# Decay profiles of a linear artificial viscosity system

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# Overview


1. Introduction
2. Theorem
3. Sketch of the proof

# The Compressible Navier-Stokes Equations

In their 1995 paper, Hoff and Zumbrun study the compressible Navier-Stokes equations over $\mathbb{R}^d$:

$$\rho_t + \text{div } m = 0$$

$$m^j_t + \text{div } \left(\frac{m^j m}{\rho}\right) + P(\rho)_{x_j} = \epsilon \, \Delta\left(\frac{m^j}{\rho}\right) + \eta \text{ div}\left(\frac{m}{\rho}\right)_{x_j}$$

These equations govern the density $\rho$ and momentum $m$ of a fluid and are particularly relevant for fluids having high Mach number.

# The Compressible Navier-Stokes Equations pt 2

They examined perturbations away from the constant state $\begin{pmatrix} \rho^* \\ m^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, showing that if $u_0 = (\rho_0, m_0)^T$ is such that

$$E = \|u_0\|_{L^1} + \|u_0\|_{H^{[d/2]+\ell}}$$

for $\ell \geq 3$ is sufficiently small, then the solution $u(x,t) = (\rho, m)^T$ satisfies

$$\|u(t)\|_{L^p} \leq C(\ell) E L(t) (1+t)^{-r_p}$$

for $2 \leq p \leq \infty$, where $r_p = \frac{d}{2}(1 - \frac{1}{p})$ is the rate of decay of the heat kernel and

$$L(t) \begin{cases} \log(1+t) & \text{if } d = 2 \\ 1 & \text{otherwise} \end{cases}$$

# The Linear Artificial Viscosity System

In the course of obtaining their result, Hoff and Zumbrun analyze the following system:

$$\rho_t + \text{div } m = \frac{1}{2}(\epsilon + \eta)\Delta\rho$$
$$m_t + c^2\nabla\rho = \epsilon\Delta m + \frac{1}{2}(\eta - \epsilon)\nabla\text{div } m$$

Hoff and Zumbrun refer to this system as the effective linear artificial viscosity system, and show that the solution $u(x,t) = (\rho, m)^T$ of the CNSE are time asymptotic to those of the artificial viscosity system:

$$\|u(t) - \tilde{G}(t) * u_0\|_{L^p} \leq C(\ell)E(1+t)^{-r_p - 1/2}$$

# Higher order asymptotics

Kagei and Okita expanded on these results by computing a higher order profile of the asymptotic behavior. For $d \geq 3$, they showed

$$\left\| u(t) - G(t) * u_0 - \sum_{i=1}^{d} \partial_i G_1(t, \cdot) \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}_i^0 \, dyds \right\|_{L^p} \leq CK(t)(1+t)^{-r_p - \frac{3}{4}}$$

We aim to provide a method by which the asymptotic behavior can be computed out to any desired order. To do so we'll study the linear artificial viscosity system.

# Fourier Analysis and Helmholtz Decomposition

We assume our solutions are such that we can take the Fourier transform. The transformed solutions satisfy

$$\hat{\rho}_t + i\xi^T \cdot \hat{m} = -\nu|\xi|^2\hat{\rho}$$
$$\hat{m}_t + ic^2\xi\hat{\rho} = -\epsilon|\xi|^2\hat{m} - \frac{1}{2}(\eta-\epsilon)\xi\left(\xi^T \cdot \hat{m}\right)$$

where $\nu = \frac{1}{2}(\epsilon+\eta)$. We then make use of the Helmholtz projection, which separates the divergence and divergence free parts of $m$:

$$\hat{m} = \frac{i\xi}{|\xi|^2}\hat{a} + \hat{b}$$

Here $\hat{a} = -i\xi^T \cdot \hat{m}$ and $\hat{b}$ satisfies $\xi^T \cdot \hat{b} = 0$.

# Hyperbolic-parabolic system

By making this projection, one must then consider the resulting system:

$$\partial_t \rho = \nu \Delta \rho - a$$
$$\partial_t a = -c^2 \Delta \rho + \nu \Delta a$$
$$\partial_t b = \epsilon \Delta b$$

The incompressible part is decoupled from the rest of the system and has already been analyzed in [Gallay and Wayne, 2002], hence we study the resulting hyperbolic-parabolic system for $\rho$ and $a$. We'll work over $\mathbb{R}^2$.

# Hyperbolic-parabolic system pt 2

To this end, we introduce the following operators:

$$W = \begin{pmatrix} 0 & -1 \\ -c^2\Delta & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} \nu\Delta & 0 \\ 0 & \nu\Delta \end{pmatrix}$$

and hence our system can be written as

$$\partial_t \begin{pmatrix} \rho \\ a \end{pmatrix} = (W + H) \begin{pmatrix} \rho \\ a \end{pmatrix} \tag{1}$$

We'll also be working in the function space $L^2(n)$ which is the closure of the smooth functions having compact support with respect the norm

$$\|f\|_{L^2(n)} = \left( \int_{\mathbb{R}^2} (1 + |x|^2)^n |f|^2 dx \right)^{1/2}$$

# Main Result

## Theorem

*Choose $n > 1$, and suppose that $\rho_0, a_0 \in C^\infty \cap L^2(n)$. If $k \in \mathbb{Z}$ is such that $k + 1 < n \leq k + 2$, then the solution $\vec{u}(x,t) = (\rho, a)^T$ of (1) with initial data $\vec{u}_0 = (\rho_0, a_0)^T$ satisfies*

$$\|\vec{u}(\cdot,t)\|_{L^\infty} \leq C\|\vec{u}_0\|_{L^2(n)}(1+\nu t)^{-5/4}$$

*and*

$$\left\|\vec{u}(\cdot,t) - \sum_{|\alpha|\leq k} \frac{1}{(1+\nu t)^{\frac{|\alpha|}{2}+1}} S(t)(H_\alpha, \vec{u}_0)\phi_\alpha\right\|_{L^\infty} \leq C\|\vec{u}_0\|_{L^2}(1+t)^{-\frac{5}{4}-\frac{k}{2}}$$

*where $S(t)$ is the semigroup generated by $W$.*

# Sketch of the proof

1. Commutativity of the $W$ and $H$ operators
2. Hermite expansion for solutions of the heat equation
3. Decay of solutions to the wave equation

# Commutativity of the W and H operators

In the Fourier domain, the system is

$$\partial_t \begin{pmatrix} \hat{\rho} \\ \hat{a} \end{pmatrix} = (\hat{W} + \hat{H}) \begin{pmatrix} \hat{\rho} \\ \hat{a} \end{pmatrix}$$

hence the solution is found to be

$$\begin{pmatrix} \rho(x,t) \\ a(x,t) \end{pmatrix} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \exp\left[(\hat{W} + \hat{H})t\right] \begin{pmatrix} \hat{\rho}_0(\xi) \\ \hat{a}_0(\xi) \end{pmatrix} d\xi$$

Since the matrices $\hat{W}$ and $\hat{H}$ commute for all $\xi \in \mathbb{R}^2$, we can write this as

$$\begin{pmatrix} \rho(x,t) \\ a(x,t) \end{pmatrix} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \exp\left[\hat{W}t\right] \exp\left[\hat{H}t\right] \begin{pmatrix} \hat{\rho}_0(\xi) \\ \hat{a}_0(\xi) \end{pmatrix} d\xi$$

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# Hermite Expansion

As shown in [Gallay and Wayne, 2002], solutions of the heat equation in $L^2(n)$ have a special form which we can make use of here. Suppose that $k \in \mathbb{Z}$ is such that $k+1 < n \leq k+2$. We define

$$\phi_0(x,t) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4(1+\nu t)}}$$
$$\phi_\alpha(x,t) = (1+\nu t)^{|\alpha|/2} \partial_x^\alpha \phi_0(x)$$

for $|\alpha| \leq k$, and let

$$H_\alpha(x) = \frac{2^{|\alpha|}}{\alpha!} e^{|x|^2/4} \partial_x^\alpha \left( e^{-|x|^2/4} \right)$$

be the $\alpha$-th Hermite polynomial.

# Hermite expansion pt 2

## Lemma

*Let* $n \geq 0$*, and suppose that* $f_0 \in L^2(n)$*. Let* $k \in \mathbb{Z}$ *be such that* $k+1 < n \leq k+2$*. If* $\tilde{S}(t)$ *is the semigroup associated with the heat equation, then the solution* $f(x,t) = \tilde{S}(t) f_0$ *has the form*

$$f(x,t) = \sum_{|\alpha| \leq k} (H_\alpha, f_0)(1+\nu t)^{-\frac{|\alpha|}{2}-1} \phi_\alpha + R(x,t)$$

*where for all* $\epsilon > 0$

$$\|R(\cdot,t)\|_{L^2(n)} \leq C \|f_0\|_{L^2(n)} \left(1+\nu t\right)^{-\frac{1}{2}(1+n-\epsilon)}$$

# Commutativity of the W and H operators

In the Fourier domain, the system is

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hence the solution is found to be

$$\begin{pmatrix} \rho(x,t) \\ a(x,t) \end{pmatrix} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \exp\left[(\hat{W} + \hat{H})t\right] \begin{pmatrix} \hat{\rho}_0(\xi) \\ \hat{a}_0(\xi) \end{pmatrix} d\xi$$

Since the matrices $\hat{W}$ and $\hat{H}$ commute for all $\xi \in \mathbb{R}^2$, we can write this as

$$\begin{pmatrix} \rho(x,t) \\ a(x,t) \end{pmatrix} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \exp\left[\hat{W}t\right] \begin{pmatrix} \hat{f}(\xi,t) \\ \hat{g}(\xi,t) \end{pmatrix} d\xi$$

# Decay of solutions to the wave equation

Following [Constantin, 2012], we make use of the method of oscillatory integrals for bounding solutions of the wave equation. Write

$$\begin{aligned} \rho(x,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\Big[\sin(ct|\xi|)|\xi|^{-1}\hat{g}(\xi) + \cos(ct|\xi|)\hat{f}(\xi)\Big]\,d\xi \\ &= \rho_+ + \rho_- \end{aligned}$$

where

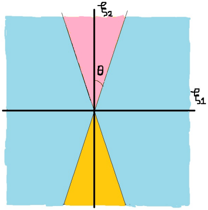
$$\rho_\pm = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} e^{i(x\cdot\xi \pm ct|\xi|)}\big(\hat{f} \mp i|\xi|^{-1}\hat{g}\big)\,d\xi$$

Without loss of generality, assume $x = (0, |x|)$.

# Decay of solutions to the wave equation pt 2

Using a smooth cutoff function $\chi$, we can write

$$\rho_+(x,t) = \frac{1}{2(2\pi)^2} \times \Big[ \int_{\mathbb{R}^2} e^{i(|x|\xi_2 + ct|\xi|)} \left[\hat{f} - i|\xi|^{-1}\hat{g}\right](1-\chi)d\xi$$
$$+ \int_0^\infty \int_0^\delta e^{icrt(\lambda\cos\theta+1)} \left[\hat{f} - ir^{-1}\hat{g}\right]\chi d\theta r dr$$
$$+ \int_0^\infty \int_{\pi-\delta}^\pi e^{icrt(\lambda\cos\theta+1)} \left[\hat{f} - ir^{-1}\hat{g}\right]\chi d\theta r dr \Big]$$
$$= T_1 + T_2 + T_3$$




where $\lambda = \frac{|x|}{ct}$, $r = |\xi|$ and $\cos\theta = \frac{\xi_2}{|\xi|}$.

# Decay of solutions to the wave equation pt 3

One can bound each of the terms separately. For $T_1$

$$T_1 = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} e^{i(|x|\xi_2+ct|\xi|)}\left[\hat{f} - i|\xi|^{-1}\hat{g}\right](1-\chi)d\xi$$

write

$$e^{ict|\xi|} = \left(\frac{1}{ict}\frac{|\xi|}{\xi_1}\right)^j \frac{\partial^j}{\partial\xi_1^j} e^{ict|\xi|}$$

By integrating by parts we can obtain the desired decay. Similarly, for $T_2$

$$T_2 = \frac{1}{2(2\pi)^2}\int_0^\infty \int_0^\delta e^{icrt(\lambda\cos\theta+1)}\left[\hat{f} - ik^{-1}\hat{g}\right]\chi d\theta r dr$$

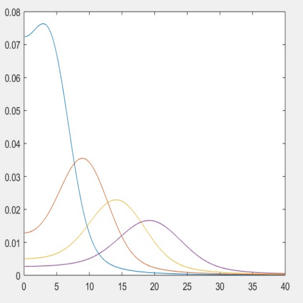
write

$$e^{icrt(\lambda\cos\theta+1)} = \left(\frac{1}{ict(\lambda\cos\theta+1)}\right)^j \frac{\partial^j}{\partial r^j} e^{icrt(\lambda\cos\theta+1)}$$

# Decay of solutions to the wave equation pt 4

For $T_3$, there are two cases: when $|x| \sim ct$ and when $|x|$ is far from the light cone.

- When $|x|$ is far from the light cone, the same integration by parts technique can be used.
- When $|x| \sim ct$, one needs to explicitly use the Hermite expansion and bound the integral using the properties of each term.



# Future Directions

1. Weaken the regularity assumptions.
2. Extend to the nonlinear artificial viscosity case.
3. Obtain results in higher dimensions.
4. Relate expansion to the compressible Navier-Stokes.

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