which the nonlinear term is quadratic and nonlocal. As we will see later, this relationship with the heat equation will play an important role in understanding solutions of (4).

Let's now look more closely at equation (4) and try to understand the influence of various terms in the equation. First consider the case in which $\nu = 0$ (the *inviscid* case). In this case, if we pretend for the moment that the velocity field is given to us, rather than being determined by the vorticity, then the equation becomes simply a transport equation in which the vorticity is carried along by the velocity field. In reality, the situation is more complicated, because as the vorticity is advected by the velocity field, the velocity field itself changes in response to the changing vorticity; and in order to obtain an accurate model of the evolution of the vorticity, one must incorporate the "feedback" of the vortex motion on the velocity field.

Helmholtz, and then later and more systematically Kirchhoff, made the assumption that the vorticity could be written as a finite sum of point vortices (i.e., delta functions) whose positions moved in response to the velocity field they created. Note that the velocity field of a point vortex can be computed explicitly from the Biot-Savart law, and using this, Helmholtz and Kirchhoff could track the dynamical evolution of the velocity field in their model and account for the feedback the vortex motion creates. Thus, if one assumes that the vorticity field can be written as $\omega(\mathbf{z}, t) = \sum_{k=1}^N \Gamma_k \delta(\mathbf{z} - \mathbf{z}_j(t))$, where Γ_j is the strength of the j^{th} vortex and $\mathbf{z}_j(t)$ is its position and substitutes this into the $\nu = 0$ case of (4) (and interprets the solution in an appropriate weak sense—see [13] for details), then one finds explicit *ordinary* differential equations for the locations of the centers of the vortices. If one sets $\mathbf{z}_j(t) = (x_j(t), y_j(t))$, then one finds that

$$\begin{aligned} \dot{x}_j(t) &= -\frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{y_j - y_k}{|\mathbf{z}_j - \mathbf{z}_k|^2}\ , \\ \dot{y}_j(t) &= \frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{x_j - x_k}{|\mathbf{z}_j - \mathbf{z}_k|^2}. \end{aligned} \tag{6}$$

These equations have explicit solutions for a number of simple arrangements of small numbers of vortices. For instance, one can easily check that two vortices of equal strength will move on a circle about the point midway between them, while two vortices of opposite strength will move on parallel lines, in a direction perpendicular to the line joining them (see Figure 4).

The set of equations (6) turns out to have a number of remarkable properties [18]. For instance,

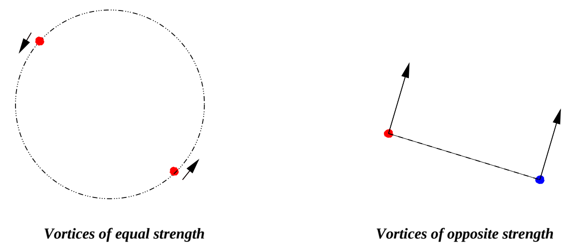

Figure 4. The dynamics of the centers of a pair of point vortices of equal strength and of opposite strength.

it is a Hamiltonian system in which the Hamiltonian is proportional to the sum of the logarithm of the distance between pairs of vortices; furthermore, if the number of vortices is three or less, it is a completely integrable Hamiltonian system. However, if one has four or more vortices, one typically finds chaotic solutions.

In spite of the special properties of the point-vortex equations, analytic solution for general initial data becomes quickly impossible for more than a small number of vortices. (An exception to this general rule are the equilibrium and relative equilibrium solutions, for which there are interesting connections with analogous solutions in the N-body problem in celestial mechanics [1].) However, given the Hamiltonian nature of the equations of motion and the chaotic nature of their solutions for large numbers of vortices, it is natural (at least in retrospect) to attempt to understand the behavior of large collections of vortices with the aid of statistical mechanics. Thus, if one considers the initial vorticity distribution in Figure 3, one can imagine it as a "gas" of point vortices that interact with the other vortices through a potential energy function in which the energy of a vortex pair is proportional to the logarithm of the distance between them. Lars Onsager may have been the first person to adopt this point of view, and it led him to a remarkable conclusion [7]. Onsager found that the statistical mechanical description of a collection of point vortices moving according to the equations of motion (6) could support states in which a parameter analogous to the absolute temperature in a traditional statistical mechanical system was actually negative. Furthermore, Onsager realized that a consequence of these negative temperature states was that vortices of like sign would coalesce and that this could explain the tendency observed in Figure 3 for large vortices of each sign to form as the fluid evolves. As Onsager himself put it ([19], quoted in [7])

> It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion.

Thus Onsager had provided a means of explaining how large vortices could form from random collections of small vortices, provided the effects of viscosity are ignored and assuming that the hypotheses that underlie the theory of statistical mechanics are satisfied.

It is not only in the theoretical understanding of fluid flows that the Helmholtz-Kirchhoff point vortex model has played an important role. Even though one cannot solve the equations (6) analytically for more than a few vortices, they are perfectly amenable to numerical solution, and this idea has formed the basis of "vortex methods" or "meshless methods" in computational fluid mechanics. In this approach, one first approximates the initial distribution of vorticity by a collection of point vortices (or, more frequently in numerical approaches, smoothed vortices with finite size cores). The key quantities are the location and strength of each of the vortices. The vortex strength is typically conserved, and thus the evolution of the fluid can be tracked by following the locations of the centers of the vortices via a system of ordinary differential equations such as (6). It can be shown that vortex methods give convergent approximations (as the number of points used in the approximation tends to infinity) to *inviscid* fluid flows (though sometimes with relatively slow convergence rate). However, one problem that can arise is related to Onsager's observation that in a large collection of vortices, those of like sign will tend to clump together. Thus, after some time, large parts of the computational domain may have only a very few vortices, which leads to a loss of information about the flow in these regions. For a further discussion of the advantages and disadvantages of using vortex methods to numerically approximate two-dimensional flows see the recent monograph of Majda and Bertozzi [12], while [2] contains a survey of recent improvements in the vortex method, with a particular focus on how one can incorporate viscous effects in the method.

Thus far, we have mainly discussed the limiting case of the Navier-Stokes equation in which the viscosity is zero. In realistic fluids (with the spectacular exception of super fluids) the viscosity may be small but is never zero, so we next consider its effects on the preceding scenarios. One can show that for finite (sometimes short) times, the solution of the weakly viscous Navier-Stokes equation with appropriate initial data is well approximated by solutions of the point-vortex model [13]. However, these short-time results cannot provide insight into the long-time phenomena that occur in the "inverse cascade". Indeed, we'll see that even within the long-time regime there are two distinct time scales, one on which the inviscid phenomena predicted by Onsager appear and a second, typically longer, time scale over which viscous effects manifest

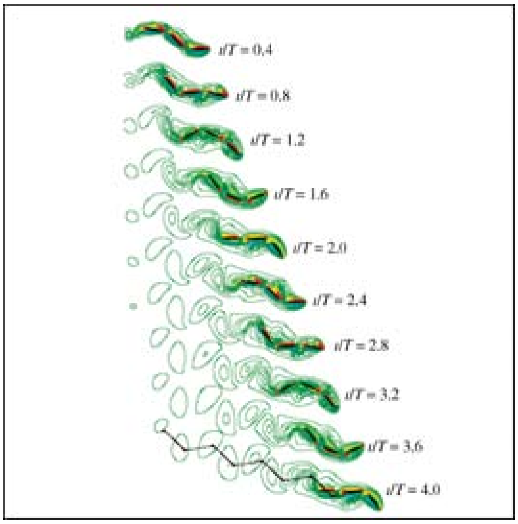

Figure 5. The vorticity field around a swimming "fish", computed with a modern vortex method [6]

themselves and which we show below lead to the formation of a single large vortex in the system.

If we ignore the nonlinear term and focus on the linear terms in the equation, we just find the two-dimensional heat equation

$$\partial_t \omega = \nu \Delta \omega \, . \tag{7}$$

In this equation we know that the effects of the Laplacian term are to spread things out—or diffuse them—at a rate proportional to ν. Indeed, if we were to take a point vortex, i.e., a Dirac δ-mass, as an initial condition, we know that the effect of the equation is just to "smear" this point out into a Gaussian. More precisely, if we take an initial condition $\omega(\mathbf{z},0) = \alpha\delta(\mathbf{z})$ for (4), and we ignore the nonlinear term in the equation, then we find immediately that the resulting solution is

$$\omega(\mathbf{z},t) = \frac{\alpha}{4\pi\nu t} e^{-|\mathbf{z}|^2/(4\nu t)} \, . \tag{8}$$

Remarkably, this explicit Gaussian turns out to be an exact solution not only of the linear heat equation but also of the nonlinear two-dimensional vorticity equation, known as the Oseen-Lamb vortex. To see the reason for this, recall that since the fluid is incompressible, its velocity field is divergence free. In this case the Helmholtz decomposition implies the existence of a stream function $\psi(\mathbf{z})$ such that $\mathbf{v}(\mathbf{z}) = (\partial_y\psi, -\partial_x\psi)$. Thus the vorticity is related to the stream function via the equation

$$\omega(\mathbf{z},t) = \partial_x \mathbf{v}_2 - \partial_y \mathbf{v}_1 = -\Delta\psi(\mathbf{z},t). \tag{9}$$

In the present example of the Oseen-Lamb vortex, in which the vorticity is a purely radial function depending only on $|z|$, solving Poisson's equation will give a stream function that is also purely radial and that in turn gives rise to a purely tangential velocity field. Since the nonlinear term in the vorticity equation consists of the dot product of the velocity field with the gradient of the vorticity, we will have the dot product of a tangential with a radial vector; i.e., the value of the nonlinear term is zero when evaluated on the velocity and vorticity fields of the Oseen-Lamb vortex. In this case the vorticity equation reduces to the heat equation, which, as we have already remarked, is solved by the Oseen-Lamb vortex.

Note that in the expression for the vorticity field of the Oseen vortex the space and time variables are linked in a special fashion. This suggests that it may be convenient to study the vorticity equation in new variables, so-called *scaling variables.* With this in mind we define new dependent and independent variables through the change of variables

$$\omega(x,y,t) = \frac{1}{1+t} w\left(\frac{x}{\sqrt{1+t}}, \frac{y}{\sqrt{1+t}}, \log(1+t)\right). \tag{10}$$

If we define $\xi = \frac{x}{\sqrt{1+t}}$, $\eta = \frac{y}{\sqrt{1+t}}$, $\boldsymbol{\zeta} = (\xi,\eta)$, and $\tau = \log(1+t)$, then in terms of these new variables the two-dimensional vorticity equation takes the form

$$\partial_t w = \nu\Delta w + \frac{1}{2}\nabla\cdot(\boldsymbol{\zeta} w) - \mathbf{u}\cdot\nabla w \equiv \mathcal{L}^\nu w - \mathbf{u}\cdot\nabla w, \tag{11}$$

where $\mathbf{u}$ is just the velocity field, rewritten in terms of the scaling variables, and the derivatives in the Laplacian and divergence are now taken with respect to $\boldsymbol{\zeta}$ instead of $\mathbf{z}$.

Note that in terms of these variables, the family of Oseen vortices are all *fixed points* of this equation, i.e., the functions

$$\begin{aligned} \omega(\xi,\eta) &= \Omega^\alpha(\xi,\eta) = \frac{\alpha}{4\pi} e^{-|\zeta|^2/4\nu} \\ &= \frac{\alpha}{4\pi} e^{-(\xi^2+\eta^2)/4\nu} \end{aligned} \tag{12}$$

are all stationary solutions for equation (11). Note that we have normalized the Gaussian so that the parameter α of the Oseen vortex gives the total vorticity (i.e., the integral of the vorticity) of the solution. Note further that both the vorticity equation, (4), and the rescaled vorticity equation, (11), conserve the total vorticity. The presence of this family of fixed points suggests that we might be able to use ideas from dynamical systems theory to study the stability of these fixed points and to try to determine whether they can explain the asymptotic behavior of some or all of the solutions of this partial differential equation.

There are (at least) two different dynamical systems approaches that can be applied to study the stability and influence on the asymptotics of the fixed points (12); namely:

- Linearization about the fixed point and the construction of invariant manifolds in the phase space corresponding to the various spectral subspaces of the linearization, or
- Lyapunov functionals.

In this article I'll focus on the Lyapunov functional approach because of its relationship to the statistical mechanics point of view that we have discussed above. However, one can also use the invariant manifold approach to analyze the asymptotic behavior of solutions of (11). In contrast to the Laplacian, whose spectrum is the entire negative real axis, the linear operator, $\mathcal{L}^{\nu}$, when considered on spaces of functions that go to zero rapidly at infinity, has a spectrum that consists of a set of discrete eigenvalues, plus a half-plane of essential spectrum. One can construct finite-dimensional, invariant manifolds corresponding to the span of the eigenfunctions of these eigenvalues that describe very precisely the asymptotics of small solutions of (11) [9]. Moreover, the eigenfunctions corresponding to these discrete eigenvalues form a convenient basis with respect to which one can systematically extend the Helmholtz-Kirchhoff point-vortex model described above to include the effects of viscosity and finite core size [16].

Recall that a Lyapunov functional for a dynamical system is a continuous function, bounded below on the phase space of the dynamical system, which when evaluated on an orbit of the dynamical system is monotonic nonincreasing and bounded below. (Very colloquially, it is a function whose value decreases with time when evaluated along solutions of the dynamical system.) Recall that our goal is to understand how the vorticity forms large structures after very long times. One way of characterizing the long-time asymptotics of a dynamical system is through the *ω-limit* set, which is the set of points in the phase space which a trajectory approaches arbitrarily closely to as time tends to infinity. If the trajectory approaches a stable fixed point, then the ω-limit set will consist just of this fixed point. However, the ω-limit set can also be a periodic orbit or even some chaotic attractor. Note that it is not immediately apparent that every trajectory will have an ω-limit. This follows if the solutions of the dynamical system satisfy certain compactness conditions. For the remainder of the article we will assume that the solutions of the vorticity equation satisfy these conditions, though proving this takes some work—the details are explained in [10].

A key tool in locating the ω-limit set is the LaSalle Invariance Principle, which says that given a Lyapunov functional for a dynamical system, the ω-limit set of a trajectory must lie in the set on which the Lyapunov function is constant (when evaluated along an orbit). More precisely, if the points in the phase space of the dynamical system are denoted by w, if the flow or semi-flow defined by the dynamical system is denoted by Φ^t, and if the Lyapunov functional is denoted by $H(w)$ (and it is differentiable), then the ω-limit set must lie in the set of points

$$E = \{w \mid \frac{d}{dt}H(\Phi^t(w))|_{t=0} = 0\}. \tag{13}$$

Let's now return to rescaled vorticity equation (11) and make use of one more analogy. So far, we have considered the equation in which we ignored the dissipative term and retained only the time derivative and the nonlinearity, and we have also ignored the nonlinear term and retained only the dissipative term. Let's finally retain all the terms in (11) but ignore for the moment the fact that the velocity field is linked to the vorticity and pretend that it is just some given, divergence-free vector field. In this case if we use the fact that $\nabla \cdot \mathbf{u} = 0$, we can write (11) as

$$\partial_\tau w = \nu \Delta w - \nabla \cdot (w \nabla \mathbf{U}) , \tag{14}$$

where $\nabla \mathbf{U}(\boldsymbol{\zeta}, \tau) = \mathbf{u}(\boldsymbol{\zeta}, \tau) - \frac{1}{2}\boldsymbol{\zeta}$. This equation is just the Fokker-Planck equation, which describes the evolution of the probability distribution of the location of a particle in a gas confined by the potential $\mathbf{U}$ and with diffusive effects modeled by the term $\nu \Delta w$. Equation (14) has been studied extensively by physicists and mathematicians for more than a century, and, in particular, motivated by Boltzmann's theory that the entropy of such a system should never decrease, Lyapunov functionals have been developed that are based on the entropy. (See [14] for a nice discussion of the interplay between physics and analysis in this problem.)

The classical entropy function for solutions of (11) would be $S[w](t) = \int_{\mathbb{R}^2} w(\boldsymbol{\zeta}) \ln w(\boldsymbol{\zeta}) d\boldsymbol{\zeta}$, but as explained in [14] it is often more useful to study the *relative entropy*, that is, the entropy relative to the expected asymptotic state of the system. In this case our candidate for the asymptotic state of the system is one of the Oseen vortices Ω^α defined in (12), which results in a relative entropy functional

$$H[w](\tau) = \int_{\mathbb{R}^2} w(\boldsymbol{\zeta}, \tau) \ln \left(\frac{w(\boldsymbol{\zeta}, \tau)}{\Omega^\alpha(\boldsymbol{\zeta})} \right) d\boldsymbol{\zeta}. \tag{15}$$

Of course, so far, there is no proof that this is a Lyapunov functional for the (rescaled) vorticity equation. It is a candidate, suggested by the analogy between (11) and the Fokker-Planck equation, but in the actual vorticity equation the fluid velocity is not independent of the vorticity (and in particular the vorticity equation, unlike the Fokker-Planck equation, is a nonlinear equation). Furthermore, a second problem is apparent from formula (15). Since solutions of the Fokker-Planck equation represent probability densities, it is natural to assume that they are nonnegative, and consequently the logarithm in (15) is well defined. However, it is quite unnatural to assume that solutions of the vorticity equation are all of one sign—typical solutions intermingle regions of positive vorticity with regions of negative vorticity, and for such solutions it becomes impossible to define the relative entropy functional.

We'll return to the second of these problems in a moment, but first consider whether or not (15) at least defines a Lyapunov functional for solutions of the vorticity equation *which are* everywhere positive. As we'll explain below, if the solution of the vorticity equation is positive at some instant of time, it will remain so for all later times. Assuming that the solution is positive for some time t and that the vorticity tends to zero sufficiently rapidly as $|\zeta| \to \infty$ so that the integral in (15) converges, we can compute the derivative of the entropy, and we find:

(16)
$$\frac{d}{d\tau}H[w](\tau) = \int_{\mathbb{R}^2} \left(1 + \ln\left(\frac{w(\zeta,\tau)}{\Omega^\alpha(\zeta)}\right)\right) \frac{\partial w}{\partial \tau} d\zeta.$$

If we now use (11) to rewrite the time derivative of the vorticity and then integrate by parts (and use the relationship between vorticity and velocity), we find

$$\frac{d}{d\tau}H[w](\tau) = -\int_{\mathbb{R}^2} w(\zeta) \left|\nabla\left(\frac{w}{\Omega^\alpha}\right)\right|^2 d\zeta. \tag{17}$$

Since w is positive, (17) shows that for such solutions the relative entropy function is a Lyapunov function. One can also show H is bounded below for w in an appropriate function space (and that the compactness properties mentioned above also hold), and hence the LaSalle Invariance Principle holds. This means that the ω-limit set for any positive solution of the vorticity equation must lie in the set where $\frac{d}{d\tau}H[w](\tau) = 0$. Since $w(\zeta, t) > 0$, (17) implies that the only way the time derivative of the entropy can vanish is if

$$\left(\frac{w}{\Omega^\alpha}\right) = \text{constant}; \tag{18}$$

i.e., if the solution is an Oseen vortex. This means that for positive solutions of the vorticity equation the only possible asymptotic states are the Oseen vortices.

As we remarked above, the assumption that solutions are positive is *a priori* a very unnatural one for the vorticity equation, and thus we now turn to the question of how to treat solutions that change sign. To deal with such solutions, we need another Lyapunov functional. The clue to finding this second Lyapunov function is the observation we made earlier about the similarity of the two-dimensional vorticity equation (4) to a nonlinear heat equation. Closer inspection shows that, just like the heat equation, solutions of (4) satisfy a maximum principle. In particular:

- A solution that is positive (for all z) for some time t_0 will remain positive for any later time $t > t_0$, and
- If the initial condition for the vorticity equation satisfies $\omega(\mathbf{z}, 0) \geq 0$ for all z, then the solution will be strictly positive for all times $t > 0$.

Note that these remarks also hold for solutions of the rescaled vorticity equation (11). As a consequence of these two observations, we find a second, surprisingly simple, Lyapunov functional, namely the $L^1(\mathbb{R}^2)$-norm of the solution. Define

$$K[w](\tau) = \int_{\mathbb{R}^2} |w(\zeta,\tau)| d\zeta. \tag{19}$$

To see that this is a Lyapunov functional, split the solution of (11) into its positive and negative parts, i.e., write $w(\xi,\eta,\tau) = w^+(\xi,\eta,\tau) - w^-(\xi,\eta,\tau)$, where $w^\pm$ are both nonnegative and satisfy the equations:

$$\partial_\tau w^\pm = \mathcal{L}^\nu w^\pm - \mathbf{u}\cdot\nabla w^\pm. \tag{20}$$

Here, $\mathbf{u}$ is the total velocity field—i.e., the velocity field associated with $w(\zeta,\tau)$ rather than that associated with either w^+ or w^-. We note that one can show (by undoing the change to scaling variables) that solutions of (20) also satisfy two properties listed above. If we choose initial conditions for $w^\pm$ so that $w^+(\zeta,0) = \sup(w(\zeta,0),0)$ and $w^-(\zeta,0) = -\inf(w(\zeta,0),0)$, then $w^\pm|_{\tau=0}$ are both nonnegative and have disjoint support. By the maximum principle, if $\omega^\pm|_{\tau=0} \not\equiv 0$, both $w^\pm$ will be strictly positive for all positive times. From this we find that

$$\begin{aligned} K[w](\tau) &= \int_{\mathbb{R}^2} |w^+(\zeta,\tau) - w^-(\zeta,\tau)| d\zeta \\ &\leq \int_{\mathbb{R}^2} w^+(\zeta,\tau)d\zeta + \int_{\mathbb{R}^2} w^-(\zeta,\tau)d\zeta \\ &= \int_{\mathbb{R}^2} w^+(\zeta,0)d\zeta + \int_{\mathbb{R}^2} w^-(\xi,\eta,0)d\zeta \\ &= \int_{\mathbb{R}^2} |w^+(\zeta,0)d\zeta - w^+(\zeta,0)|d\zeta \\ &= K[w](0)\ , \end{aligned} \tag{21}$$

where the equality in the middle of (21) follows from the fact that solutions of (20) conserve "mass" (i.e., the integral of the solution), as do solutions of the vorticity equation. From (21) we see that K is a Lyapunov functional for solutions of the vorticity equation. Furthermore, recalling that the maximum principle implies that both $w^\pm$ are strictly positive, we see that the first inequality in (21) will be a strict inequality unless either w^+ or w^- is identically zero. Thus the Lyapunov functional K is strictly decreasing except on the set of functions that is either strictly positive or strictly negative, and, appealing again to the LaSalle invariance principle, we see that the ω-limit set of solutions must lie in the set where K is not strictly decreasing—i.e., in the set of either everywhere positive or everywhere negative solutions.

If we now put together our two Lyapunov functionals, we have the following conclusion, namely, for general solutions, the Lyapunov functional K implies that the ω-limit set must lie in the set of solutions of one sign. However, for solutions of one sign, the relative entropy functional, H, implies that the ω limit set must be an Oseen vortex. So far, we have been somewhat vague about the function space on which we work, but in fact these results hold for any solution whose initial vorticity is absolutely integrable (for this and other technical details we refer the reader to [10]) and thus we have

Theorem 1.1. *Any solution of the two-dimensional vorticity equation whose initial vorticity is in $L^1(\mathbb{R}^2)$ and whose total vorticity $\int_{\mathbb{R}^2} \omega(\mathbf{z}, 0) d\mathbf{z} \neq 0$ will tend, as time tends to infinity, to the Oseen vortex with parameter $\alpha = \int_{\mathbb{R}^2} \omega(\mathbf{z}, 0) d\mathbf{z}$.*

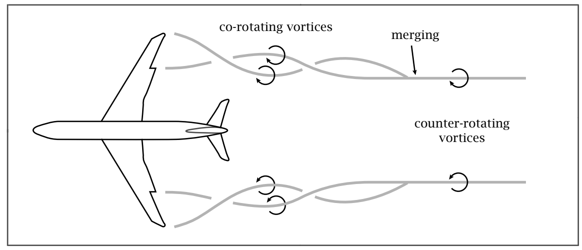

Figure 6. The merger of wingtip vortices behind an airplane (from [15]). Although the flow is obviously three-dimensional, much of the process of vortex merger can be understood by considering cross-sections of the flow as if they were two-dimensional vortices.

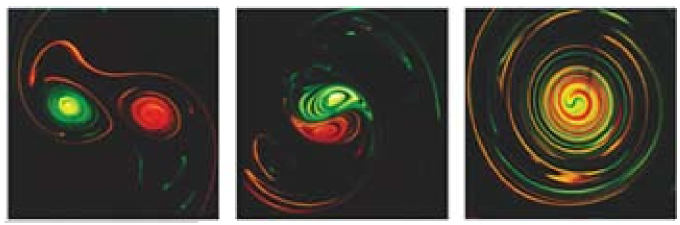

Figure 7. An experimental dye visualization of the merger of two-dimensional vortices, from [15]

Extensions and Conclusions

Theorem 1.1 implies that with even the slightest amount of viscosity present, two-dimensional fluid flows will eventually approach a single large vortex. However, if the viscosity is small, this convergence may take a very long time. Furthermore, Onsager's original calculations of vortex coalescence were for an inviscid fluid model, which suggests that some sort of coalescence should occur independent of the viscosity—and, in particular, on a time scale that does not depend on the viscosity. Thus, while Theorem 1.1 says that eventually all two-dimensional viscous flows will approach an Oseen vortex, there should be a variety of interesting and important behaviors that manifest themselves in the fluid prior to reaching the asymptotic state described in the theorem.

One of the most important physical effects, and one of the hardest to understand from a mathematical point of view, concerns the merger of two or more vortices. Clearly such mergers must take place in order for the multitude of small vortices present in an initially turbulent flow to coalesce into the small number of large vortices predicted by Onsager. Furthermore, as discussed in [15], this process plays a key role in many nonturbulent flows such as the wingtip vortices that form behind an airplane wing, as illustrated in Figure 6. As the authors of [15] explain, although the phenomenon is obviously a three-dimensional one, and three-dimensional effects undoubtedly influence the details of the flow, the two-dimensional dynamics "contain all the ingredients necessary to explore and understand the physics involved in vortex merging". While the Oseen vortex that characterizes the long-time asymptotics of the flow has the property that the effects of the nonlinear terms in the vorticity equation vanish, both numerical and experimental studies show that the merger process is highly nonlinear and involves the filamentation and interpenetration of the two vortices into one another, as shown in Figure 7. While physically based criteria exist to predict when merger will occur, a rigorous mathematical understanding of this phenomenon is so far almost completely absent.

A second interesting phenomenon that is particularly noticeable in the numerical simulations of two-dimensional flows on bounded domains is the creation and persistence of metastable structures. For square domains with periodic boundary conditions, the total vorticity is forced to be zero, and as a consequence the asymptotic state is the zero state. Nonetheless, a number of different, very long-lived, metastable states are observed [22]. The origin and properties of these states in the two-dimensional Navier-Stokes equation is still not understood, but statistical mechanical ideas have again been used to propose an explanation associated with the different time scales on which energy and entropy are dissipated [4]. Similar metastable phenomena also occur in the weakly viscous Burgers' equation, which is often used as a simplified testing ground for understanding the Navier-Stokes equations. There, the long-time asymptotics are again governed by a family of self-similar states, analogous to the Oseen vortices in the two-dimensional Navier-Stokes equations. However, for very long times (exponentially long in the reciprocal of the viscosity!), one observes not the self-similar state but rather a special family of solutions known as "diffusive N-waves" [11], [3]. As in the expected behavior of the two-dimensional Navier-Stokes equation, the metastable states in Burgers' equation are closely related to the N-waves of the inviscid equation, while the self-similar asymptotic state depends crucially on the presence of dissipation in the systems. Because of the simpler nature of Burgers' equation, one can show that the metastable states form a one-dimensional attractive invariant manifold in the phase space of the equation, and one can speculate that a similar dynamical systems explanation might account for the metastable behavior observed in the two-dimensional Navier-Stokes equation, as it has for the long-time asymptotics of solutions.

In summary, two-dimensional fluid motions present interesting differences with three-dimensional fluids from both the mathematical and physical points of view. In spite of the fact that we live in a three-dimensional world, in many situations a two-dimensional fluid model is appropriate. One important situation in which this is the case and for which a two-dimensional fluid model is often used is the earth's atmosphere. In two dimensions, it is particularly convenient to study the evolution of the vorticity, rather than work directly with the velocity field of the fluid. Ever since Helmholtz and Kirchhoff developed an ordinary differential equation model to describe the evolution of point vortices, dynamical systems ideas have played an important role in understanding the evolution of the vorticity in two-dimensional flows, a theme that continues to pay dividends to the present day.

A distinctive feature of two-dimensional flows is the "inverse cascade" of energy from small scales to large ones. Lars Onsager first sought to explain this phenomenon by studying the statistical mechanics of large collections of inviscid point vortices. While Onsager's observation about *inviscid* flows remains unexplained, dynamical systems ideas—in this case Lyapunov functionals inspired by kinetic theory—have been used to show that in the presence of an arbitrarily small amount of viscosity, essentially any two-dimensional flow whose initial vorticity field is absolutely integrable will evolve as time goes to infinity toward a single, explicitly computable vortex.

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