STRUCTURE OF FINE SELMER GROUPS

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Abstract. In this talk, we explain and provide more evidence on the relationship between the classical Iwasawa $\mu = 0$ conjecture and the $\mu = 0$ conjecture for Fine Selmer groups, first observed in [CS05] under some strict hypotheses. We give sufficient conditions to prove the classical $\mu = 0$ conjecture that improves upon the result of [Ces15]. We also prove isogeny invariance of Conjecture A in previously unknown cases.

1. Classical Iwasawa Theory

Classical Iwasawa theory is concerned with the growth of arithmetic objects in towers of number fields. More precisely, one studies the growth of $p$-parts of class groups in towers of number fields of $p$-power degree. This growth was shown by Iwasawa to often exhibit certain regularity that can be described by a $p$-adic invariant of a complex-valued $L$-function.

Consider a tower of number fields

$$F = F_0 \subset F_1 \subset F_2 \subset \ldots F_n \ldots \subset F_\infty = \bigcup_{n=0}^{\infty} F_n$$

where $F_n/F$ is cyclic of degree $p^n$. The Galois group $\Gamma = \text{Gal}(F_\infty/F)$ is defined as the inverse limit of the Galois groups $\Gamma_n = \text{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$. Thus,

$$\Gamma := \lim_{\leftarrow \atop n} \Gamma_n = \lim_{\leftarrow \atop n} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

Here, $\mathbb{Z}_p$ is the additive group of $p$-adic integers and is therefore a compact group when given the $p$-adic topology. Every number field, $F$, has a cyclotomic $\mathbb{Z}_p$-extension which is the unique $\mathbb{Z}_p$-extension of $F$ contained in $\bigcup_n F(\zeta_{p^n})$.

Let $F_\infty/F$ be a fixed $\mathbb{Z}_p$-extension. In [Iwa59], it was shown that the growth of the $p$-part of the class group of $F_n$, denoted by $A_n$, is regular.

**Theorem. (Iwasawa)** There exist non-negative integers $\lambda$ and $\mu$, and an integer $\nu$ such that for large enough $n$,

$$|A_n| = p^{\mu p^n + \lambda n + \nu}.$$

The integers $\lambda, \mu, \nu$ are independent of $n$.

Serre observed that the Iwasawa algebra, $\Lambda(\Gamma) = \Lambda := \lim_{\leftarrow \atop n} \mathbb{Z}_p[\Gamma_n] = \mathbb{Z}_p[[\Gamma]]$ is isomorphic to the power series ring, $\mathbb{Z}_p[[T]]$. This isomorphism of compact $\mathbb{Z}_p$-algebras is given by $\gamma_0 - 1 \mapsto T$, where $\gamma_0$ is a topological generator of $\Gamma$.

The structure theorem of finitely generated modules over $\Lambda$ mimics the theory of finitely generated modules over a PID if one treats $\Lambda$-modules as being defined

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1This work is a part of my PhD thesis. Feel free to contact me if you want details of the proof.

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up to finite submodules and quotient modules. The notion of pseudo-isomorphism, i.e., a homomorphism with a finite kernel and cokernel, gives an equivalence relation on any set of finitely generated, torsion $\Lambda$-modules.

**Theorem.** For any finitely generated, torsion $\Lambda$-module, $M$, there is a pseudo-isomorphism

$$M \rightarrow \bigoplus_{i=1}^{s} \Lambda/(f_i^{l_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(p^{m_j})$$

where $s, t \geq 0$, each $f_i$ is an irreducible distinguished polynomial and $l_i, m_j$ are positive integers.

Set the notation,

$$\lambda(M) = \sum_i l_i \deg(f_i), \quad \mu(M) = \sum_j m_j$$

and the characteristic polynomial $f_M(T) = p^{\mu(M)} \prod_i f_i^{l_i}$. This polynomial generates the characteristic ideal, i.e.,

$$\text{char}(M) = \left( p^{\mu(M)} \prod_i f_i^{l_i} \right).$$

In the classical case, the Artin isomorphism identifies $A_n$ with the Galois group, $X_n := \text{Gal}(L_n/F_n)$, of the maximal unramified abelian $p$-extension $L_n$ of $F_n$. The inverse limit of Artin isomorphisms then identifies $\lim_{\leftarrow} A_n$ with the Galois group, $X_\infty := \text{Gal}(L_\infty/F_\infty)$. Here, $L_\infty = \bigcup_n L_n$. Note that $X_\infty$ is a pro-$p$ group. $X_\infty$ is a finitely generated torsion $\Lambda$-module; thus the structure theorem applies. With the notation as above, $\lambda = \lambda(X_\infty)$ and $\mu = \mu(X_\infty)$.

**Conjecture.** (Iwasawa) When $F_\infty = F_{\text{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension, $\mu = 0$.

This is known for the case when $F$ is an Abelian extension of $\mathbb{Q}$ [FW79]. A different proof using $p$-adic L-functions is given in [Sin84].

2. **Iwasawa Theory of Elliptic Curves**

In [Maz72], the Iwasawa theory of Selmer groups was introduced. Using this theory, it is possible to describe the growth of the size of the $(p$-part of) Selmer group in $\mathbb{Z}_p$ towers.

There are several equivalent definitions of the $p^\infty$ Selmer group. We use the definition as in [Wu04].

**Definition 2.1.** The usual $(p^\infty)$-Selmer group of $A/F$ for a fixed prime $p$ is the following direct limit, $\text{Sel}(A/F) := \lim_{\rightarrow} k \text{Sel}^k(A/F)$ where

$$\text{Sel}^k(A/F) := \ker \left( H^1(G_S(F), A[p^k]) \rightarrow \bigoplus_{v \in S} H^1(F_v, A[p^k]) \right)$$

where $v$ runs over all the primes of $F$. For any $G$-module, $M$, we use the notation $H^*(F_v, M)$ for the Galois cohomology of the decomposition group at $v$. 

Recall that
\[ \bigoplus_{v \in S} H^1(F_v, A[p^k]) \simeq \bigoplus_{v \in S} \frac{H^1(F_v, A[p^k])}{\operatorname{Im}(\kappa_{A,p^k})} \]
where \(\kappa_{E,p^k} : A(F_v)/p^k A(F_v) \rightarrow H^1(F_v, A[p^k])\). One can check that \(\operatorname{Sel}(A/F)\) is in fact a \(p\)-primary subgroup, ie \(\operatorname{Sel}(A/F) = \operatorname{Sel}(A/F)_p\). For a detailed explanation one may check [Gre01, §2]. Set the notation,
\[ \operatorname{Sel}(A/F) = \lim_{\rightarrow L} \operatorname{Sel}(A/L) \]
where the inductive limit is over all finite extensions \(L/F\) contained in \(F_\infty\). Since \(p\) is fixed, we drop it from the notation.

Mazur proved that the Pontryagin dual of the Selmer group, denoted by \(X(A/F_\infty)\), is a finitely generated \(\Lambda\)-module. However it need not always be torsion.

**Conjecture.** ([Maz72]) Let \(A\) be an Abelian variety and \(p\) be a prime of good ordinary reduction. \(X(A/F_\infty)\) is \(\Lambda\)-torsion.

When \(X(A/F_\infty)\) is \(\Lambda\)-torsion, there is a structure theorem as before. However, even for the cyclotomic \(\mathbb{Z}_p\)-extension of \(\mathbb{Q}\) there are examples of elliptic curves where \(\mu(X(E/Q_\infty)) > 0\) where \(p\) is a prime of good ordinary reduction.

In the fundamental paper of [CS05], they studied a certain subgroup, called the fine Selmer group. They made the following conjecture.

**Conjecture.** ([CS05, Conjecture A]) Let \(p\) be an odd prime and \(E/F\) be an elliptic curve. When \(F_\infty = F_{\text{cyc}}\), the Pontryagin dual of the fine Selmer group, denoted by \(Y(E/F_{\text{cyc}})\) is a finitely generated \(\mathbb{Z}_p\)-module ie \(Y(E/F_{\text{cyc}})\) is \(\Lambda\)-torsion and \(\mu(Y(E/F_{\text{cyc}})) = 0\).

This conjecture is far from being proven. Even isogeny invariance of this conjecture is yet unknown.

Note that there is no restriction on the reduction type at \(p\). Conjecture A is equivalent to the vanishing of \(H^2(G_S(F_{\text{cyc}}), E[p^\infty])\) and the associated \(\mu\)-invariant [CS05, Cor 3.3]. The vanishing of \(H^2(G_S(F_{\text{cyc}}), E[p^\infty])\) is called the elliptic curve analogue of the weak Leopoldt conjecture and is proved for elliptic curves over Abelian number fields in [Kat04].

Let \(A/F\) be a \(d\)-dimensional Abelian variety and \(S\) be a finite set of primes of \(F\) including the Archimedean primes, the primes of \(F\) above \(p\) and the primes where \(A\) has bad reduction. Fix an algebraic closure \(\overline{F}/F\) and set \(F_S\) to be the maximal subfield of \(\overline{F}\) containing \(F\) which is unramified outside \(S\). Denote the absolute Galois extension \(\operatorname{Gal}(\overline{F}/F)\) by \(G\) and the Galois group, \(\operatorname{Gal}(F_S/F)\) by \(G_S(F)\).

**Definition 2.2.** The \((p^\infty)\text{-fine Selmer group}\) is defined as
\[ R(A/F) := \ker \left( H^1(G_S(F), A[p^\infty]) \to \bigoplus_{v \in S} H^1(F_v, A[p^\infty]) \right). \]

With the inductive limit over all finite extensions \(L/F\) contained in \(F_\infty\), set
\[ R(A/F_\infty) = \lim_{\rightarrow L} R(A/L). \]
The above definition is independent of the choice of $S$. Indeed, it follows from the exact sequence
\[ 0 \to R(A/F) \to \text{Sel}(A/F) \to \bigoplus_{v|p} A(F_v) \otimes Q_p/Z_p. \]

A priori it is not obvious there is a relation between Galois modules coming from class groups and those coming from elliptic curves. In [CS05] they showed that under certain strict conditions, the $p$-part of fine Selmer group grows like the $p$-part of the class group in a cyclotomic tower.

**Theorem.** [CS05, Theorem 3.4] Let $p$ be an odd prime such that the extension $F(E[p^\infty])/F$ is pro-$p$. Then conjecture A holds for $E$ over $F_{\text{cyc}}$ if and only if the classical Iwasawa $\mu = 0$ conjecture holds for $F_{\text{cyc}}$.

A different proof of a slight variant of this theorem was given by [LM16] by comparing $p$-ranks of class groups and fine Selmer groups. Instead of assuming that $F(E[p^\infty])/F$ is pro-$p$, they assume $F(E[p])/F$ is a finite $p$-extension.

An important corollary of the above theorem is the following.

**Corollary.** Assume $F$ is an Abelian extension of $\mathbb{Q}$ and that $p$ is an odd prime such that $E(F)[p] \neq 0$. Then Conjecture A is valid for $E/F_{\text{cyc}}$.

The proof of this corollary crucially uses that the Iwasawa $\mu = 0$ conjecture is known for all Abelian extensions of $\mathbb{Q}$.

### 3. Results

It is in fact possible to prove a stronger statement using only the Iwasawa $\mu = 0$ conjecture for the fixed number field $F$.

**Theorem 3.1.** Assume either
1. $F$ contains $\mu_p$ or
2. $F$ is a totally real field.

In either of the cases, suppose the classical Iwasawa $\mu = 0$ is true. If $E$ is an elliptic curve over $F$ such that $E(F)[p] \neq 0$, then Conjecture A holds for $Y(E/F_{\text{cyc}})$.

**Remark.** The first case subsumes the corollary above. Indeed, if $F/\mathbb{Q}$ is Abelian then so is $F(\mu_p)/\mathbb{Q}$ and the Iwasawa $\mu = 0$ Conjecture is known to be true.

Using similar machinery, it is possible to prove a (stronger) converse of the above theorem.

**Theorem 3.2.** Let $E$ be an elliptic curve defined over the number field $F$. Let $p$ be any odd prime. Further assume that $E(F)[p] \neq 0$. If Conjecture A holds for $Y(E/F_{\text{cyc}})$ then the classical $\mu = 0$ conjecture holds for $F_{\text{cyc}}/F$.

Theorem 3.2 implies that given a number field $F$, it is enough to provide one example of an elliptic curve $E/F$ such that $E(F)[p] \neq 0$ for which conjecture A holds to prove the classical $\mu = 0$ conjecture for $F_{\text{cyc}}/F$. By a result of Merel, our theorem can at best show the classical Iwasawa $\mu = 0$ conjecture for finitely many primes. However, it improves upon the following theorem of [Fes15].

**Theorem.** For a prime $p$ and a number field $F$, to prove the classical Iwasawa $\mu = 0$ conjecture, it suffices to find an Abelian $F$-variety, $A$ such that...
(1) A has good ordinary reduction at all places above p,
(2) A has $\mathbb{Z}/p\mathbb{Z}$ as an $F$-subgroup,
(3) $X(A/F_{cyc})$ is $\Lambda$-torsion and has $\mu$-invariant 0.

This is an improvement because

(1) There is no condition on the reduction type at primes above $p$, unlike when one is working with the Selmer group.
(2) There are no known examples where $\mu(Y) > 0$ but there are several examples where $\mu(X) > 0$ even when the base field is $\mathbb{Q}$ and $p$ is a prime of good ordinary reduction.
(3) $F(E[p])/F$ need not be a pro-$p$-extension as in [CS05]. This is a relatively strict condition to impose. The only requirement is $E(F)[p] \neq 0$.

The two main theorems mentioned above can prove isogeny invariance of Conjecture A in some previously unknown cases.

**Corollary 3.3.** Let $F$ be a number field that contains $\mu_p$ or be a totally real number field. Let $E$ and $E'$ be isogenous elliptic curves such that both $E$ and $E'$ have nontrivial $p$-torsion points over $F$. Then, Conjecture A holds for $Y(E/F_{cyc})$ if and only if Conjecture A holds for $Y(E'/F_{cyc})$.

**Remark.** All statements hold for Abelian varieties of dimension $d$. The only property of the cyclotomic $\mathbb{Z}_p$-extension we use in the proofs is that primes in $S$ decompose finitely. The theorems go through for a more general class of $\mathbb{Z}_p$-extensions.

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**References**
