On finite multiple zeta values

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Multiple Zeta Values

$$\zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

Here, $k_i \in \mathbf{N}$, $k_r > 1$ (for convergence).

Just simple real numbers!

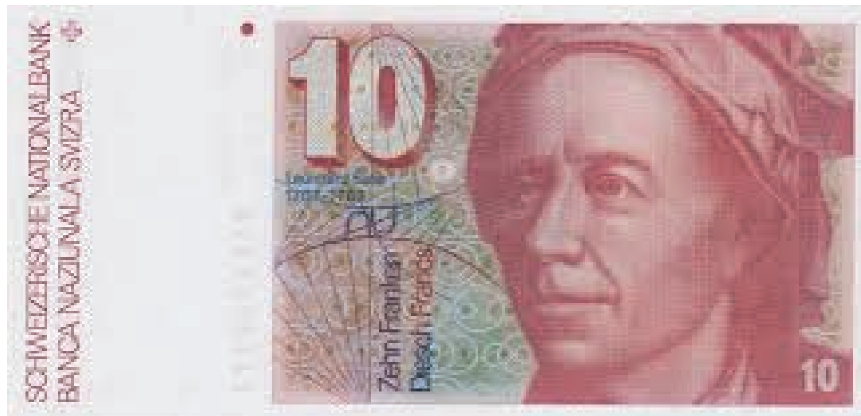
→ Knot invariants, KZ equation, Quantum groups

Galois rep'n on the fund. gp. of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$

Mixed Tate motives

Conformal field theory, Feynman amplitude etc.

Euler



L. Euler (1707 — 1783)

$r = 1$

- $\zeta(2n) = \frac{(-1)^{n-1}}{2}\frac{B_{2n}}{(2n)!}(2\pi)^{2n} \quad (n = 1, 2, 3, \ldots)$
- $\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (n = 0, 1, 2, \ldots)$

Here B_n is **Bernoulli number** : $\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$

$r = 2$

Letter from Goldbach to Euler (Dec.1742)

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GOLDBACH TO EULER

Moscow. December (13th) 24th, 1742[1]

Sir,

when I recently reconsidered the supposed sums of the two series mentioned at the end of my last letter,[2] I perceived at once that they had arisen by a mere writing mistake. But of this indeed the proverb says "If he had not erred, he should have achieved less";[3] for on that occasion I came upon the summations of some other series which otherwise I should hardly have looked for, much less discovered.

I daresay it is a problem among problems to determine the sum of

$$1+\frac{1}{2^n}\left(1+\frac{1}{2^m}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}\right)+\dots$$

in those cases where m and n are not equal even integers;[4] all the same there are some cases for which the sum can be indicated: if, e. g., $m = 1$, $n = 3$, then

$$1+\frac{1}{2^3}\left(1+\frac{1}{2}\right)+\frac{1}{3^3}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{4^3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\dots=\frac{\pi^4}{72}$$

(if, as usual, π is taken to be the circumference of the circle whose diameter equals 1). On the other hand, I do not yet know the sums of the series

$$A\ \dots\ 1+\frac{1}{2^5}\left(1+\frac{1}{2}\right)+\frac{1}{3^5}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{4^5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\dots$$

and

(English translation from German)

Euler's responce (Jan. 1743)

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EULER TO GOLDBACH
Berlin, January 19th, 1743

Sir,

indeed you have to consider the writing mistake you reported to be very fortunate, Sir,[1] as it gave rise to such marvellous inventions. It cost me many hours and extensive calculations to understand the truth of the summations that you kindly communicated to me; however, I cannot lay claim to any further glory than that I proved them. The method by which I arrived there is rather far-fetched and of such a nature that I should never have discovered those sums even with its help if they had not been known to me beforehand from your letter, Sir. I made use of the following series:

$$A = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad B = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$$

$$C = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots \qquad D = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \ldots$$

$$E = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots \qquad F = 1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \ldots$$

$$G = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \ldots \qquad H = 1 + \frac{1}{2^9} + \frac{1}{3^9} + \frac{1}{4^9} + \ldots$$

etc.,

of which the sums A, C, E, G, etc. are known. With great pains I finally elicited from them the following:

$$\alpha = 1 + \frac{1}{2^3}\left(1 + \frac{1}{2}\right) + \frac{1}{3^3}\left(1 + \frac{1}{2}\right.$$

$$= \frac{1}{2}A^2$$

$$\beta = 1 + \frac{1}{2^5}\left(1 + \frac{1}{2}\right) + \frac{1}{3^5}\left(1 + \frac{1}{2}\right.$$

$$= AC - \frac{1}{2}B^2$$

$$\gamma = 1 + \frac{1}{2^7}\left(1 + \frac{1}{2}\right) + \frac{1}{3^7}\left(1 + \frac{1}{2}\right.$$

$$= AE - BD + \frac{1}{2}C^2$$

$$\delta = 1 + \frac{1}{2^9}\left(1 + \frac{1}{2}\right) + \frac{1}{3^9}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4^9}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \ldots$$

$$= AG - BF + CE - \frac{1}{2}D^2$$

Euler's paper
(1776)

MEDITATIONES CIRCA SINGVLARE SERIERVM GENVS

Auctore:

L. EVLERO.

In commercio litterario, quod olim cum Illustrissimo *Goldbachio* coluerum, inter alias varii argumenti speculationes, circa series in hac forma generali:

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

comprehensas sumus versati, earumque summas scrutati. Tametsi autem huiusmodi series raro occurrere solent, parumque vtilitatis polliceri videntur, inuestigationes tamen, ad quas earum consideratio nos perduxerat, eo magis dignae videntur, vt ab obliuione et interitu vindicentur, quod methodi, qui-

Zagier's conjecture
(1994)

The dimension of the $\mathbf{Q}$ vector space

$$\mathcal{Z}_k := \sum_{k_1+\cdots+k_r=k} \mathbf{Q}\cdot\zeta(k_1,\ldots,k_r)$$

is equal to d_k given by

$$d_k = d_{k-2} + d_{k-3},\ \ d_0 = 1, d_{<0} = 0 \qquad ?$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28
2^{k-2}	–	–	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192

Th (Goncharov, Terasoma, 2002)

$$dim_{\mathbf{Q}} \mathcal{Z}_k \leq d_k.$$

→ there should exsit many relations among MZVs.

Example of relations (double shuffle relations)

$$\zeta(2)\zeta(3) = \sum_{0<m} \frac{1}{m^2} \sum_{0<n} \frac{1}{n^3} = \sum_{\substack{0<m \\ 0<n}} \frac{1}{m^2 n^3}$$

$$= \left(\sum_{0<m<n} + \sum_{0<n<m} + \sum_{0<m=n} \right) \frac{1}{m^2 n^3}$$

$$= \zeta(2,3) + \zeta(3,2) + \zeta(5)$$

$$\zeta(2)\zeta(3) = \zeta((2) * (3))$$ "stuffle" product

$$(2) * (3) = (2,3) + (3,2) + (5)$$

integral expression of MZV

$$\zeta(2) = \int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

$$\zeta(3) = \int_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}$$

$$\zeta(2,3)$$

$$= \int_{0<t_1<t_2<t_3<t_4<t_5<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{t_4}\frac{dt_5}{t_5}$$

$$
\begin{aligned}
\zeta(2)^2 &= \int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\int_{0<s_1<s_2<1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2} \\
&= \int_{\substack{0<t_1<t_2<1\\0<s_1<s_2<1}} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\int_{0<t_1<t_2<s_1<s_2<1} + \int_{0<t_1<s_1<t_2<s_2<1} + \int_{0<t_1<s_1<s_2<t_2<1} + \int_{0<s_1<t_1<t_2<s_2<1} \\
&\qquad + \int_{0<s_1<t_1<s_2<t_2<1} + \int_{0<s_1<s_2<t_1<t_2<1}\Big) \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2} \\
&= \int_{0<t_1<t_2<s_1<s_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2} + \int_{0<t_1<s_1<t_2<s_2<1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{dt_2}{t_2}\frac{ds_2}{s_2} \\
&+ \int_{0<t_1<s_1<s_2<t_2<1} \frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} + \int_{0<s_1<t_1<t_2<s_2<1} \frac{ds_1}{1-s_1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_2}{s_2} \\
&+ \int_{0<s_1<t_1<s_2<t_2<1} \frac{ds_1}{1-s_1}\frac{dt_1}{1-t_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2} + \int_{0<s_1<s_2<t_1<t_2<1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \\
&= \zeta(2,2) + \zeta(1,3) + \zeta(1,3) + \zeta(1,3) + \zeta(1,3) + \zeta(2,2) \\
&= 2\zeta(2,2) + 4\zeta(1,3).
\end{aligned}
$$

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(1,3)$$
$$= \zeta((2) ш (2))$$

shuffle product

$$(2) ш (2) = 2(2,2) + 4(1,3)$$

$$\zeta(2)^2 = \zeta((2) * (2)) = 2\zeta(2,2) + \zeta(4)$$
$$= \zeta((2) ш (2)) = 2\zeta(2,2) + 4\zeta(1,3)$$

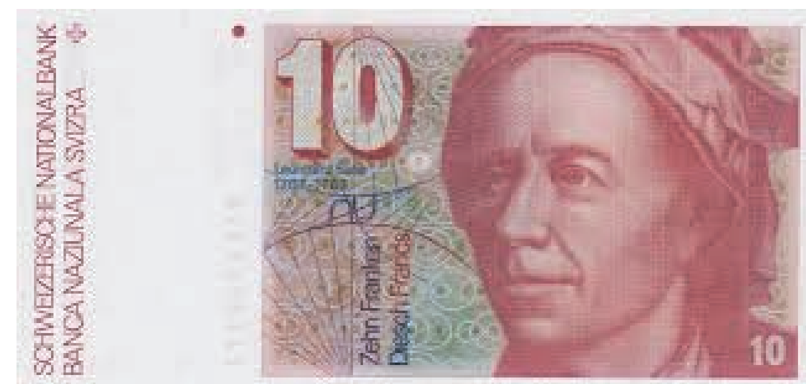
$$\Longrightarrow \quad \zeta(4) = 4\zeta(1,3)$$

double shuffle relation

regularized double shuffle relations

$$\zeta(1)\zeta(2) = \zeta((1) * (2)) = \zeta(1,2) + \zeta(2,1) + \zeta(3)$$
$$= \zeta((1) \sqcup\!\sqcup (2)) = 2\zeta(1,2) + \zeta(2,1)$$

$$\Longrightarrow \quad \zeta(3) = \zeta(1,2)$$



regularized values

$$\zeta^{*}(k_1, k_2, \ldots, k_r) \qquad \zeta^{\sqcup\!\sqcup}(k_1, k_2, \ldots, k_r)$$

for $k_r = 1$

Conjecture (folklore)

Regularized double shuffle relations are enough to describe all linear relations among multiple zeta values.

Various other relations are known which are conjectured to give all relations of multiple zeta values.

- — associator relations
- — Kawashima relations
- — confluence relations
- — integral-series relations

Hoffman's conj. (1997)

$$\mathcal{Z} = \sum_{k=0}^{\infty} \mathcal{Z}_k$$ is generated by $\zeta(k_1, \ldots, k_r)$'s with all k_i

being either 2 or 3.

Brown's Th. (2012)

This is indeed so.

→

Many important consequence like the Deligne-Ihara conjecture

Finite multiple zeta values (with D. Zagier)

For a prime p

$$\sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p \in \mathbf{Z}/p\mathbf{Z}$$

We consider the collection of this for all p in the ring

$$\mathcal{A} := \prod_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z} \Big/ \bigoplus_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z}$$

- This ring is a $\mathbf{Q}$ algebra (of char. 0).

Definition

$$\zeta^{\mathcal{A}}(k_1,\ldots,k_r) = \left(\zeta_p(k_1,\ldots,k_r) \bmod p\right)_{p \text{ prime}} \in \mathcal{A}$$

where

$$\zeta_p(k_1,\ldots,k_r) = \sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

- May consider similar problems as in the classical case.
- Similar relations hold (rather surprisingly).

Ex 1) If $k \neq 0$ then $\zeta^{\mathcal{A}}(k) = 0$.

$\because)$
$$p-1 \nmid k \Rightarrow \sum_{m=1}^{p-1} \frac{1}{m^k} \equiv 0 \pmod{p}.$$

2) $$\zeta^{\mathcal{A}}(k_1, k_2) = (-1)^{k_2} \binom{k_1+k_2}{k_1} Z(k_1+k_2).$$

with $$Z(k) := \left(\frac{B_{p-k}}{k} \bmod p \right)_p \in \mathcal{A}.$$

$\because)$ Seki-Bernoulli formula for sum of powers

$$\sum_{0<m<n<p} \frac{1}{m^{k_1} n^{k_2}} \equiv \sum_{0<m<n<p} m^{p-1-k_1} n^{-k_2} \pmod{p}.$$

Dimension conjecture

The $\mathbf{Q}$ vector space

$$\mathcal{Z}_{\mathcal{A},k} := \sum_{k_1+\cdots+k_r=k} \mathbf{Q} \cdot \zeta^{\mathcal{A}}(k_1, \ldots, k_r)$$

is of dimension d_{k-3} $(= d_k - d_{k-2})$?

Example of relations

As an analog of the classical sum formula

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_r\geq 2}} \zeta(k_1,\ldots,k_r) \;=\; \zeta(k)$$

we have (Saito-Wakabayashi)

$$\sum_{\substack{k_1+\cdots+k_r=k \\ k_r\geq 2}} \zeta^{\mathcal{A}}(k_1,\ldots,k_r) \;=\; \left(1+(-1)^r\binom{k-1}{r-1}\right)Z(k)$$

$$\left(Z(k)=\left(\frac{B_{p-k}}{k} \bmod p\right)_p \in \mathcal{A}\right)$$

Is this $Z(k)$ an $\mathcal{A}-$analog of $\zeta(k)$?

"Explanation"

$$\zeta(k) \text{ ``} \underset{\text{Fermat}}{\equiv} \text{'' } \zeta(k-(p-1)) \underset{\text{Euler}}{=} -\frac{B_{p-k}}{p-k}$$

When k is even, $Z(k) = 0.$

- Is $Z(k) \neq 0$ when k is odd?

Theorem—Conjecture

For any index sets $\mathbf{k}, \mathbf{l}$, the relations

$$\zeta^{\mathcal{A}}(\mathbf{k})\zeta^{\mathcal{A}}(\mathbf{l}) = \zeta^{\mathcal{A}}(\mathbf{k} * \mathbf{l})$$

$$\zeta^{\mathcal{A}}(\mathbf{k} ш \mathbf{l}) = (-1)^{|\mathbf{l}|}\zeta^{\mathcal{A}}(\mathbf{k}, \bar{\mathbf{l}})$$

hold.

All relations of fMZVs over $\mathbf{Q}$ can be deduced from these ?

Finite “real” MZVs

Real analog of $\zeta^{\mathcal{A}}(k_1, \ldots, k_r)$?

- $\zeta(k_1, \ldots, k_r)$ itself is not really an analog, as for instance $\zeta^{\mathcal{A}}(k) = 0$ shows.
- We have found a correct analog of $\zeta^{\mathcal{A}}(k_1, \ldots, k_r)$ in the real world, which seems to satisfy exactly the same relations.

Definition

$$S^\bullet(k_1,\dots,k_r) := \sum_{i=0}^{r}(-1)^{k_{i+1}+\cdots+k_r}\zeta^\bullet(k_1,\dots,k_i)\zeta^\bullet(k_r,\dots,k_{i+1})$$

Here, $\bullet = *$ or ш, and ζ^*, $\zeta^{\text{ш}}$ are two regularized values.

$$S^*(k_1,\dots,k_r) = \text{“}\sum_{\substack{m_1\prec\cdots\prec m_r\\ m_i\neq 0}}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\text{”}$$

with

$$1\prec 2\prec 3\prec\cdots\prec(\infty=-\infty)\prec\cdots-3\prec -2\prec -1$$

Proposition

$$S^*(k_1,\ldots,k_r) \equiv S^{ш}(k_1,\ldots,k_r) \mod \pi^2.$$

Definition

Define $\zeta^S(k_1,\ldots,k_r) \in \mathcal{Z}/\pi^2\mathcal{Z}$ by

$$\zeta^S(k_1,\ldots,k_r) := S^\bullet(k_1,\ldots,k_r) \mod \pi^2$$

(may $k_r = 1$, and $\mathcal{Z}$ is the $\mathbf{Q}$ algebra generated by classical multiple zeta values.)

Ex 1)

$$S^{\bullet}(k) = (-1)^k \zeta^{\bullet}(k) + \zeta^{\bullet}(k) = \begin{cases} 2\zeta(k) & k\colon \ \text{even}, \\ 0 & k\colon \ \text{odd}. \end{cases}$$

Hence $\zeta^{S}(k) = 0$ (Euler) .

2)

$$S^{\bullet}(k_1, k_2)$$

$$\equiv \begin{cases} 0 \pmod{\pi^2} & k_1 + k_2\colon \ \text{even}, \\ (-1)^{k_2} \binom{k_1+k_2}{k_1} \zeta(k_1+k_2) \pmod{\pi^2} & k_1 + k_2\colon \ \text{odd}. \end{cases}$$

Hence

$$\zeta^{S}(k_1, k_2) = (-1)^{k_2} \binom{k_1+k_2}{k_1} \zeta(k_1+k_2) \bmod \pi^2.$$

Recall

1) If $k \neq 0$ then $\zeta^{\mathcal{A}}(k) = 0.$

2) $$\zeta^{\mathcal{A}}(k_1, k_2) = (-1)^{k_2} \binom{k_1 + k_2}{k_1} Z(k_1 + k_2).$$

with $$Z(k) := \left(\frac{B_{p-k}}{k} \bmod p \right)_p \in \mathcal{A}.$$

"Main Conjecture"

The $\mathbf{Q}$ algebra generated by $\zeta^{\mathcal{A}}(k_1, \ldots, k_r)$ is

isomorphic to $\mathcal{Z}/\pi^2\mathcal{Z}$ under

$$\zeta^{\mathcal{A}}(k_1, \ldots, k_r) \longleftrightarrow \zeta^{S}(k_1, \ldots, k_r)$$

- The dimension of weight k piece $= d_k - d_{k-2} = d_{k-3}$.

- If $Z(k) \longleftrightarrow \zeta(k)$, then

$$Z(k) \neq 0 \Longleftrightarrow \zeta(k) \bmod \pi^2 \neq 0$$

for odd k ?

Ex

$$\sum_{\substack{k_1+\cdots+k_r=k\\ k_r\geq 2}} \zeta^S(k_1,\ldots,k_r) = \left(1+(-1)^r\binom{k-1}{r-1}\right)\zeta(k) \mod \pi^2$$

cf.

$$\sum_{\substack{k_1+\cdots+k_r=k\\ k_r\geq 2}} \zeta^{\mathcal{A}}(k_1,\ldots,k_r) = \left(1+(-1)^r\binom{k-1}{r-1}\right)Z(k)$$

- Many other relations hold in exactly the same form for

$\zeta^{\mathcal{A}}(k_1,\ldots,k_r)$ and $\zeta^S(k_1,\ldots,k_r)$.

Theorem

For any index sets $\mathbf{k}, \mathbf{l}$, the relations

$$\zeta^S(\mathbf{k})\zeta^S(\mathbf{l}) = \zeta^S(\mathbf{k} * \mathbf{l})$$

$$\zeta^S(\mathbf{k} ш \mathbf{l}) = (-1)^{|\mathbf{l}|}\zeta^S(\mathbf{k}, \bar{\mathbf{l}})$$

hold.

$$\left(\begin{array}{l} \zeta^{\mathcal{A}}(\mathbf{k})\zeta^{\mathcal{A}}(\mathbf{l}) = \zeta^{\mathcal{A}}(\mathbf{k} * \mathbf{l}) \\ \zeta^{\mathcal{A}}(\mathbf{k} ш \mathbf{l}) = (-1)^{|\mathbf{l}|}\zeta^{\mathcal{A}}(\mathbf{k}, \bar{\mathbf{l}}) \end{array} \right)$$

Recent progress

Theorem Akagi-Hirose-Yasuda, Jarossay

$$dim_{\mathbf{Q}} \mathcal{Z}_{\mathcal{A},k} \leq d_{k-3}.$$

They relate $\zeta^{\mathcal{A}}(k_1, \ldots, k_r)$ to " p-adic MZV".

おわり

Thank you!