Dynamical Systems and the Two-dimensional Navier-Stokes Equations

C. E. Wayne

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Abstract

Two-dimensional fluid flows exhibit a variety of coherent structures such as vortices and dipoles which can often serve as organizing centers for the flow. These coherent structures can, in turn, sometimes be associated with the existence of special geometrical structures in the phase space of the equations and in these cases the evolution of the flow can be studied with the aid of dynamical systems theory.

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Introduction

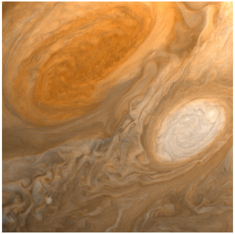
1. Understanding the long-time evolution of fluid motion is often facilitated by studying the coherent structures of the flow.
2. In physical flows, these structures are often vortices.
3. From a mathematical point of view these structures may be invariant manifolds in the phase space of the system.

Two-dimensional fluids

1. Although we live in a three dimensional world, many fluid flows behave in an essentially two-dimensional way.
   (a) In many physical circumstances (e.g. the ocean or the atmosphere), one dimension of the domain is much smaller than either the other two dimensions, or the dimensions of typical features of interest.
   (b) This effect is compounded by the effects of stratification and rotation.
2. There are fascinating physical and mathematical differences between two and three dimensional flows.

Two-dimensional fluids

What are some typical phenomena in two dimensional fluids?



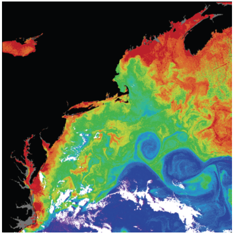


Figure: A variety of atmospheric and oceanic vortices. (All images from NASA)

Three-dimensional turbulence

This is in marked contrast to three-dimensional systems where the vorticity field tends to concentrate in small space time structures:

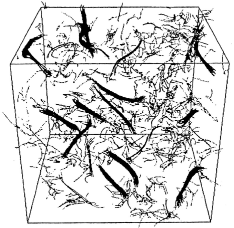


Figure: Numerical simulation of a 3D flow from She, Jackson, and Orzag; Proc. R. Soc. London, Ser. A 434101-124 (1991)

Note that the vorticity is concentrated in very small filaments.

Two-dimensional vortices

One our goals is to explain the characteristic tendency of two-dimensional flows to form large vortices regardless of the initial state of the fluid.

- This is in marked contrast to three-dimensional fluids where energy flows from large scales to small scales.
- This is an example of the "inverse cascade" of energy in two-dimensional fluids.

How do we characterize vortices?

The Navier-Stokes Equations

A system of nonlinear partial differential equations which describe the motion of a viscous, incompressible fluid.

If $\mathbf{u}(x,t)$ describes the velocity of the fluid at the point x and time t then the evolution of $\mathbf{u}$ is described by:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p\ , \quad \nabla\cdot\mathbf{u} = 0\ ,$$

The first of these equations is basically Newton's Law; $F = ma$ while the second just enforces the fact that the fluid is incompressible.

Vorticity

The velocity of the fluid is not the best way to visualize or characterize vortices, however, for that it is better to use the vorticity!

Roughly speaking, the vorticity describes how much "swirl" there is in the fluid.

$$\boldsymbol{\omega}(x,t) = \nabla \times \mathbf{u}(x,t)$$

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$$\boldsymbol{\omega}(x,t) = \nabla \times \mathbf{u}(x,t)$$

Note that in general, the vorticity is a vector quantity, but for two-dimensional fluid flows, $\mathbf{u}(x,t) = (u_1(x,y), u_2(x,y), 0)$, so

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = (0, 0, \partial_x u_2 - \partial_y u_1) \ .$$

Thus, in two dimensions we can treat the vorticity as a scalar!

The Vorticity Equation

To find out how the vorticity evolves in time we can take the curl of the Navier-Stokes equation. We find quite different equations, depending on whether we are in two or three dimensions. In three dimensions one has the systems of equations

$$\partial_t \boldsymbol{\omega}(x,t) - \boldsymbol{\omega} \cdot \nabla \mathbf{v}(x,t) + \mathbf{v} \cdot \nabla \boldsymbol{\omega}(x,t) = \nu \Delta \boldsymbol{\omega}(x,t)$$

while in two dimensions one has only the single, scalar equation

$$\partial_t \omega(x,t) + \mathbf{v} \cdot \nabla \omega(x,t) = \nu \Delta \omega(x,t) \ .$$

Vortex Stretching

The presence of the “vortex stretching” term

$$-\boldsymbol{\omega} \cdot \nabla \mathbf{v}$$

in the three-dimensional equation is a crucial physical as well as mathematical difference - it is literally the million dollar term. Because of its presence it is not known whether or not solutions of the three-dimensional Navier-Stokes even exist for all time.

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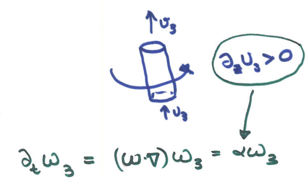
Why is this such a hard problem?

Physically, there exist mechanisms which *could* lead the solution to blow up in a finite time.

Vortex Stretching

Imagine a small cylinder of fluid in which the vorticity is pointed upwards and the vertical component of the velocity is larger at the bottom than the top:



Note that such a leads to a positive coefficient in front of the third component of the vorticity on the RHS of this equation, and this in turn leads to a positive feedback which causes this component of the vorticity to grow, so that our cylinder of fluid is stretched. When the cylinder of fluid is stretched, it gets thinner (by conservation of mass), but then by conservation of angular momentum, it must spin faster, so the vorticity increases.

The two dimensional vorticity equation

For the remainder of the lecture I'll focus on the two-dimesional vorticity equation

$$\partial_t \omega(x,t) + \mathbf{v} \cdot \nabla \omega(x,t) = \nu \Delta \omega(x,t) \ .$$

For this equation, proving the existence and uniqueness of solutions is possible even for initial vorticity distributions that have little regularity.
A complicating factor is the presence of the velocity field in the equation for the vorticity:

1. One can recover the velocity field from the vorticity via the Biot-Savart operator - a linear, but nonlocal, operator.
2. As a consequence, we can think of the two-dimensional vorticity equation as the heat equation, perturbed by a quadratic nonlinear term.

Vortex formation in two-dimensional fluids

Let's begin by looking at typical phenomena present in solutions of the two-dimensional vorticity equation - or at least in the numerical approximation of solutions of this equation.

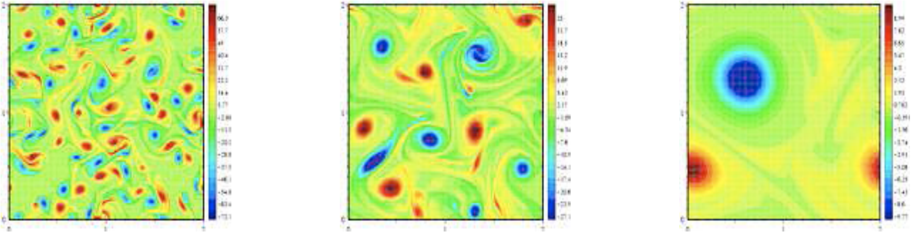


Figure: A numerical simulation of a two-dimensional turbulent flow. The figures display the vorticity field (with blue and red representing fluid "swirling" in opposite directions) at successively later and later times and clearly indicate the tendency of regions of vorticity of like sign to coalesce into a smaller and smaller number of larger vortices. From the Technical University of Eindhoven; Fluid mechanics lab

Emergence of Vortices

Our goal will be to try and understand the emergence and stability of these large vortices from very general initial conditions for two-dimensional flows – – or more poetically,

> *When little whirls meet little whirls,*
> *they show a strong affection;*
> *elope, or form a bigger whirl,*
> *and so on by advection.*

This is quoted without attribution on

`http://www.fluid.tue.nl/WDY/vort/2Dturb/2Dturb.html`

The 2D Vorticity Equation

Let's see what insight we can obtain into the behavior of the 2D vorticity equation by considering two different limiting cases:

$$\partial_t \omega(x,t) + \mathbf{v} \cdot \nabla \omega(x,t) = \nu \Delta \omega(x,t) \ .$$

1. First limiting case - ignore the dissipative term:

$$\partial_t \omega(x,t) + \mathbf{v} \cdot \nabla \omega(x,t) = 0 \ .$$

   This is known as Euler's equation - but note that if we "forget" that the velocity is in fact determined by the vorticity, it is just the transport equation which says that the vorticity is carried along by the background velocity field.

2. Second limiting case - ignore the nonlinear term:

$$\partial_t \omega(x,t) = \nu \Delta \omega(x,t) \ .$$

   In this case we just have the heat equation.

The point vortex model

Helmholtz and Kirchhoff studied the equation without dissipation and assumed that the vorticity could be written as a sum of finitely many point vortices (... not always a good assumption, but let's see where it leads ...)

In this case, the vortices are just swept along by the velocity field - however, the velocity field itself must respond to the alteration in the vorticity field caused by the motion of the vortices.

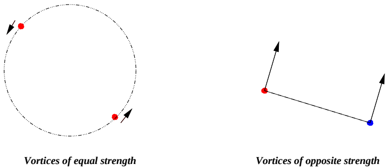
It turns out that one can compute this response and one finds a simple and explicit system of equations for the motion of the centers $z_j = (x_j, y_j)$ of the vortices:

$$\dot{x}_j(t) = -\frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{y_j - y_k}{|z_j - z_k|^2} \ , \quad \dot{y}_j(t) = \frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{x_j - x_k}{|z_j - z_k|^2}$$

The Helmholtz-Kirchhoff model

The Helmholtz-Kirchoff model has a number of remarkable properties

1. It is a Hamiltonian system.
2. It is completely integrable for two or three vortices.



3. Four or more vortices typically form a chaotic system and analytic solution of the H-K equations becomes impossible for more than a small number of vortices.

Point vortices and Hurricanes

Although point vortices may seem like mathematical abstractions, they appear to give remarkably good approximations to real world systems, like .... hurricanes!.

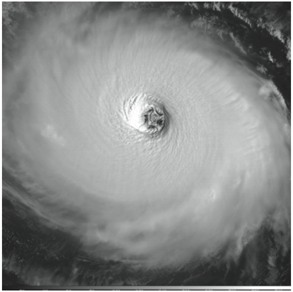


Figure: The core of Hurricane Isabel, from Kossin, James P., Wayne H. Schubert, (2001) J. Atmos. Sci., 58, 21962209.

(See: "Vortex Crystals in Fluids" by Anna Barry.)

Onsager's idea

Given the Hamiltonian nature of the equations of motion and the chaotic nature of their solutions for large numbers of vortices, it is natural (at least in retrospect) to attempt to understand the behavior of large collections of vortices with the aid of statistical mechanics.

Lars Onsager seems to have been the first person to adopt this point of view and it lead him to a remarkable conclusion.

- Onsager found that the statistical mechanical description of a collection of point vortices moving according to the H-K equations could support states of negative absolute temperature.
- He then realized that a consequence of these negative temperature states was that vortices of like sign would tend to attract each other and that this could explain the tendency of large vortices to form, regardless of the initial conditions.

Drawbacks

The limitation of Onsager's idea is that even now, sixty years after Onsager first proposed this method of explaining the formation of large vortices, we have no idea of whether or not the hypotheses that underly the theory of statistical mechanics are actually satisfied by the dynamical system defined by the H-K equations.

There is also the problem that the H-K model itself applies only to an idealization in which

1. The vorticity of the fluid is approximated by a sum of delta-functions.
2. The viscosity is assumed to be zero.

The heat equation

Let's turn to the opposite extreme, in which we ignore the nonlinear term and focus just on the linear terms in the vorticity equation - this yields the heat equation:

$$\partial_t \omega(x,t) = \nu \Delta \omega(x,t) \ .$$

If we assume again that the initial vorticity is concentrated in a delta-function, it will not remain a point vortex - the viscosity will cause it to spread with time. In fact, if we assume that the initial vorticity is given by

$$\omega(z,0) = \alpha \delta(z)$$

the solution at later times is found to be

$$\omega(z,t) = \frac{\alpha}{4\pi\nu t} e^{-|z|^2/(4\nu t)} \ .$$

Oseen vortices

Remarkably, this explicit Gaussian turns out to be an exact solution of the full, 2D vorticity equation, not just the linear approximation.

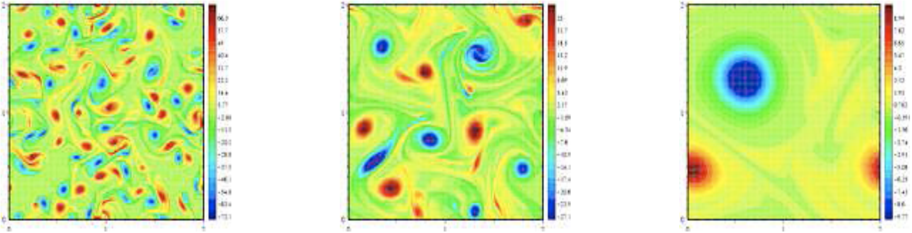
1. Note that the Gaussian solution corresponds to a vorticity distribution that depends only on the radial variable.
2. Inserting this into the Biot-Savart law yields a purely tangential velocity field.
3. This combination insures that the nonlinear term in the vorticity equation

$$\mathbf{v} \cdot \nabla \omega = 0$$

Thus, the Gaussian vorticity profile is an exact solution of the 2D vorticity equation known as the Oseen-Lamb vortex.

Oseen vortices (cont)

Recall that in the numerical simulations we considered earlier, the system seemed to tend to a small number of large vortices which increase in size with time:



Scaling variables

Note that the formula for the Oseen vortices shows that the size of the vortex increases with time (like $\sqrt{t}$). This is consistent with the simulations we looked at above and suggests that the analysis of these vortices may be more natural in rescaled coordinates. With this in mind we introduce "scaling variables" or "similarity variables":

$$\xi = \frac{\mathbf{x}}{\sqrt{1+t}} \; , \qquad \tau = \log(1+t) \; .$$

Scaling variables (cont.)

Also rescale the dependent variables. If $\omega(\mathbf{x}, t)$ is a solution of the vorticity equation and if $\mathbf{v}(t)$ is the corresponding velocity field, we introduce new functions $w(\xi, \tau)$, $\mathbf{u}(\xi, \tau)$ by

$$\omega(\mathbf{x}, t) = \frac{1}{1+t} w\left(\frac{\mathbf{x}}{\sqrt{1+t}}, \log(1+t)\right) ,$$

and analogously for $\mathbf{u}$.

Scaling variables (cont.)

In terms of these new variables the vorticity equation becomes

$$\partial_\tau w = \mathcal{L}w - (\mathbf{u} \cdot \nabla_\xi) w \ ,$$

where

$$\mathcal{L}w = \Delta_\xi w + \frac{1}{2}\xi \cdot \nabla_\xi w + w$$

Note that the Oseen vortices take the form

$$W^A(\xi, \tau) = AG(\xi) = \frac{A}{4\pi} e^{-\frac{\xi^2}{4}} \ ,$$

in these new variables. Thus, they are fixed points of the vorticity equation in this formulation.

Dynamical Systems

It is natural to inquire whether or not these fixed points are stable. It turns out (somewhat remarkably) that they are actually globally stable. Any solution of the two-dimensional vorticity equation whose initial velocity is integrable will approach one of these Oseen vortices.

There are (at least) two approaches that we could use to study the stability of these vortex solutions:

- A local approach, based linearization about the fixed point.
- A global approach based on Lyapunov functionals.

Global Stability

Recall that a Lyapunov function is a function that decreases along solutions of our dynamical system. In the present case it will be a functional of the vorticity field $w(\xi, \tau)$ which is monotonic non-increasing as a function of time.

We'll look for the ω-limit set of solutions of the 2D vorticity equation.

1. Describes the long-time behavior of solutions.
2. Can be a fixed point, periodic orbit, or even a chaotic attractor.
3. Always exists provided the system satisfies certain compactness properties.

LaSalle Invariance Principle

A key tool in determining the ω-limit set is the LaSalle Invariance Principle. - i.e. the ω-limit set of a trajectory must lie in the set on which the Lyapunov function is constant (when evaluated along an orbit). More precisely, if the points in the phase space of the dynamical system are denoted by w, if the flow, or semi-flow defined by the dyanamical system is denoted by Φ^t and if the Lyapunov functional is denoted by $H(w)$ (and it is differentiable), then the ω-limit set must lie in the set of points

$$E = \{w \mid \frac{d}{dt} H(\Phi^t(w))|_{t=0} = 0\} \tag{1}$$

The Lyapunov functionals

We choose two Lyapunov functions, each motivated by one of the two different points of view:

1. The H-K model, and Onsager's idea of treating it with statistical mechanics ideas, suggests a Lyapunov function based on the entropy.
2. The linearization which yields the heat equation suggests a Lyapunov function based on the maximum principle.

The (relative) entropy

The classical entropy function is $S[w](\tau) = \int_{\mathbb{R}^2} w(\xi,\tau) \ln w(\xi,\tau) d\xi$. However, this would typically be unbounded for the sorts of solutions we wish to consider. Thus, we study the relative entropy

$$H[w](\tau) = \int_{\mathbb{R}^2} w(\xi,\tau) \ln \left(\frac{w(\xi,\tau)}{AG(\xi)} \right) d\xi$$

where G is the Gaussian that describes the Oseen vortex.

Note that $H[w]$ is only defined for vorticity distributions which are everywhere positive. This is not a problem in statistical mechanics (where w would typically be a probability distribution) but it is a very unnatural restriction in fluid mechanics.

The relative entropy (cont)

To show that $H[w]$ is a Lyapunov function compute:

$$\frac{d}{d\tau} H[w](\tau) = \int_{\mathbb{R}^2} \left(1 + \ln \left(\frac{w(\xi,\tau)}{AG(\xi)} \right) \right) \frac{\partial w}{\partial \tau} d\xi$$

Now use the vorticity equation to rewrite $\frac{\partial w}{\partial \tau}$, integrate by parts (several times!) and we find:

$$\frac{d}{d\tau} H[w](\tau) = -\int_{\mathbb{R}^2} w(\xi) \left| \nabla \left(\frac{w}{AG} \right) \right|^2 d\xi$$

The ω-limit set of positive solutions

$$\frac{d}{d\tau} H[w](\tau) = -\int_{\mathbb{R}^2} w(\xi) \left| \nabla \left(\frac{w}{AG} \right) \right|^2 d\xi$$

Let's now consider the implications of this calculation for non-negative solutions. If we assume that $w(\xi, \tau)$ we see:

1. $\frac{d}{d\tau} H[w](\tau) \leq 0$ (so H is a Lyapunov function.)
2. $\frac{d}{d\tau} H[w](\tau) = 0$ only if w is a constant multiple of G.

Recalling the LaSalle invariance principle, we see that the only possibility for the ω-limit set of positive solutions of the vorticity equation is some multiple of the Gaussian - i.e. one of the Oseen vortices.

The same result also holds for solutions that are everywhere negative, but what about solutions that change sign?

The maximum principle for the vorticity equation

One of the most powerful qualitative properties of solutions of the heat equation is the maximum principle. Closer inspection shows that just like the heat equation, solutions of the 2D vorticity equation also satisfy a maximum principle. In particular:

- A solution that is positive for some time t_0 will remain positive for any later time $t > t_0$, and
- If the initial condition for the vorticity equation satisfies $\omega(z, 0) \geq 0$ then the solution will be strictly positive for all times $t > 0$.

Note that these remarks also hold for solutions of the rescaled vorticity equation. As a consequence of these two observations, we find a second, surprisingly simple, Lyapunov functional, namely the $L^1(\mathbb{R}^2)$-norm of the solution!

The L^1 norm as a Lyapunov function

To show that the L^1 norm is a Lyapunov function one splits a solution that changes sign into two pieces the positive part and the negative part. Applying the maximum principle to each piece, one can conclude:

1. The L^1 norm of the solution cannot increase with time.
2. In fact, the L^1 norm is strictly decreasing unless the solution is either everywhere positive, or everywhere negative.

Once again, we appeal to the LaSalle Principle and conclude that the ω-limit set of a solution whose initial condition changes sign, must lie in the set of functions that is either everywhere positive or everywhere negative.

Putting the pieces together

Putting together our two Lyapunov functionals we have the following conclusion,

1. For general solutions the ω-limit set must lie in the set of solutions that are everywhere positive or everywhere negative.
2. However, for such solutions, the relative entropy function implies that the ω-limit set must be a multiple of the Oseen vortex.

Thus, we conclude:

Theorem (Th. Gallay and CEW) Any solution of the two-dimensional vorticity equation whose initial vorticity is in $L^1(\mathbb{R}^2)$ and whose total vorticity $\int_{\mathbb{R}^2} \omega(z,0)dz \neq 0$ will tend, as time tends to infinity, to the Oseen vortex with parameter $\alpha = \int_{\mathbb{R}^2} \omega(z,0)dz$.

Extensions and Conclusions

This theorem implies that with even the slightest amount of viscosity present, two-dimensional fluid flows will eventually approach a single, large vortex.

1. However, if the viscosity is small, this convergence may take a very long time. (Much longer than observed in the numerical experiments, for example.)
2. Furthermore, Onsager's original calculations of vortex coalescence were for an inviscid fluid model which suggests that some sort of coalescence should occur independent of the viscosity

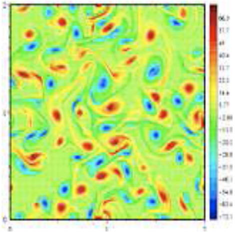
Thus, while Gallay's and my theorem says that eventually, all two-dimensional viscous flows will approach an Oseen vortex, there should be a variety of interesting and important behaviors that manifest themselves in the fluid prior to reaching the asymptotic state described in the theorem.

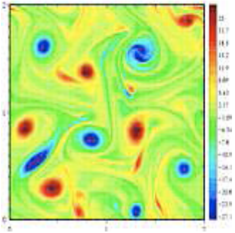
Metastability

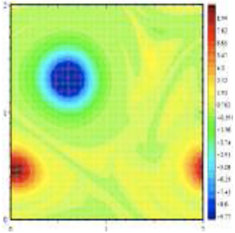
One interesting phenomenon that is particularly noticeable in the numerical simulations of two-dimensional flows is the creation and persistence of metastable structures.

- This refers to structures which appear on time scales much shorter than the time scale over which the long-time asymptotic behavior appears.
- These structures then dominate the evolution of the flow for extremely long times.

Metastability







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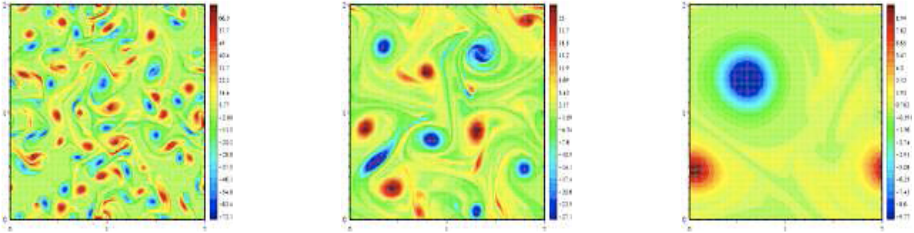

The origin and properties of these states in the two-dimensional Navier-Stokes equation is still not understood but statistical mechanical ideas have again been used to propose an explanation associated with the different time scales on which energy and entropy are dissipated.

Metastability in Burgers Equation

(Joint work with Margaret Beck at BU.)

Similar metastable phenomena also occur in the weakly viscous Burgers equation which is often used as a simplified testing ground for understanding the Navier-Stokes equations.

$$\partial_t u = \nu \partial_x^2 u - u \partial_x u$$

As with Navier-Stokes, one again introduces scaling variables, and reduces the equation to:

$$\partial_\tau w = \mathcal{L} w - w \partial_\xi w = (\nu \partial_\xi^2 w + \frac{\xi}{2} \partial_\xi w + \frac{1}{2} w) - w \partial_\xi w \tag{2}$$

Metastability in Burgers Equation

- One can show that there is a family of self-similar fixed-point solutions similar to the Oseen vortices, though in this case, they are no longer solutions of the linear equation.

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- We can analyze the stability at each of these fixed points and show that they are stable, with one zero eigenvalue (corresponding to motion along the family of fixed points) and an eigenvalue $-1/2$ corresponding to the slowest rate of approach to the fixed point. (And the remainder of the spectrum has real part less than or equal to -1.

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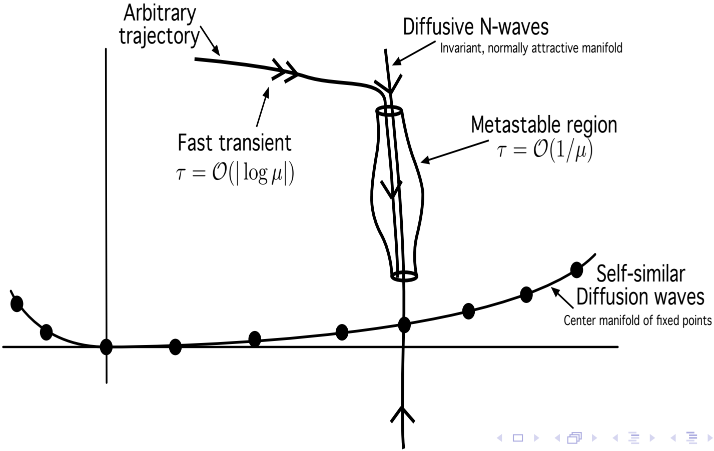
- Using the Cole-Hopf transformation we can then extend the one-dimensional manifold tangent to the eigenspace with eigenvalue $-1/2$ to a global "metastable manifold".

Metastability in Burgers Equation



Metastability in Burgers Equation

Interestingly enough, very similar metastable phenomena arise in weakly damped Hamiltonian systems.

- In this case, the undamped system has a multitude of periodic orbits.
- If one damps the system, eventually, all trajectories tend to zero.
- However, on intermediate time scales, the "ghosts" of a small collection of periodic orbits seem to capture the system and persist for a very long time.

(Ongoing work with Noé Cuneo and Jean-Pierre Eckmann.)

Summary

A distinctive feature of two-dimensional flows is the "inverse cascade" of energy from small scales to large ones. Lars Onsager first sought to explain this phenomenon by studying the statistical mechanics of large collections of inviscid point vortices. While Onsager's observation about *inviscid* flows remains unexplained, dynamical systems ideas - in this case Lyapunov functionals inspired by kinetic theory - have been used to show that in the presence of an arbitrarily small amount of viscosity, essentially any two-dimensional flow whose initial vorticity field is absolutely integrable will evolve as time goes to infinity toward a single, explicitly computable vortex.