Global well-posedness and scattering for the defocusing quintic nonlinear Schrödinger equation in two dimensions

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Outline

1. Introduction
   - Well-posedness problem and Scattering
   - Schrödinger equations
   - Conservation laws
   - Strichartz estimates
   - Symmetries
   - Local well-posedness
   - Gobal well-posedness
2. Main result
3. Road map
4. Outline of proof
   - Impossibility of quasi-soliton solutions
   - Impossibility of finite-time blow-up solutions
5. Generalization

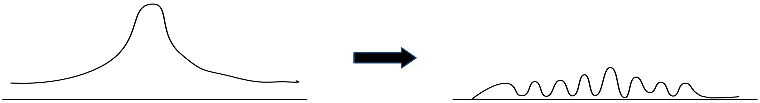
Well-posedness problem and Scattering

- Well-posedness problem

$$\begin{cases} \text{a solution } \textbf{exists} \text{ in short time or long time,} \\ \text{the solution is } \textbf{unique}, \\ \text{the solution's behavior changes } \textbf{continuously with the initial data}. \end{cases}$$

- Scattering

  the solutions of the nonlinear problem behave asymptotically like the solutions of the associated linear problem.



Introduction of Schrödinger equations

- Initial value problem of the **linear Schrödinger equations**:

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Fourier transformation gives the following solution:

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi), \quad \text{then } u(t,x) = e^{it\Delta}u_0.$$

- The power-type nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u, \\ u(0,x) = u_0(x), \end{cases} \tag{pNLS}$$

  - $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space, $p > 1$
  - Duhamel formula:

$$u(t) = e^{it\Delta}u_0 \mp i \int_0^t e^{i(t-s)\Delta}\left(|u|^{p-1}u\right)(s,x)\, ds$$

Conservation laws

This equation (pNLS) conserves

- **Mass**,

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx = M(u_0),$$

- **Total energy** or **Hamiltonian**,

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t,x)|^2 \pm \frac{1}{p+1} |u(t,x)|^{p+1} \, dx = E(u_0),$$

$$\begin{cases} + : \text{defocusing} & \longleftarrow \text{We center the discussion below here } \star \\ - : \text{focusing} \end{cases}$$

- and **Momentum**,

$$\mathcal{P}(u(t)) := \int_{\mathbb{R}^d} \mathrm{Im}[\bar{u}(t,x) \nabla u(t,x)] \, dx = \mathcal{P}(u_0).$$

Strichartz estimates

We call a pair of exponents (q, r) **admissible** if

$$2 \le q, r \le \infty, \ \frac{2}{q} + \frac{d}{r} = \frac{d}{2} \ \text{ and } \ (q, r, d) \ne (2, \infty, 2).$$

Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have

- the homogeneous Strichartz estimate

$$\left\| e^{it\Delta} u_0 \right\|_{L_t^q L_x^r(\mathbb{R}^d)} \le C \left\| u_0 \right\|_{L^2(\mathbb{R}^d)},$$

- and the inhomogeneous Strichartz estimate

$$\left\| \int_{s<t} e^{i(t-s)\Delta} F(s)\, ds \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \le C \left\| F \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Symmetries

- Time and space translation invariance, spatial rotation symmetry, phase rotation symmetry, time reversal symmetry
- Pseudo-conformal symmetry, Galilean invariance: only at $\frac{d}{2} = \frac{2}{p-1}$
- **Scaling symmetry**
  - If u solves (pNLS), then for any $\lambda \in \mathbb{R}$

$$u_\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \text{ with } u_{\lambda,0}(x) = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda}) \text{ solves (pNLS).}$$

  - Initial data under scaling:

$$\|u_{\lambda,0}\|_{\dot{H}^s} \sim \lambda^{-s+s_c} \|u_0\|_{\dot{H}^s}, \text{ where } s_c = \frac{d}{2} - \frac{2}{p-1}.$$

  - Different regimes:

$$\begin{cases} s > s_c & \textbf{subcritical} & \to \text{size of initial data} \downarrow \text{ while time of existence} \uparrow \\ s = s_c & \textbf{critical} & \to \text{initial data remains invariant} \\ s < s_c & \textbf{supercritical} & \to \text{size of initial data} \uparrow \text{ while time of existence} \downarrow \end{cases}$$

Local well-posedness?

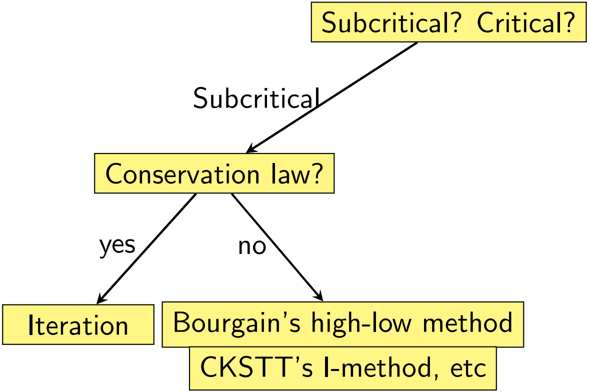
Global well-posedness?

Local (in time) well-posedness (LWP)

- Both subcritical and critical cases were solved by Cazenave and Weissler.
- Tools: Strichartz estimates + fixed point argument
- Time of existence:

$$\begin{cases} \text{Subcritical: depends } \textcolor{red}{\text{only}} \text{ on the } H^s \text{ norm of the data} \\ \text{Critical: depends } \textcolor{red}{\text{also}} \text{ on the profile of the data} \end{cases}$$

Global (in time) well-posedness (GWP)



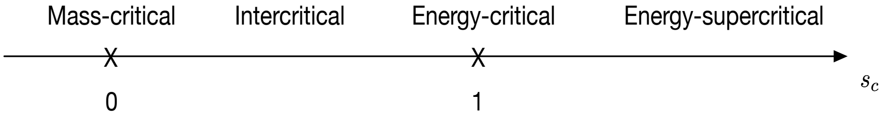
Defocusing critical NLS

Recall the Cauchy problem for the defocusing $\dot{H}^{s_c}$-critical NLS in $\mathbb{R}^{1+d}$:

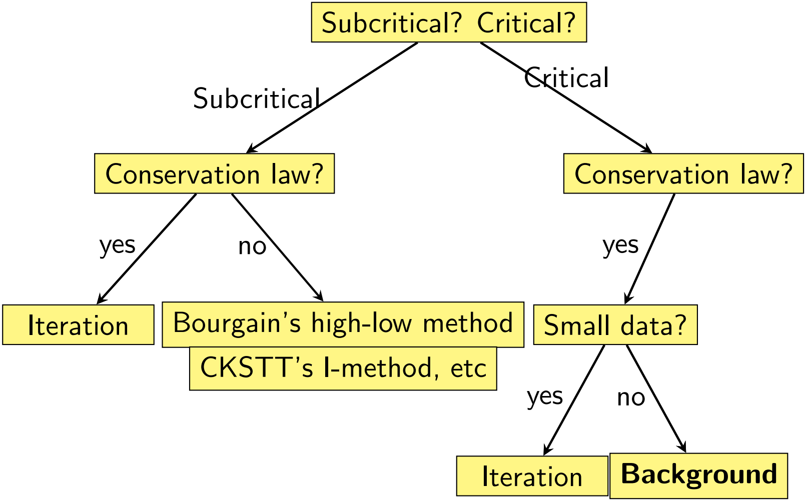
$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, \\ u(0,x) = u_0(x) \in \dot{H}^{s_c}(\mathbb{R}^d). \end{cases}$$

with $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, and $s_c = \frac{d}{2} - \frac{2}{p-1}$.

$$\begin{cases} s_c = 0 & \text{mass-critical} \\ s_c = 1 & \text{energy-critical} \\ s_c \in (0,1) & \text{intercritical} \\ s_c > 1 & \text{energy-supercritical} \end{cases}$$



Global (in time) well-posedness (GWP)



Background

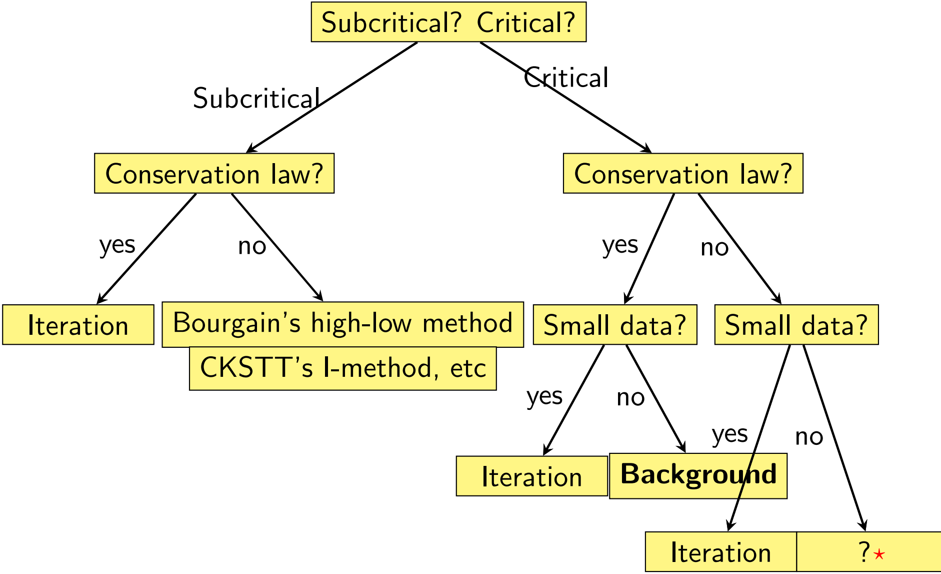
- Energy-critical regime ($s_c = 1$) Large data

	$d = 3$	$d = 4$	$d \geq 5$
Radial	Bourgain, Grillakis		
General	Colliander-Keel-Staffilani-Takaoka-Tao	Ryckman-Visan	Visan

- Mass-critical regime ($s_c = 0$) Large data

	$d = 1$	$d = 2$	$d \geq 3$
Radial		Killip-Tao-Visan	Tao-Visan-Zhang
General	Dodson		

Global (in time) well-posedness (GWP)



Background

- Intercritical regime and energy-supercritical regime ($s_c \neq 0, 1$)

Large data

- No conservation laws
- Assume $\dot{H}^{s_c}$ norm bounded

	$d = 2$	$d = 3$	$d \geq 4$
$s_c = \frac{1}{2}$	$\star$	Kenig-Merle	Murphy
$s_c > 1$	X	Killip-Visan, Miao-Murphy-Zheng, Murphy, Dodson-Miao-Murphy-Zheng,...	
$0 < s_c < 1$		Murphy, Xie-Fang,...	
$s_c < 0$		Killip-Masaki-Murphy-Visan,...	

The quintic $\dot{H}^{\frac{1}{2}}$-critical result in dimensions two remained open, because:

- the interaction Morawetz estimates in $d = 2$ are significantly different from those in $d \geq 3$,
- the endpoint Strichartz estimates fail.

Outline

Main theorem

Let's focus on the Cauchy problem for the defocusing $\dot{H}^{\frac{1}{2}}$-critical quintic Schrödinger equation in $\mathbb{R}^{1+2}$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u, \\ u(0,x) = u_0(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2). \end{cases} \tag{5NLS}$$

We show that if a solution remains bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ in its maximal interval of existence, then the interval is infinite and the solution scatters.

Theorem (Y. 2018)

Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a maximal-lifespan solution to (5NLS) *such that $u \in L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^2)$. Then u is global and scatters, with*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^8 \, dxdt \leq C\left(\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \right)$$

for some function $C : [0,\infty) \to [0,\infty)$.

Outline

Road map (Bourgain, CKSTT, Kenig-Merle)

Argue by contradiction:

- Step 1: existence of minimal blow-up solutions, that is,

  $\exists$ a solution u s.t. u blows up in $L^8_{t,x}$ norm with minimal $L^\infty_t \dot{H}^{\frac{1}{2}}_x$ norm.

$$\begin{cases} \text{for large data} & \text{assumption: } L^8_{t,x} \text{ norm is unbounded} \\ \text{for small data} & \text{fact: GWP and } L^8_{t,x} \text{ norm is bounded} \end{cases}$$

$\Rightarrow$ existence of **minimal blow-up solutions**

Moreover, u is **almost periodic**. (u concentrates in both space with radius $\frac{1}{N(t)}$ and frequency with radius $N(t)$.)

Definition (almost periodicity)

There exist functions: $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^2$, $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$\int_{|x-x(t)|\geq \frac{C(\eta)}{N(t)}} \left| |\nabla|^{\frac{1}{2}} u(t,x) \right|^2 dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi| \, |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$.

Road map (Bourgain, CKSTT, Kenig-Merle)

- Step 2: preclude the existence of minimal blow-up solutions

  Main tools:

$$\begin{cases} \text{conservation laws} \\ \text{suitable (frequency-localized interaction) Morawetz estimates} \\ \text{long-time Strichartz estimates} \end{cases}$$

Outline

Outline of proof (Step 2)

Now we prove this kind of u does not exist. Here, we classify u into the following 4 classes:

	$T_{max} < \infty$	$T_{max} = \infty$
$\int_0^{T_{max}} N(t)\,dt < \infty$	I	II
$\int_0^{T_{max}} N(t)\,dt = \infty$	III	IV

where

- I, III: finite-time blow-up solutions
- I, II: frequency cascade solutions
- III, IV: quasi-soliton solutions

Note that in $\dot{H}^{\frac{1}{2}}$ critical regime, $\int_0^{T_{max}} N(t)\,dt < \infty$ implies $T_{max} < \infty$, hence there is **No case II** in this setting.

Impossibility of quasi-soliton solutions $\int_0^{T_{max}} N(t)\,dt = \infty$

In dimensions two, Planchon-Vega and Colliander-Grillakis-Tzirakis proved the following interaction Morawetz estimates:

Theorem (Interaction Morawetz estimates in 2D)

$$\left\| |\nabla|^{\frac{1}{2}} |u(t,x)|^2 \right\|_{L^2_{t,x}(I\times\mathbb{R}^2)}^2 \lesssim \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^2)}^2 \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)}^2 .$$

- Morawetz inequality above scales like $\int_I N(t)\,dt$.

To preclude the this case, we use interaction Morawetz estimates (FLME).

$$K \leq \text{ ME } \leq o(K)$$

where $K = \int_0^T N(t)\,dt$.
Then take $K \to \infty$, contradiction!

Impossibility of finite-time blow-up solutions $T_{max} < \infty$

To preclude the finite-time blow-up solutions, we consider the following quantity:

$$y^2(t, R) := \int_{\mathbb{R}^2} \chi_R(x) \left|u(t,x)\right|^2 \, dx,$$

where $\chi_R(x) = \chi(\frac{x}{R})$ is a smooth cutoff function, such that

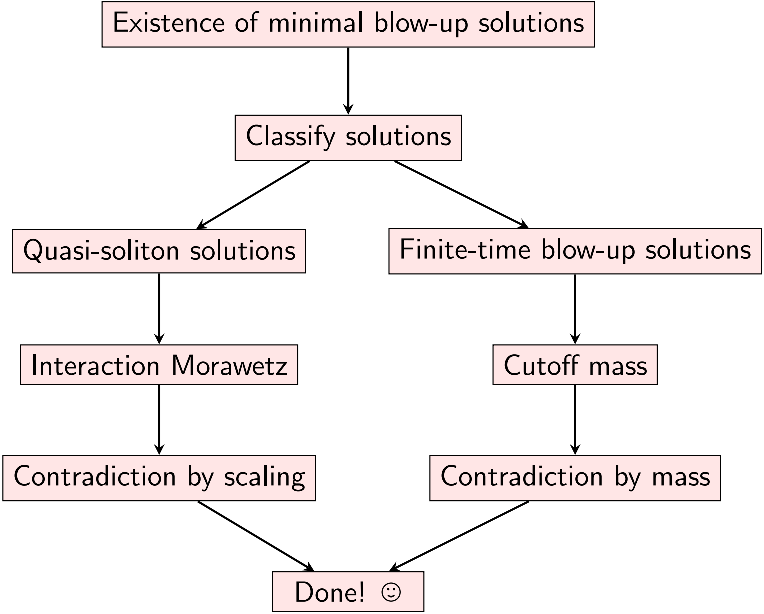
$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 2. \end{cases}$$

Then we can compute the derivative of y^2 w.r.t. time t (the rate of change in time)

$$\left|\frac{\partial y^2}{\partial t}\right| \lesssim \frac{1}{\sqrt{R}}.$$

The fact $\lim_{t \to T_{max}} y^2(t, R) = 0$ implies that $y^2(0, R) \lesssim \frac{T_{max}}{\sqrt{R}}$.
Next by taking $R \to \infty$, $\|u_0\|_{L_x^2}^2 = \lim_{R \to \infty} y^2(0, R) = 0$ implies $u_0 \equiv 0$.
Contradiction!

Recap


Existence of minimal blow-up solutions
Classify solutions
Quasi-soliton solutions
Finite-time blow-up solutions
Interaction Morawetz
Cutoff mass
Contradiction by scaling
Contradiction by mass
Done! ☺

Outline

Generalization

Let's focus on the Cauchy problem for the defocusing $\dot{H}^{s_c}$-critical Schrödinger equation in $\mathbb{R}^{1+2}$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^{2k} u, \\ u(0,x) = u_0(x) \in \dot{H}^{s_c}(\mathbb{R}^2), \end{cases} \qquad ((2k+1)\text{NLS})$$

where $s_c = 1 - \frac{1}{k}$.

Theorem (Y.-Haitian Yue 2018)

$k = 3, 4, 5, 6, \dots$

Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a maximal-lifespan solution to ((2k+1)NLS) such that $u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^2)$. Then u is global and scatters, with

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^{4k} \, dxdt \le C \left(\|u\|_{L_t^\infty \dot{H}_x^{s_c}(\mathbb{R}\times\mathbb{R}^2)} \right)$$

for some function $C : [0,\infty) \to [0,\infty)$.

Outline of proof (Step 2)

Now we prove this kind of u does not exist. Here, we classify u into the following 4 classes:

	$T_{max} < \infty$	$T_{max} = \infty$
$\int_0^{T_{max}} N(t)^{\frac{4}{k}-1}\,dt < \infty$	I	II
$\int_0^{T_{max}} N(t)^{\frac{4}{k}-1}\,dt = \infty$	III	IV

where

- I, III: finite-time blow-up solutions
- I, II: frequency cascade solutions
- III, IV: quasi-soliton solutions

Note that

- When $k = 3, 4$, $\int_0^{T_{max}} N(t)^{\frac{4}{k}-1}\,dt < \infty$ implies $T_{max} < \infty$, hence there is **No case II** in this setting.
- When $k = 5, 6, 7, \dots$, we have all 4 cases.

Differences

- Impossibility of quasi-soliton solutions
  Recall the Interaction Morawetz estimates in 2D

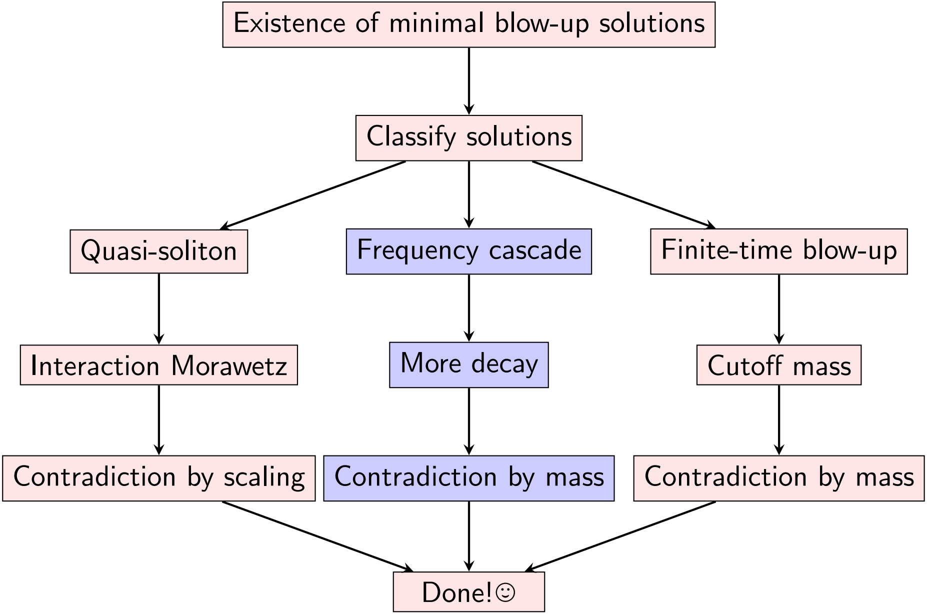
$$\left\| |\nabla|^{\frac{1}{2}} |u(t,x)|^2 \right\|^2_{L^2_{t,x}(I\times\mathbb{R}^2)} \lesssim \|u\|^2_{L^\infty_t L^2_x(I\times\mathbb{R}^2)} \|u\|^2_{L^\infty_t \dot{H}^{\frac{1}{2}}_x(I\times\mathbb{R}^2)}$$

- Impossibility of frequency cascade solutions for $k \geq 5$

$$\left.\begin{array}{l} \begin{cases} \|u\|_{\dot{H}^{-s}} \lesssim 1 \\ \left\|P_{\leq c(\eta)N(t)} u\right\|_{\dot{H}^{s_c}_x} \leq \eta \end{cases} \Rightarrow \left\|P_{\leq c(\eta)N(t)} u\right\|_{L^2_x} \lesssim \eta^\alpha \\ \left\|P_{\geq c(\eta)N(t)} u\right\|_{L^2_x} \lesssim \frac{1}{(c(\eta)N(t))^{s_c}} \left\| |\nabla|^{s_c} P_{\geq c(\eta)N(t)} u\right\|_{L^2_x} \lesssim \varepsilon \end{array}\right\} \Rightarrow \|u\|_{L^2_x} = 0$$

- Impossibility of finite-time blow-up solutions

Recap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Done!☺

Thank you for your attention!