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LECTURE NOTES: PERIOD POLYNOMIALS FOR PICARD MODULAR FORMS

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Abstract. The relations satisfied by period polynomials associated to modular forms yield a way to count dimensions of spaces of cusp forms. After showing how these relations arise from those on the mapping class group $PSL(2,\mathbb{Z})$ of the moduli space $\mathcal{M}_{0,4}$ of genus 0 curves with 4 marked points, in this lecture I will go on to define period polynomials associated to Picard modular forms. Relations on these Picard period polynomials will then be given, and via an embedding of a monodromy representation of the moduli space $\mathcal{M}_{0,5}$ of genus 0 curves with 5 marked points in $PU(2,1;\mathbb{Z}[\rho])$ (where ρ denotes a third root of unity), they will be related to the geometry of $\mathcal{M}_{0,5}$.

Contents

1. From elliptic curves to period polynomials

1.1. The Eichler-Shimura theorem and the Manin relations.

Suppose that f is a cusp form for $SL(2,\mathbb{Z})$ of weight k over $\mathbb{R}$. Recall that this means that f is a holomorphic function on the upper half space $\mathfrak{H}$ which is also holomorphic at the cusps $\{i\infty\} \cup \mathbb{Q}$ and which transforms under the action of

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

on $\mathfrak{H}$ via

$$f(\sigma z) = (cz+d)^k f(z).$$

Denote the space of these cusp forms of weight k for $SL(2,\mathbb{Z})$ over $\mathbb{R}$ by $M_k^0(\mathbb{R})$.

The periods with moment associated to $f \in M_k^0(\mathbb{R})$ and $x_1, x_2 \in \{i\infty\} \cup \mathbb{Q}$ are the complex numbers

$$\int_{x_1 \to x_2} f(\tau)\tau^s d\tau$$

for $s = 0, 1, \ldots, k-2$, where the integral is taken along the geodesic arc $x_1 \to x_2$ from x_1 to x_2.

Of special interest is the case $x_1 = i\infty$ and $x_2 = 0$, giving integrals along the imaginary axis. We set

$$r_s(f) := \int_{i\infty \to 0} f(\tau)\tau^s d\tau.$$

These periods are examples of the modular symbols studied extensively by Manin, as discussed in the lecture by Professor Stevens.

Notice that when s is even, $r_s(f)$ is purely imaginary (since f is defined over $\mathbb{R}$) while s odd implies that $r_s(f)$ is real. Hence writing

$$r^+(f) := \frac{1}{i}\left(r_0(f), r_2(f), \ldots, r_{k-2}(f)\right)$$

and

$$r^-(f) := \left(r_1(f), r_3(f), \ldots, r_{k-3}(f)\right),$$

each of these vectors lies in a Euclidean space, say $\mathbb{R}^+_{k-1}$ and $\mathbb{R}^-_{k-1}$ respectively; and the relations with integer coefficients which they satisfy determine subspaces, say V^+ of $\mathbb{R}^+_{k-1}$ and V^- of $\mathbb{R}^-_{k-1}$ defined over $\mathbb{Q}$. We would like to find these linear relations.

Let us denote generators of $PSL(2,\mathbb{Z})$ by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1}$$

It turns out that corresponding to each of the two relations $S^2 = I$ and $(ST)^3 = I$ on $PSL(2,\mathbb{Z})$ we get a relation on the periods:

$$\int_{i\infty \to 0} f(\tau)\tau^j d\tau + \int_{S(i\infty \to 0)} f(\tau)\tau^j d\tau = 0$$

and

$$(2)\quad \int_{i\infty\to 0} f(\tau)\tau^j d\tau + \int_{ST(i\infty\to 0)} f(\tau)\tau^j d\tau + \int_{(ST)^2(i\infty\to 0)} f(\tau)\tau^j d\tau = 0$$

There is a nice geometric way to understand these so-called Manin relations: first, notice that S acts on $\mathfrak{H}$ as an involution about the circle of radius 1 centered at the cusp 0. Consequently, S acts on the imaginary line as a reflection about i, and hence merely switches the direction of the geodesic path $i\infty \to 0$. Next, one checks that ST maps $i\infty \to 0$ to the geodesic arc from 0 to -1, and $(ST)^2$ maps $i\infty \to 0$ to the vertical line from -1 to $-1 + i\infty = i\infty$. These three arcs form a triangle, and - roughly speaking - because the cusp form f is holomorphic at the vertices, the integral taken around the boundary of the triangle (which is the sum in (2)) is zero. One can write down a careful proof checking these details (cf. Lang's book [14] for example), but I will shortly explain this from a slightly different perspective since the triangle in fact corresponds to the lower hemisphere of $\mathbb{P}^1\backslash\{0, 1, \infty\}$ and by considering this viewpoint we will see that these relations can be connected to relations satisfied by the Drinfel'd associator.

We are now in a position to state the Eichler-Shimura-Manin Theorem. Usually it is framed in terms of cohomology; for this perspective see the original works of Eichler and Shimura from the fifties. The version I'm giving follows Lang's discussion in [14], in which the proof is attributed to David Rohrlich. The injectivity of the map is the hardest part of the proof.

Theorem 1.1. *The mapping*

$$f \mapsto r^-(f)$$

is an isomorphism of $M_k^0(\mathbb{R})$ with V^-, while the mapping

$$f \mapsto r^+(f)$$

is an isomorphism of $M_k^0(\mathbb{R})$ with a subspace of V^+ of codimension 1 which does not contain $(1, 0, \ldots, 0, -1)$.

As a nice corollary we get a dimension count of the space of cusp forms.

Now let X and Y be formal variables. We can define an action of $PSL(2,\mathbb{Z})$ on ${}^t(X, Y)$ via a left action of the inverse transpose.[1] Then if $f \in M_k^0(\mathbb{R})$ as above, the 1-form $f(\tau)(X\tau-Y)^{k-2}d\tau$ is invariant under the $PSL(2,\mathbb{Z})$ action. We define the period polynomial associated to

[1]This is not the usual action one will find in the literature but it works just as well here and generalizes to the Picard period polynomial case - see below.

the cusp form f of weight k to be

$$P_f(X,Y) = \int_{i\infty\to 0} f(\tau)(X\tau - Y)^{k-2} d\tau.$$

This is a sort of generating function for the periods with moment in that $r_s(f)$ appears as a term "labelled" by X^sY^{k-2-s} multiplied by $\begin{pmatrix} k-2 \\ s \end{pmatrix}$.

The Manin relations for the period polynomials are then:

$$P_f(X_0,X_1) = -P_f(-X_1,X_0) \tag{3}$$

and

$$P_f(X_0,X_1) + P_f(-X_0-X_1,X_0) + P_f(X_1,-X_0-X_1) = 0 \tag{4}$$

1.2. **Extending the monodromy representation of $\mathbb{P}^1\backslash\{0,1,\infty\}$.** The Legendre form of the Weierstrass equation of an elliptic curve E_λ is

$$y^2 = x(x-1)(x-\lambda)$$

and for each $\lambda \in \mathbb{P}^1\backslash\{0,1,\infty\}$ the curve E_λ is isomorphic to the curves $E_{f(\lambda)}$ where $f(\lambda)$ is an element of the anharmonic group of transformations of $\mathbb{P}^1\backslash\{0,1,\infty\}$ given by

$$\Lambda_\lambda := \left\{\lambda, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{\lambda}, \frac{\lambda-1}{\lambda}\right\}.v \tag{5}$$

There are six such curves for a given choice of λ, unless two or more of the elements of Λ_λ are equal (which only happens if λ is a sixth root of unity or lies in $\{-1,2,\frac{1}{2}\}$).

Fix $\lambda \in \mathbb{P}^1\backslash\{0,1,\infty\}$. Then we can regard E_λ as a two-sheeted cover of the x-sphere by making branch cuts on two respective copies P_0 and P_1 of $\mathbb{P}^1\backslash\{0,1,\infty\}$ along the geodesics γ_0^j between 0 and ∞ and γ_1^j between 1 and λ in P_j for $j = 0,1$, and gluing along these cuts. Now let A_λ denote a loop in E_λ about 0 and 1 which crosses each of the four geodesics γ_k^j exactly once, and let B_λ denote a loop in P_0 about 1 and λ which does not cross any of the four geodesics γ_j^k. Then $\{A_\lambda, B_\lambda\}$ may be regarded as a basis of the homology group $H_1(E_\lambda,\mathbb{Z})$.

The periods of E_λ are then

$$\omega_1 = \int_{A_\lambda} \frac{dx}{y} \qquad \text{and} \qquad \omega_2 = \int_{B_\lambda} \frac{dx}{y},$$

and $\tau_\lambda := \omega_2/\omega_1$ lies in $\mathfrak{H}$, the upper half plane.

If we now let λ vary in $\mathbb{P}^1\backslash\{0,1,\infty\}$, then $\lambda \mapsto \tau_\lambda$ is a multi-valued map Ψ of $\mathbb{P}^1\backslash\{0,1,\infty\} \to \mathfrak{H}$, the inverse of which is the classical lambda function giving the uniformization of $\mathbb{P}^1\backslash\{0,1,\infty\}$ by $\mathfrak{H}$.

Given $\gamma \in \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\},\lambda_0)$ for some basepoint λ_0, the analytic continuation of the restriction of Ψ to some simply connected neighborhood of λ_0 along γ gives rise to a map $\gamma \mapsto N(\gamma)$ via the transformation of Ψ to $N(\gamma)\circ\Psi$ where $N(\gamma)$ is some automorphism of $\mathfrak{H}$ - i.e. $N(\gamma) \in PSL(2,\mathbb{R})$. In other words, we have a map $\pi_1(\mathbb{P}^1\backslash\{0,1,\infty\},\lambda_0) \to PSL(2,\mathbb{R})$, the image of which is called the monodromy group of Ψ.

Denote loops based at λ_0 about 0 and 1 respectively by σ_0 and σ_1. These can be viewed as the generators of $\pi_1(\mathbb{P}^1\backslash\{0,1,\infty\},\lambda_0)$. Now $N(\sigma_j)$ can be computed via the effect of the analytic continuation along σ_j on the homology basis $\{A_\lambda, B_\lambda\}$. One sees that

$$N(\sigma_0)\ {}^t[B_\lambda\ \ A_\lambda] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B_\lambda \\ A_\lambda \end{bmatrix}$$

while

$$N(\sigma_1)\ {}^t[B_\lambda\ \ A_\lambda] = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} B_\lambda \\ A_\lambda \end{bmatrix}.$$

One finds that $N(\sigma_0) = T^2$ while $N(\sigma_1) = ST^2S$. Since the latter matrices generate the congruence subgroup $\Gamma(2) \le PSL(2,\mathbb{Z})$, which is a free group on the two generators, in this case the monodromy representation is faithful.

Now as Schneps explains in §2.5 of [2], the fundamental group of the moduli space $\mathcal{M}_{0,n}$ of genus 0 curves with n marked points may be identified with the kernel $K(0,n)$ of the projection of the mapping class group $M(0,n)$ to S_n (where n should be replaced by $n-1$ when $n=4$); and this identification may be extended to a mapping from $M(0,n)$ itself to a certain set of path classes emanating from a so-called real point of $\mathcal{M}_{0,n}$. When the real point in question admits no automorphisms (which could arise from the orbifold structure on $\mathcal{M}_{0,n}$) the map is an isomorphism and the path space thus acquires a group structure.

Schneps gives her map for the configuration space model of $\mathcal{M}_{0,n}$ but an explicit version of this map in the complex model would be very useful for computations. In [11] I gave such a map for the $n=4$ case. This yields a correspondence between the elements of all of $M(0,3) = PSL(2,\mathbb{Z})$, and certain homotopy classes of paths in $\mathbb{P}^1\backslash\{0,1,\infty\}$. In order to describe this correspondence, first we need to introduce the notion of tangential basepoint, which is due to Deligne [4]. To this end, suppose that $X = \overline{X}\backslash S$ is a smooth curve over $\mathbb{C}$ where S denotes some finite set of points. Then at any omitted point $a \in S$, it is possible to define the fundamental group of X in the direction of a specified tangent vector to $\overline{X}$ at a. A classical way to do this (cf. for example

[8]) is to set

$$P_{v_0,v_1} := \{\gamma : [0,1] \to \overline{X} \mid \gamma'(0) = v_0; \gamma'(1) = -v_1; \gamma((0,1)) \subset X\}$$

(where $v_j \in T_{a_j}$ is a tangent vector at $a_j \in S$ for $j = 0,1$), and then denote the set of path components of P_{v_0,v_1} by $\pi_1(X, v_0, v_1)$. When $v_0 = v_1$ this gives the fundamental group based at v_0, denoted $\pi_1(X, v_0)$.

Now the anharmonic group $\Lambda_\lambda \simeq S_3$ of (5) both permutes $0, 1$ and ∞, and gives the group of all linear fractional transformations of $\mathbb{P}^1\backslash\{0,1,\infty\}$. Therefore given a homotopy class $\alpha \in \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{01}, \overrightarrow{ab})$ where a, b are distinct elements of $\{0,1,\infty\}$ and $\overrightarrow{ab}$ denotes the unit vector based at a in the direction of b along the real line in $\mathbb{P}^1(\mathbb{C})$, the permutation $\sigma_{ab} \in S_3$ sending 0 to a and 1 to b corresponds to a unique element of Λ_λ, say λ_{ab}, so given any $\beta \in \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{01}, \overrightarrow{cd})$ for any distinct $c, d \in \{0,1,\infty\}$ then $\lambda_{ab}(\beta) \in \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{ab}, \overrightarrow{\sigma_{ab}c\,\sigma_{ab}d})$. But then we can define the product of α and β in

$$\mathcal{G}_{01} := \bigcup_{a,b\in\{0,1,\infty\}, a\neq b} \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{01}, \overrightarrow{ab})$$

as the path α followed by $\lambda_{ab}(\beta)$. Now let s denote the tangential path along the unit interval from $\overrightarrow{01}$ to $\overrightarrow{10}$, and let t denote the tangential path from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$ which is a loop tangential to the real line lying in the upper half plane. In [11], $\mathcal{G}_{01}$ with the group operation described above is shown to be isomorphic to $PSL(2,\mathbb{Z})$ under the association of s and t respectively with the matrices S and T of (1). With this notation a presentation for $\mathcal{G}_{01}$ is:

$$\mathcal{G}_{01} = < s, t \mid s^2 = (st)^3 = 1 > .$$

Notice that σ_0 above corresponds to t^2, and σ_1 corresponds to s followed by t^2 followed by s, so $N(\sigma_0) = T^2$ and $N(\sigma_1) = ST^2S$ is exactly the restriction of the isomorphism $\mathcal{G}_{01} \to PSL(2,\mathbb{Z})$, to $\pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{01}) \to \Gamma(2) = K(0,3)$. In other words, the short exact sequence

$$1 \to K(0,3) \to PSL(2,\mathbb{Z}) \to S_3 \to 1 \tag{6}$$

is then faithfully represented on

$$1 \to \pi_1(\mathbb{P}^1\backslash\{0,1,\infty\}, \overrightarrow{01}) \to \mathcal{G}_{01} \to S_3 \to 1, \tag{7}$$

where the multiplication on $\mathcal{G}_{01}$ is "twisted" by the action of S_3 on $\mathbb{P}^1\backslash\{0,1,\infty\}$ as described above.

1.3. **Period polynomials and the geometry of $\mathbb{P}^1\backslash\{0,1,\infty\}$.** The classical lambda function λ which gives the uniformization of $\mathbb{P}^1\backslash\{0,1,\infty\}$ by $\mathfrak{H}$ maps the path of integration $i\infty \to 0$ to the tangential path s, which we associate with the element S of $PSL(2,\mathbb{Z})$.

Given any contractible path in $\mathbb{P}^1\backslash\{0, 1\infty\}$, there is a corresponding relation in $PSL(2, \mathbb{Z})$ coming from the decomposition of the path in terms of σ and τ. The path also lifts to a (contractible) path in a fundamental domain for $\Gamma(2)$ in the $PSL(2, \mathbb{Z})$ action on $\mathfrak{H}$, for example (cf. [3]) the set

$$\mathcal{F} := \{\tau \in \mathfrak{H} \mid -1 < \Re(\tau) \leq 1;\ \left|\tau - \frac{1}{2}\right| \geq \frac{1}{2};\ \left|\tau + \frac{1}{2}\right| > \frac{1}{2}\},$$

which, by prepending $\sigma \circ \sigma$ (the identity) if necessary, starts with the path $i\infty \to 0$. This path in $\mathfrak{H}$ can then be regarded as a succession of images of $i\infty \to 0$ and in this way we get a relation on our period polynomial.

Example. Consider the trivial path $(\sigma\tau^{-1})^3 = 1$ on $\mathcal{G}_{01}$. This is the tangential path from 0 to 1 (s) followed by a loop $(\lambda_{10}(\tau^{-1}))$ in the upper hemisphere from $\overrightarrow{10}$ to $\overrightarrow{1\infty}$, then the tangential path $(\lambda 1\infty\sigma)$ from 1 to ∞ along the real line, a loop in the upper hemisphere $(\lambda_{\infty 1}(\tau^{-1})$ from $\overrightarrow{\infty 1}$ to $\overrightarrow{\infty 0}$, the path along the real axis from ∞ to 0, and finally the loop in the upper hemisphere from $\overrightarrow{0\infty}$ to $\overrightarrow{01}$. This path is merely a contractible loop in the upper hemisphere of $\mathbb{P}^1\backslash\{0, 1, \infty\}$. In the lifting thereof to the fundamental domain $\mathcal{F}$ for $\Gamma(2)$ given above, the images of τ^{-1} lift to non-geodesic loops from tangential basepoints at the cusps of the reflection of the triangle considered before over the imaginary axis. The two tangential basepoints at each relevant cusp are in fact equal. For example, at the cusp 0, there is a tangential basepoint pointing in the direction of $i\infty$ along the imaginary axis, and a second tangential basepoint in the direction of the geodesic arc from 0 to 1. The relevant tangent vectors coincide. Consequently, the lifts of τ^{-1} to $\mathcal{F}$ are loops over which the images of the period integral are zero. Essentially then, these serve to cut out the cusps from the path, and as before the integral over the entire path is zero. In this way, we find that the hexagonal path $(\sigma\tau^{-1})^3$ in $\mathbb{P}^1\backslash\{0, 1, \infty\}$ gives rise to the Manin relation on the period polynomials. (For a more careful proof see Theorem 1.1 in [12].)

Integrating over the same path gives a way to compute the analytic continuation of a flat section of the universal prounipotent bundle with connection[2] on $\mathbb{P}^1\backslash\{0, 1, \infty\}$. Such a flat section is given by the polylogarithm generating function $Li(z, A, B)$, and I showed in [11] that this procedure carried out along a path α in $\mathcal{G}_{01}$ always transforms

[2]Here the connection is the Knizhnik-Zamolodchikov equation

$$dG(z, A, B) = \frac{Adz}{z} + \frac{Bdz}{1-z}$$

mentioned in Professor Kaneko's talk.

$Li(z, A, B)$ to $F_\alpha(A, B) \cdot Li(z, A, B)$ where $F_\alpha(A, B)$ is some power series in the non-commuting variables A and B, and the association of paths in $\mathcal{G}_{01}$ (and hence of elements of $PSL(2, \mathbb{Z})$) with power series is *injective*. When interpreted suitably, (working with a unipotent completion of the topological fundamental group), this mapping can be identified with Deligne's Betti-de Rham comparison isomorphism of fundamental groups of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, (cf. [4] and §4 of Kedlaya's notes [13] from Deligne's 2003 Arizona Winter School lecture); and group-like elements[3] of the image lie in the Grothendieck-Teichmüller group GT - see Appendix A.

Now the power series arising from analytic continuation along σ is the Drinfel'd associator $\Phi(A, B)$ and the relation on power series coming from the hexagonal path $(\sigma\tau^{-1})^3$ is none other than the hexagonal relation

$$\Phi_{KZ}(A, B) \exp(i\pi B) \Phi_{KZ}(-B, A-B) \exp(i\pi(A-B)) \Phi_{KZ}(B-A, -A) \exp(i\pi(-A)) = 1$$

on GT.

The Manin relation (4) coincides with the Lie algebra version of the hexagonal relation in GT.

Now a five term "pentagonal relation" for GT arises similarly in the context of the moduli space $\mathcal{M}_{0,5}$ of genus zero curves with 5 marked points. In some way, this relation must be more fundamental than the hexagonal relation: in [7], Hidekazu Furusho showed the quite astonishing fact that the pentagonal relation implies the hexagonal relation.

The obvious question which arises here is then: Can we develop an analogue of the period polynomials in the context of $\mathcal{M}_{0,5}$, and would such objects satisfy a linearized pentagonal relation?

In an attempt to answer this question, we begin by studying a monodromy representation for $\mathcal{M}_{0,5}$ arising from a family of genus 3 curves.

But first, let us mention an amazing recent result of Brown and Hain: they proved that the comparison isomorphism for de Rham and singular (Betti) cohomology for the moduli space $\mathcal{M}_{1,1}$ of elliptic curves yields a version of the Eichler-Shimura-Manin Theorem for weakly holomorphic modular forms of level 1. See their work [1] for details. It's noteworthy that in our situation, the logarithm (Lie algebra mapping) of the image of the Betti-deRham comparison isomorphism for fundamental groups can be seen to give the relations on period polynomials which are at the heart of the Eichler-Shimura-Manin Theorem for cusp forms.

[3]Group like elements satisfy $\Delta\Psi = \Psi \otimes \Psi$ under a coproduct Δ on the image.

2. The Picard curve case

2.1. **From Picard curves to a representation of the fundamental groupoid of $\mathcal{M}_{0,5}$ on $PU(2,1;\mathbb{Z}[\rho])$.** The curves $C(x_0, y_0)$ given in affine coordinates (z, w) by the equation

$$w^3 = z(z-1)(z-x_0)(z-y_0)$$

for $[x_0 : y_0 : 1]$ in the space

$$M := \{[x_0 : y_0 : 1] \in \mathbb{P}^2(\mathbb{C}) \ : \ x_0 y_0 (x_0 - 1)(y_0 - 1)(x_0 - y_0) \neq 0\}$$

are genus 3 compact curves known as Picard curves. Notice that if $\rho = \exp(2\pi i/3)$, $(z, w) \mapsto (z, \rho w)$ gives an automorphism of $C(x_0, y_0)$. The parameter space for this family is none other than the moduli space $\mathcal{M}_{0,5}$ of genus 0 curves with 5 marked points since (as in the case of the complex model of $\mathcal{M}_{0,4}$) one can use the cross ratio to see that it is isomorphic to

$$(\mathbb{P}^1 \backslash \{0, 1, \infty\})^2 \backslash \Delta$$

where $\Delta = \{(z, z) \in \mathbb{P}^2(\mathbb{C}) \ : \ z \in \mathbb{P}^1 \backslash \{0, 1, \infty\}\}$. Now, as in the elliptic curve case discussed in §1.2 above, analytic continuation along paths in the parameter space $\mathcal{M}_{0,5}$ gives rise to an action on periods of the Picard curves, which could also be viewed as an action on a homology basis for $H_1(C(x_0, y_0), \mathbb{Z})$ for some fixed $(x_0, y_0) \in \mathcal{M}_{0,5}$. In particular, suppose that $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ is such a basis with the standard intersection matrix, and $\{\omega_1, \omega_2, \omega_3\}$ is a basis of abelian differentials of the first kind on $C(x_0, y_0)$ for which

$$\int_{A_j} \omega_i = \delta_{ij}.$$

The period matrix $\Omega = \left(\int_{B_j} \omega_i\right)$ was computed by Picard. This gives a map of $\mathcal{M}_{0,5}$ into the Siegel upper half space $\mathfrak{H}_3$ of symmetric matrices with positive definite imaginary part.

An example of a suitable basis for $H_1(C(x_0, y_0), \mathbb{Z})$ was given explicitly by Shiga in [16]. Using his basis, $\frac{dw}{z}$ is a holomorphic differential on $C(x_0, y_0)$ which is not among the ω_j, $j = 1, 2, 3$. Then set

$$\eta_0 = \int_{A_1} \frac{dw}{z} \qquad \eta_1 = -\rho^2 \int_{B_1} \frac{dw}{z} \qquad \eta_2 = \int_{A_2} \frac{dw}{z}$$

and write

$$v = \frac{\eta_1}{\eta_0} \qquad \text{and} \qquad u = \frac{\eta_2}{\eta_0}.$$

Picard gave the period matrix in terms of (u, v) and in this way we can view the period mapping $(x_0, y_0) \mapsto (u, v)$ as a map from $\mathcal{M}_{0,5}$ into the complex hyperbolic space (the complex 2-ball) $\mathbb{H}^2_{\mathbb{C}}$ which is given by

$$\{(u, v) \in \mathbb{C}^2 \ : \ 2Re(v) + |u|^2 < 0\}.$$

It is known that the image is open and dense in $\mathbb{H}^2_{\mathbb{C}}$, cf. [16].

As in the elliptic curve case, analytic continuation deforms this period mapping by post-composition by an automorphism of the covering space - in this case by an element of $PU(2,1;\mathbb{Z}[\rho])$, the projectivization of the unitary group of matrices g with ${}^t\overline{g}Jg = J$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{8}$$

with coefficients in $\mathbb{Z}[\rho]$.

Here the image of the monodromy representation is the subgroup

$$\Gamma_1 := \{g \in PU(2,1;\mathbb{Z}[\rho]) \ : \ g \equiv I \mod (\sqrt{3}iI).$$

A major difference from the $\mathcal{M}_{0,4}$ case is that this representation is not faithful.

Now the automorphism group of $\mathcal{M}_{0,4}$ is the anharmonic group $\Lambda_\lambda \simeq S_3$ while that of $\mathcal{M}_{0,5}$ is S_5. But there is also a natural S_4 action on $\mathcal{M}_{0,5}$ induced by the S_4 action on the points $p_1 = [0:0:1]$, $p_2 = [0:1:0]$, $p_3 = [1:0:0]$ and $p_4 = [1:1:1]$ of $\mathbb{P}^2$, and this turns out to be the suitable action to consider in giving a sort of extension of the monodromy representation to mimic that in the $\mathcal{M}_{0,4}$ case, because here we have the short exact sequence

$$1 \to \Gamma_1 \to PU(2,1;\mathbb{Z}[\rho]) \to S_4 \to 1.$$

(It's also the case that the blowup of $\mathbb{P}^2$ at the four points p_1, p_2, p_3 and p_4 coincides with a certain compactification of $\mathcal{M}_{0,5}$.) Based upon this S_4 action, in [12] an *ad hoc* S_4 action on those tangential basepoints of $\mathcal{M}_{0,5}$ based at the "points at infinity" p_1, p_2, p_3 and p_4 is given. Using a presentation for $PU(2,1;\mathbb{Z}[\rho])$ given in [6], the S_4 action is paired with an explicit version of the surjection $PU(2,1;\mathbb{Z}[\rho]) \to S_4$ to define an explicit group of paths on $(\mathbb{P}^1\backslash\{0,1,\infty\})^2\backslash\Delta$ which surjects onto $PU(2,1;\mathbb{Z}[\rho])$, in an analogue of the mapping to the group of paths $\mathcal{G}_{01}$ in $\mathbb{P}^1\backslash\{0,1,\infty\}$ described above. For the details see [12]. Unfortunately the group of paths we obtain is free, so an analogue of the association of trivial paths (like $(st^{-1})^3 = 1$) in $\mathcal{G}_{01}$ with relations on the period polynomials would not have any content in the $\mathcal{M}_{0,5}$ setting using this explicit group of paths.

2.2. **Picard period polynomials.** Following [16], define a Picard modular form (respectively a Picard meromorphic modular form) of weight k relative to a subgroup G of $PU(2,1;\mathbb{Z}[\rho])$ to be any holomorphic (respectively meromorphic) function $f(z_1,z_2)$ on $\mathbb{H}^2_{\mathbb{C}}$ which satisfies

$$f(g(z_1,z_2)) = (\det g)^{-k}(g_{20}z_1 + g_{21}z_2 + g_{22})^{3k} f(z_1,z_1) \tag{9}$$

for any $g = (g_{ij})_{i,j=0,1,2} \in G$.Here $g(z_1,z_2)$ is defined via the action of g on $[\eta_0 : \eta_1 : \eta_2] \in \mathbb{P}^2(\mathbb{C})$ with the identifications $z_1 = \frac{\eta_0}{\eta_2}$ and $z_2 = \frac{\eta_1}{\eta_2}$. As Shiga points out, one checks that

$$\frac{\partial(g(z_1,z_2))}{\partial(z_1,z_2)} = \frac{\det g}{(g_{20}z_1 + g_{21}z_2 + g_{22})^3} =: j_g(z_1,z_2). \tag{10}$$

In other words, (9) is of the form

$$f(z_1,z_2) = j_g^k(z_1,z_2) f(g(z_1,z_2)) \tag{11}$$

where j_g denotes the Jacobian determinant given by (10).

The (left) action induced by the transpose of the inverse ${}^tg^{-1}$, multiplied by a cubed root of $\det g$, turns out to give a suitable action of $g = (g_{ij})_{i,j=0,1,2} \in PU(2,1;\mathbb{Z}[\rho])$ on the polynomial ring $\mathbb{Z}[\rho][X_0,X_1,X_2]_{3n}$ of homogeneous polynomials in the variables X_0, X_1 and X_2 of degree $3n$ for some fixed $n \geq 1$ with coefficients in $\mathbb{Z}[\rho]$. Given a general element $A = (a_{ij})_{i,j=0,1,2}$ of $PU(2,1;\mathbb{Z}[\rho])$ the inverse is

$$A^{-1} = \begin{bmatrix} \overline{a}_{22} & \overline{a}_{12} & \overline{a}_{02} \\ \overline{a}_{21} & \overline{a}_{11} & \overline{a}_{01} \\ \overline{a}_{20} & \overline{a}_{10} & \overline{a}_{00} \end{bmatrix}$$

-i.e. $A^{-1} = (b_{ij})_{i,j=0,1,2}$ where $b_{ij} = \overline{a}_{(2-j)(2-i)}$. Hence this action is given by:

$$X_k \mapsto (\det g)^{1/3} \sum_{j=0}^{2} \overline{g}_{(2-k)(2-j)} X_j \tag{12}$$

for $k = 0,1,2$. Here $\det g$ is some sixth root of unity for any $g \in PU(2,1;\mathbb{Z}[\rho])$ but notice that since we are working only with the action on homogeneous polynomials of degree divisible by 3, the choice of the cubed root is irrelevant. Also, since the sixth roots of unity lie in $\mathbb{Z}[\rho]$ we don't need to enlarge the ring of coefficients.

Suppose that $f(z_1,z_2)$ denotes a Picard modular form of weight k relative to a subgroup G of $PU(2,1;\mathbb{Z}[\rho])$. By combining (9), (10) and (12) one checks that

$$\underline{f}(z_1,z_2;X_0,X_1,X_2)dz_1 \wedge dz_2 := f(z_1,z_2)(z_1X_0 + z_2X_1 + X_2)^{3k-3} dz_1 \wedge dz_2$$

is invariant under the action of G.

To determine Picard modular forms as defined above for the full group $PU(2,1;\mathbb{Z}[\rho])$ and not merely subgroups thereof Runge in [15] used a method he developed to determine rings of Siegel modular forms. In doing so, it was necessary to introduce a new feature: his rings of modular forms are each associated to some specific element, say M, of finite order in $Sp(3,\mathbb{Z})$, and accordingly are called Picard modular forms of Picard type M. Using the period mapping and a correspondence given by Holzapfel in [9] between $PU(2,1;\mathbb{Z}[\rho])$ and a subgroup of $Sp(3,\mathbb{Z})$ one can show that Runge's modular forms of a certain Picard type M_{Pic} satisfy the condition of (11) for all g in $PU(2,1;\mathbb{Z}[\rho])$. Runge found the ring of Picard modular forms of type M_{Pic} is a polynomial ring in certain homogeneous polynomials in the theta constants

$$f_{\mathbf{k}}(\Omega) := \theta \begin{bmatrix} \mathbf{k} \\ \mathbf{0} \end{bmatrix} (0, 2\Omega) := \sum_{\mathbf{n} \in \mathbb{Z}^3} \exp(2\pi i \, {}^t(\mathbf{n}+\mathbf{k})\Omega(\mathbf{n}+\mathbf{k})) \tag{13}$$

where $\Omega \in \mathfrak{H}_3$ and $\mathbf{k}$ is one of

$$\mathbf{1} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{2} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{or} \quad \mathbf{3} := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Again thanks to the period mapping, these functions may be regarded as functions of $\mathbb{H}^2_{\mathbb{C}}$.

Runge also gave a method to determine the ring of Picard modular forms of Picard type M_{Pic} explicitly, which I carried out using Macaulay2 and reported on in the Appendix to [12].

The final ingredient before we can define the Picard period polynomials is the domain D in $\mathbb{H}^2_{\mathbb{C}}$ over which we will integrate - in other words the analogue of the path $i\infty \to 0$ in $\mathcal{H}$.

There is a representation of $PSL(2,\mathbb{Z})$ on a subgroup of $PU(2,1;\mathbb{Z}[\rho])$ given by the following map:

$$S \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =: R \quad T \mapsto \begin{pmatrix} 1 & 1 & \rho \\ 0 & \rho & -\rho \\ 0 & 0 & 1 \end{pmatrix} =: P$$

Adjoining

$$R_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to this representation, we find that $PU(2,1;\mathbb{Z}[\rho])$ is generated by R, P and R_1 with the presentation

$$PU(2,1;\mathbb{Z}[\rho]) =< R, P, R_1 | R^2 = (RP)^6 = R_1^6 = [R_1, R] = PR_1^{-1}P^{-1}R_1^{-1}P = I >$$

(see Proposition 5.10 in [6]).

Now R is a reflection about the isometric sphere $\mathcal{S}_R$ of R:

$$\mathcal{S}_R := \{z \in \mathbb{H}^2_{\mathbb{C}} \ : \ | < q_\infty, z > | = | < q_\infty, Rz > |\}.$$

where as in [6] q_∞ denotes the point $[1:0:0]$ at infinity for $\mathbb{H}^2_{\mathbb{C}}$ and $< \cdot, \cdot >$ is the Hermitian form $< r, s > := s^* J r$ with J as in (8) above. This is a precise analogue of the action of S on $PSL(2, \mathbb{Z})$ with respect to the geodesic arc $|z| = 1$ in the upper half space $\mathfrak{H}$: indeed, define the Hermitian form $< w, z >= \overline{w} z$ for w, z in $\mathfrak{H}$, consider 1 as a cusp of $\mathfrak{H}$ and then set

$$\mathcal{U}_S := \{ z \in \mathfrak{H} \ : < 1, z >=< 1, Sz > \}.$$

It's clear that $\mathcal{U}_S$ is the upper half of the circle $|z| = 1$ and that S, which acts on $\mathfrak{H}$ by $z \mapsto -\frac{1}{z}$ is a reflection across this arc.

One checks that

$$\mathcal{S}_R = \{ [z_1 : z_2 : 1] \in \mathbb{H}^2_{\mathbb{C}} \ : \ |z_1| = 1; \ |z_2| \leq \sqrt{2} \}$$

and R fixes the set i_3 of points of $\mathcal{S}_R$ with $z_1 = -1$. Not only is $i \in \mathcal{U}_S$ fixed by S in the same way that the points of $i_3 \in \mathcal{S}_R$ are fixed by R; but the S-action interchanges the center 0 of $\mathcal{U}_S$ and $i\infty$, while the R-action switches the center $[0:0:1]$ of the isometric sphere $\mathcal{S}_R$ and q_∞.

Next we consider fixed points of the other generators. In the $PSL(2, \mathbb{Z})$ case, T fixes $i\infty$ and the path of integration for the period polynomials is comprised of the path from $i\infty$ (the fixed point of T) to i (the fixed point of S on $\mathcal{U}_S$), together with the image thereof under S itself. In the $PU(2, 1; \mathbb{Z}[\rho])$ setting, one checks that P similarly fixes q_∞, and R_1 fixes both $[0:0:1]$ and q_∞, and shares with R the fixed point $[-1:0:1] \in i_3$. This might suggest that we should integrate over the geodesic from q_∞ to $[-1:0:1]$ together with the image of the geodesic from $[-1:0:1]$ to q_∞ under R. The geometry resulting from this choice is not very interesting. Instead, we will integrate over a 2-simplex found by considering fixed points of other elements of $PU(2, 1; \mathbb{Z}[\rho])$.

To this end, we set $R_2 = R^{-1} R_1^{-1} P R$ and $R_3 = P R_1^{-1}$. Under the homotopy representation computed by Shiga, $R R_2^2 R^{-1}$ corresponds to a loop in $\mathcal{M}_{0,5}$ based at x_0 about y_0 for some fixed $(x_0, y_0) \in \mathcal{M}_{0,5}$ with $1 < x_0 < y_0 < \infty$ and $R R_3^2 R^{-1}$ corresponds to a loop about 0 based at x_0 (cf. [16]). Then $\{R_1, R_2, R_3\}$ is also a generating set for $PU(2, 1; \mathbb{Z}[\rho])$ and we also have that $R_1^2 = R R_1^2 R^{-1}$ corresponds to a loop about 1 based at x_0. Then R_2 fixes $[0:0:1]$ and $[-1:-\rho:1] \in i_3$ while R_3 fixes q_∞ and $[-1:1:1] \in i_3$.

Within i_3 consider the path j_3 from the fixed point $[-1:-\rho:1]$ of R_3 to the fixed point $[-1:0:1]$ of R_1, followed by the geodesic from $[-1:0:1]$ to the fixed point $[-1:1:1]$ of R_2.

Now define the domain D in $\mathbb{H}^2_{\mathbb{C}}$, which is to serve as the analogue of the line from $i\infty$ through i to 0 in $\mathfrak{H}$, to consist of the union of all geodesics from q_∞ to the points of j_3, together with the union of the images of these lines under the R-action, each oriented from a point of j_3 to $[0:0:1]$. Using the explicit description of geodesics in $\mathbb{H}^2_{\mathbb{C}}$ given in [6], it is possible to visualize D in affine coordinates as two sides of the surface of a triangular prism in $\mathbb{R}\times\mathbb{C}$ which extends from infinity along the negative real axis towards the omitted vertex $(0,0)$.

Finally we define the Picard period polynomial associated to the Picard modular form f of type M and weight k as

$$P_f(X_0,X_1,X_2) := \int_D f(z_1,z_2)(z_1X_0+z_2X_1+X_2)^{3k-3}\,dz_1\wedge dz_2.$$

Then we can prove the

Proposition 2.1. *The integral $P_f(X_0,X_1,X_2)$ converges.*

(See [12] for the proof.)

Guided by the way in which the usual period polynomials satisfy relations arising from those on $PSL(2,\mathbb{Z})$, one can now determine the relations on the Picard period polynomials. Again in [12] I prove the following theorem, working in the covering space $\mathbb{H}^2_{\mathbb{C}}$ rather than in $\mathcal{M}_{0,5}$ itself (since the monodromy representation is not faithful):

Theorem 2.2. *The period polynomial $P_f(X_0,X_1,X_2)$ for the Picard modular form f of type M and weight k satisfies the following relations corresponding to defining relations on $PU(2,1;\mathbb{Z}[\rho])$:*

$$P_f(X_0,X_1,X_2) = -P_f(X_2,-X_1,X_0) \tag{14}$$

corresponding to $R^2=I$;

$$\begin{gathered}P_f(X_0,X_1,X_2)+(-\rho^2)^{k-1}P_f(X_0,-\rho X_1,X_2)+\rho^{k-1}P_f(X_0,\rho^2X_1,X_2)\\ +(-1)^{k-1}P_f(X_0,-X_1,X_2)+\rho^{2k-2}P_f(X_0,\rho X_1,X_2)+(-\rho)^{k-1}P_f(X_0,-\rho^2X_1,X_2)=0\end{gathered} \tag{15}$$

corresponding to $R_1^6=I$;

$$\begin{aligned}P_f(X_0,X_1,X_2)&+\rho^{k-1}P_f(\rho^2X_0+X_1+X_2,\rho^2X_0-\rho^2X_1,X_0)\\ &+\rho^{2k-2}P_f(\rho^2X_2,\rho^2X_2-X_1,\rho^2X_0+X_1+X_2)=0\end{aligned} \tag{16}$$

corresponding to $(RP)^3=I$;

$$P_f(X_2, \rho X_1, X_0) + P_f(X_0, -\rho X_1, X_2) = 0 \tag{17}$$

coming from $[R, R_1] = I$*; and the pair of relations*

$$\begin{aligned} P_f(X_0, X_1, X_2) + \rho^{k-1} P_f(X_0, -\rho^2 X_0 + \rho^2 X_1, \rho^2 X_0 + X_1 + X_2) \\ = (-\rho^2)^{k-1} P_f(X_0, \rho X_0 - \rho X_1, \rho^2 X_0 + X_1 + X_2) \end{aligned} \tag{18}$$

and

$$\begin{aligned} P_f(X_0, X_1, X_2) + \rho^{2k-2} P_f(X_0, X_0 + \rho X_1, \rho X_0 - \rho X_1 + X_2) \\ = (-\rho)^{k-1} P_f(X_0, -\rho X_0 - \rho^2 X_1, \rho X_0 - \rho X_1 + X_2) \end{aligned} \tag{19}$$

coming from $P R_1^{-1} P^{-1} R_1^{-1} P = I$.

Notice that the equation (17) is clearly equivalent to (14).

Equation (16) is the Picard period polynomial analogue of the Manin equation (4).

Equations (18) and (19) are hexagonal-type equations similar to the Manin relation (16) but which come out of the pentagonal equation (**??**) on $PU(2,1;\mathbb{Z}[\rho])$. There is a geometric reason that we would not expect a pentagonal equation to be faithfully represented on the symmetry relations satisfied by the $P_f(X_0, X_1, X_2)$: as Ihara explains in [10], the pentagonal relation on paths in $\mathcal{M}_{0,5}$ runs between 5 specific tangential basepoints which are permuted by the action of a 5-cycle in S_5 on the canonical S_5-action on the space. Consequently the S_4-orbit of any of these tangential basepoints does not include all of the others. Because $PU(2,1;\mathbb{Z}[\rho])$ is the image of the monodromy representation of the orbifold fundamental group of the orbifold $[\mathcal{M}_{0,5}/S_4]$ (cf. [16]), any pentagonal path in $\mathcal{M}_{0,5}$ which gives rise to the pentagonal relation on GT will not give rise to a five term relation on the Picard period polynomials.

The computations needed for the proof show that the 2-simplex $P(D)$ and the 2-simplex $(P^{-1}R_1^{-1}P)(D)$ are complex conjugates of one another. This gives the following further (non-trivial) relation on period polynomials for Picard modular forms:

$$P_f(X_0, -\rho^2 X_0 + \rho^2 X_1, \rho^2 X_0 + X_1 + X_2) = \overline{P_f}(X_0, -\rho X_0 - \rho^2 X_1, \rho X_0 - \rho X_1 + X_2)$$

where $\overline{P_f}$ denotes complex conjugation.

Finally we remark that since we know that every relation on $PU(2,1;\mathbb{Z}[\rho])$ yields a relation on the period polynomials for Picard modular forms, the relations coming from the presentation of $PU(2,1;\mathbb{Z}[\rho])$ using the generators R_1, R_2 and R_3 do so as well. For example, since $R_3 = PR_1^{-1}$, one can deduce $R_3^6 = I$ from $R_1^6 = I$ along with repeated application of $PR_1^{-1}P^{-1}R_1^{-1}P = I$ - indeed, $P^{-1}R_1 = PR_1^{-1}P^{-1}$ so

$$\begin{aligned}
I &= P^{-1}R_1^6P = (P^{-1}R_1)R_1^5P \\
&= (PR_1^{-1}P^{-1})R_1^5P = (PR_1^{-1})(PR_1^{-1}P^{-1})R_1^4P \\
&= \cdots = (PR_1^{-1})^6 = R_3^6.
\end{aligned}$$

Consequently, through repeated application of 18 and 19 to 15, the equation corresponding to $R_3^6 = I$ would result, namely

$$\begin{aligned}
0 = & P_f(X_0, X_1, X_2) + P_f(X_0, -\rho^2 X_0 - \rho X_1, \rho^2 X_0 - \rho^2 X_1 + X_2) \\
& + P_f(X_0, (1-\rho^2)X_0 + \rho^2 X_1, (\rho^2 - 1)X_0 + (1-\rho^2)X_1 + X_2) \\
& + P_f(X_0, 2X_0 - X_1, -2X_0 + 2X_1 + X_2) \\
& + P_f(X_0, (1-\rho)X_0 + \rho X_1, (\rho - 1)X_0 + (1-\rho)X_1 + X_2) \\
& + P_f(X_0, -\rho X_0 - \rho^2 X_1, \rho X_0 - \rho X_1 + X_2).
\end{aligned}$$

Combining Theorem 2.2 with the monodromy mapping from §2.1 gives the

Corollary 2.3. *Every contractible path in $\pi_1((\mathbb{P}^1\backslash\{0,1,\infty\})^2\backslash\Delta, (\overrightarrow{01}, \overrightarrow{01})$ gives a relation on $P_f(X_0, X_1, X_2)$ in the sense that one or more of the equations in Theorem 2.2 can be associated with the path once it is mapped into $PU(2,1;\mathbb{Z}[\rho])$.*

Appendix A. The Grothendieck-Teichmüller Group

The absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) =: G_{\mathbb{Q}}$ of the rationals, which is the group of automorphisms of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, is one of the central objects of study in number theory. Because it encodes information pertaining to all polynomial equations in integer coefficients, at once, it is immensely complicated. A lot is known about $G_{\mathbb{Q}}$: for example, the abelian extensions of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ are handled by the beautiful edifice of class field theory; and the study of Galois representations gives a very productive approach to understanding the group. Yet, there is still much to learn about $G_{\mathbb{Q}}$. In fact the only known description of any element of this profinite group other than the identity automorphism and complex conjugation, would have to reference the restriction to each of infinitely many field extensions of $\mathbb{Q}$.[4] We can "name" an absolute Frobenius element above a prime p of $\mathbb{Z}$ but this is defined as an element of $G_{\mathbb{Q}}$ only up to its action on an algebraic closure of the finite field of p elements (viewed as a quotient of the

[4]This is essentially due to a Theorem of Artin which asserts that up to conjugation, the identity and complex conjugation are the only elements of $G_{\mathbb{Q}}$ of finite order.

integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ by a maximal ideal $\mathfrak{p}$ dividing p). Other than such approximations explicit descriptions of elements of $G_{\mathbb{Q}}$ are not known.

In the 1980s, through work of Belyi, Grothendieck, Drinfel'd, Ihara and others, it became known that $G_{\mathbb{Q}}$ embeds into a finitely presented group with geometric origins, the so-called Grothendieck-Teichmüller group GT. Whether this embedding is a surjection remains an important open question, but in any event GT is a fundamental object which could shed light on $G_{\mathbb{Q}}$.

While GT is an essentially geometric object, serving as it does as the group of automorphisms of the tower of (profinite completions of) fundamental groupoids with basepoints at infinity of the moduli spaces $\mathcal{M}_{0,n}$ of genus 0 curves with n marked points, in [5], Drinfel'd motivated the definition of GT based on purely algebraic constructions, namely via perturbations of the structures on quasitriangular quasi-Hopf algebras. A quasi-Hopf algebra on a set A is a Hopf algebra in which the coassociativity axiom is replaced by a weaker condition coming from a so-called associator Φ which lies in $A \otimes A \otimes A$; and a quasitriangular quasi-Hopf algebra has the comultiplication adjusted by an element R of $A \otimes A$ (called an R-matrix). It is possible to perturb these structures in such a way that the defining axioms are unchanged - in other words to produce a new quasitriangular quasi-Hopf algebra which differs from the old only in that Φ and R are changed. This imposes conditions on the automorphisms of tensor products of representations of the quasitriangular quasi-Hopf algebra $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes V_2$, which amount to the following *hexagonal relation:*

$$f(X_3, X_1)X_3^m f(X_2, X_3)X_2^m f(X_1, X_2)X_1^m = 1 \tag{20}$$

together with

$$f(X, Y) = f(Y, X)^{-1} \tag{21}$$

where $f \in F_2$, the free group on two generators X and Y, $X_1X_2X_3 = 1$ and $m \in \mathbb{Z}$.

In addition, preserving the usual type of pentagonal commutative diagram for coassociativity in representations of Hopf algebras in this context of perturbation of structures gives rise to the *pentagonal relation:*

$$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}). \tag{22}$$

Here the x_{ij} with $1 \leq i < j \leq 4$ are standard generators of the colored braid group $K(0, 4)$ (the kernel of the natural map of the Artin braid group B_4 to S_4). These satisfy the relations

$$(x_{12}, x_{34}) = (x_{14}, x_{23}) = 1$$

$$(x_{13}, x_{12}^{-1} x_{24} x_{12}) = 1$$

and

$$(a_{ijk}, x_{ij}) = (a_{ijk}, x_{ik}) = (a_{ijk}, x_{jk}) = 1$$

where $i < j < k$ and $a_{ijk} = x_{ij} x_{ik} x_{jk}$. For our purposes we may regard the x_{ij} and x_{ij}^{-1} as formal variables satisfying the above relations along with $x_{ij} x_{ij}^{-1} = 1$.

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