ℓ-torsion in class groups of certain families of D_4-quartic fields

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Outline

Objects:

- certain families of D_4-quartic fields
- ℓ-torsion in class groups

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Goal: An unconditional nontrivial upper bound on ℓ-torsion in class groups of the families of fields

Tools:

- A lower bound for the number of fields in the families
- A new effective Chebotarev density theorem for these families of D_4-quartic fields

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For any such Q, we define the family

$$\mathscr{F}_4(Q) = \{K : K \text{ is a } D_4\text{-quartic field}, \widetilde{K} \text{ contains } Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})\}$$

and denote

$$\mathscr{F}_4(Q; X) = \{K : K \in \mathscr{F}_4(Q), D_K \leq X\}.$$

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- Conjectural bound: for any $\varepsilon > 0$,

$$|\mathrm{Cl}_K[\ell]| \ll_{d,\ell,\varepsilon} D_K^{\varepsilon}.$$

- Starting point

Theorem (Ellenberg & Venkatesh, 2007)

Let K be a number field of degree d and fix a positive integer ℓ. Set $\eta < \frac{1}{2\ell(d-1)}$ and suppose that there are at least M rational primes with $p \leq D_K^{\eta}$ that are unramified and split completely in K. Then

$$|\mathrm{Cl}_K[\ell]| \ll_{d,\ell,\varepsilon_1} D_K^{\frac{1}{2}+\varepsilon_1} M^{-1},$$

for any $\varepsilon_1 > 0$.

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- Nontrivial bound under GRH: (Ellenberg & Venkatesh, 2007)

$$|\mathrm{Cl}_K[\ell]| \ll_{d,\ell,\varepsilon} D_K^{\frac{1}{2}-\frac{1}{2\ell(d-1)}+\varepsilon},$$

for all $\varepsilon > 0$.

- Nontrivial bound without assuming GRH:
  - K imaginary quadratic: (Heath-Brown & Pierce, 2017)
    Let $\ell \geq 5$ be prime. For all but a possible zero-density exceptional family of imaginary quadratic fields $K/\mathbb{Q}$, we have

    $$|\mathrm{Cl}_K[\ell]| \ll_{\ell,\varepsilon} D_K^{\frac{1}{2}-\frac{3}{2\ell+2}+\varepsilon},$$

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    for all $\varepsilon > 0$.
  - $d = 2, 3, 4(K \text{ non-}D_4), 5$: Ellenberg, Pierce, & Wood, 2017
    For all but a possible zero-density exceptional family of fields $K/\mathbb{Q}$, we have

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- Nontrivial bound without assuming GRH:
  - $\mathrm{Gal}(\widetilde{K}/\mathbb{Q}) = C_n, S_n, A_n,$ or D_p, $n \geq 2$, p prime, d=n or p respectively: Pierce, Turnage-Butterbaugh, & Wood, 2017
    Under some restriction on tamely ramified primes, for all but a possible zero-density exceptional subfamily of fields $K/\mathbb{Q}$, we have
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Notably, D_4-quartic fields have not been treated in these works.

Main theorem

Recall that
$\mathscr{F}_4(Q) = \{K : K\ D_4\text{-quartic}, \widetilde{K} \text{ contains } Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})\}$ and
that $\mathscr{F}_4(Q; X) = \{K : K \in \mathscr{F}_4(Q), D_K \leq X\}$.

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Theorem (A., arXiv 2018)

Let $Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ be such that $\mathscr{F}_4(Q) \neq \emptyset$. For every $0 < \varepsilon < \frac{1}{4}$ sufficiently small, and every integer $\ell \geq 1$, there exists a parameter $B_1 = B_1(\ell, \varepsilon)$ such that for every $X \geq 1$, aside from at most $B_1 X^{\varepsilon}$ fields in $\mathscr{F}_4(Q; X)$, every field $K \in \mathscr{F}_4(Q; X)$ satisfies

$$|\mathrm{Cl}_K[\ell]| \ll_{\ell,\varepsilon} D_K^{\frac{1}{2} - \frac{1}{6\ell} + \varepsilon}.$$

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$$|\mathrm{Cl}_K[\ell]| \ll_{\ell,\varepsilon} D_K^{\frac{1}{2} - \frac{1}{6\ell} + \varepsilon}.$$

This theorem provides the first unconditional nontrivial bound for ℓ-torsion in class groups of infinite families of D_4-quartic fields.

Question: assuming that $\mathscr{F}_4(Q) \neq \emptyset$, we want a lower bound of $|\mathscr{F}_4(Q; X)|$ as $X \to \infty$.

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In the following of this talk, we will assume that for $K \in \mathscr{F}_4(Q)$, the unique quadratic subfield of K is $\mathbb{Q}(\sqrt{a})$. The general case follows by rearranging a, b, $\frac{ab}{\gcd(a,b)^2}$.

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Lemma

For $K \in \mathscr{F}_4(Q)$ such that the unique quadratic subfield of K is $\mathbb{Q}(\sqrt{a})$, there exists $g \in \mathbb{Z}$, $h \in \mathbb{Z} - \{0\}$ s.t. $K = \mathbb{Q}(\sqrt{g + h\sqrt{a}})$.

A lower bound for the number of fields in $\mathcal{F}_4(Q; X)$

Lemma

There exists a well-defined function $\varphi : (a, b) \mapsto (g_0, h_0, n_0)$*, the image being an ordered triple of positive integers satisfying*

$$g_0^2 - h_0^2 a = n_0^2 b$$

or

$$g_0^2 - h_0^2 a = n_0^2 \frac{ab}{\gcd(a, b)^2}.$$

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Lemma

Let $K = \mathbb{Q}(\sqrt{g + h\sqrt{a}})$ as before. Then we have $D_K \leq C_{a,b}|g^2 - h^2 a|$.

A lower bound for the number of fields in $\mathcal{F}_4(Q;X)$

- $K_{[m]} = \mathbb{Q}(\sqrt{g_0 m + h_0 m\sqrt{a}})$ where m is any positive integer.
- $M(X) = \{m \in \mathbb{Z}_{>0}$ square-free $\mid \gcd(m, |ab|) = 1, m \leq \frac{1}{n_0 C_{a,b}^{1/2}} X^{1/2}\}$
- $T(X) = \{K_{[m]} : m \in M(X)\}$

Theorem

(1) We have $T(X) \subseteq \mathscr{F}_4(Q;X)$.
(2) If $m_1, m_2 \in M(X)$, $m_1 \neq m_2$, then $K_{[m_1]} \neq K_{[m_2]}$.
(3) We have

$$|T(X)| \gg_Q X^{1/2}.$$

A lower bound for the number of fields in $\mathcal{F}_4(Q; X)$

Theorem (A., 2018)

Let $Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ be such that $\mathscr{F}_4(Q) \neq \emptyset$. Then we have

$$|\mathscr{F}_4(Q; X)| \gg_Q X^{1/2}.$$

Hence, the family of at most $B_1 X^{\varepsilon}$ possible exceptions to our ℓ-torsion bound is truly zero density.

For a D_4-quartic field K and its Galois closure $\widetilde{K}$, and for any fixed conjugacy class $\mathscr{C}$ in $G \cong D_4$, we define the prime counting function as

$$\pi_{\mathscr{C}}(x, \widetilde{K}/\mathbb{Q}) = |\{p \text{ prime}: \ p \text{ is unramified in } \widetilde{K}, \left[\frac{\widetilde{K}/\mathbb{Q}}{p}\right] = \mathscr{C}, p \leq x\}|,$$

where $\left[\frac{\widetilde{K}/\mathbb{Q}}{p}\right]$ is the Artin symbol, i.e., the conjugacy class of the Frobenius element corresponding to the extension $\widetilde{K}/\mathbb{Q}$ and the prime p.

For $K \in \mathscr{F}_4(Q)$, we consider

$$\zeta_{\widetilde{K}}(s) = \prod_{\rho \in \mathrm{Irr}(D_4)} L(s, \rho, \widetilde{K}/\mathbb{Q})^{\dim \rho} = \zeta(s) L(s, \chi_1) L(s, \chi_2) L(s, \chi_3) L^2(s, \rho_{\widetilde{K}}).$$

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is fixed. Therefore, as K varies in $\mathscr{F}_4(a, b)$, the only varying L-factor in $\zeta_{\widetilde{K}}(s)$ is $L(s, \rho_{\widetilde{K}})$. It is crucial in our study together with the fact that $\rho_{\widetilde{K}}$ is faithful.

Theorem (Effective Chebotarev density theorem with assumed zero-free region)

Let $0 < \varepsilon_0 < \frac{1}{4}$ be sufficiently small. Suppose that $\mathscr{F}_4(Q) \neq \emptyset$. Suppose also that for $K \in \mathscr{F}_4(Q)$ such that $D_{\widetilde{K}} \geq C_7$ for an absolute constant C_7, $\zeta_{\widetilde{K}}(s)/\zeta_Q(s) = L^2(s, \rho_{\widetilde{K}})$ (hence $L(s, \rho_{\widetilde{K}})$) has no zero in

$$[1-\delta, 1] \times [-(\log D_{\widetilde{K}})^{2/\delta}, (\log D_{\widetilde{K}})^{2/\delta}],$$

where $\delta = \frac{\varepsilon_0}{42+4\varepsilon_0}$, then for every conjugacy class $\mathscr{C} \subset G = D_4$,

$$\left| \pi_{\mathscr{C}}(x, \widetilde{K}/\mathbb{Q}) - \frac{|\mathscr{C}|}{|G|}\mathrm{Li}(x) \right| \leq \frac{|\mathscr{C}|}{|G|} \frac{x}{(\log x)^2}$$

for all

$$x \geq \kappa_1 \exp\left[\kappa_2 (\log\log(D_{\widetilde{K}}^{\kappa_3}))^2\right]$$

for parameters $\kappa_i = \kappa_i(a, b, \varepsilon_0)$.

We note that $\kappa_1 \exp\left[\kappa_2(\log\log(D_{\widetilde{K}}^{\kappa_3}))^2\right] \ll D_K^{\varepsilon}$ for any $\varepsilon > 0$. Hence, for the field K above, we are able to count unramified primes p that splits completely in K and satisfies $D_K^{\varepsilon} \leq p \leq D_K^{\frac{1}{2\ell(d-1)}-\iota}$ for any $\varepsilon > 0$, $\iota > 0$. By the Theorem of Ellenberg and Venkatesh, we obtain our ℓ-torsion bound for Cl_K.

Next, we show via work of Kowalski and Michel (2002) that almost all fields in our family are zero-free in the described region.

Theorem

Suppose that $Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ is such that $\mathscr{F}_4(Q) \neq \emptyset$. For every $0 < \varepsilon_0 < \frac{1}{4}$, there are $\ll_{\varepsilon_0} X^{\varepsilon_0}$ fields $K \in \mathscr{F}_4(Q; X)$ such that $\zeta_{\widetilde{K}}(s)/\zeta_Q(s) = L^2(s, \rho_{\widetilde{K}})$ could have a zero in the region

$$[1-\delta, 1] \times [-(\log D_{\widetilde{K}})^{2/\delta}, (\log D_{\widetilde{K}})^{2/\delta}].$$

Theorem (A., 2018)

Suppose that $Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ *is such that* $\mathscr{F}_4(Q) \neq \emptyset$. *For every* $0 < \varepsilon_0 < \frac{1}{4}$ *sufficiently small, there exists a constant* $B_2 = B_2(\varepsilon_0)$ *such that for every* $X \geq 1$, *aside from at most* $B_2 X^{\varepsilon_0}$ *fields in* $\mathscr{F}_4(Q; X)$, *each field* $K \in \mathscr{F}_4(Q; X)$ *has the property that for every conjugacy class* $\mathscr{C} \subset G = D_4$,

$$\left| \pi_{\mathscr{C}}(x, \widetilde{K}/\mathbb{Q}) - \frac{|\mathscr{C}|}{|G|} \mathrm{Li}(x) \right| \leq \frac{|\mathscr{C}|}{|G|} \frac{x}{(\log x)^2}$$

for all

$$x \geq \kappa_1 \exp\left[\kappa_2 (\log\log(D_{\widetilde{K}}^{\kappa_3}))^2\right]$$

for parameters $\kappa_i = \kappa_i(a, b, \varepsilon_0)$.

Review of the main theorem

Theorem

Let $Q = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ be such that $\mathscr{F}_4(Q) \neq \emptyset$. For every $0 < \varepsilon < \frac{1}{4}$ sufficiently small, and every integer $\ell \geq 1$, there exists a parameter $B_1 = B_1(\ell, \varepsilon)$ such that for every $X \geq 1$, aside from at most $B_1 X^{\varepsilon}$ fields in $\mathscr{F}_4(Q; X)$, every field $K \in \mathscr{F}_4(Q; X)$ satisfies

$$|\mathrm{Cl}_K[\ell]| \ll_{\ell,\varepsilon} D_K^{\frac{1}{2} - \frac{1}{6\ell} + \varepsilon}.$$