A relative endoscopic fundamental lemma for unitary groups

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Main objects of interest: Relative orbital integrals

- E/F be an unramified quadratic extension of p-adic fields,
- (V, Φ) (split) Hermitian space with unitary group $U(V)$.
- Study action $U(V) \times U(V)$ on $\mathrm{End}(V)$ by

$$(g, h) \cdot X = gXh^{-1}.$$

Relative orbital integrals:

$$\mathrm{RO}(X, f) := \int_{T_X \backslash U(V) \times U(V)} f(g^{-1}Xh) \frac{d(g, h)}{dt},$$

for $f \in C_c^\infty(\mathrm{End}(V))$ and X is “regular semi-simple” in $\mathrm{End}(V)$.

Case of primary interest: $f = \mathbf{1}_{\mathrm{End}(\Lambda)}$ where $\Lambda \subset V$ is a self-dual lattice.

Stable orbital integrals

This action is not “stable”: Two types of orbit

- **rational orbits:** $X' = gXh^{-1}$ for $(g, h) \in U(V) \times U(V)$
- **stable orbit**: $X' = gXh^{-1}$ for $(g, h) \in (U(V) \times U(V))_{\overline{F}}$.

$\mathrm{RO}(X, f)$ only knows rational orbits, but stable orbits are (somehow) more natural. We set

$$\mathrm{SRO}(X, f) := \sum_{X \sim_{st} X'} \mathrm{RO}(X', f)$$

to be the **stable relative orbital integral.**

- these rational orbits are parametrized by cohomology classes

$$inv(X, X') \in H^1(F, T_X).$$

What's the difference?: κ-orbital integrals

Definition

For any character $\kappa : H^1(F, T_X) \to \mathbb{C}^\times$ and any $f \in C_c^\infty(\mathrm{End}(V))$, define the κ-**relative orbital integral** to be

$$\mathrm{RO}^\kappa(X, f) := \sum_{X \sim_{st} X'} \kappa(inv(X, X'))\mathrm{RO}(X', f).$$

When $\kappa = 1$, we have $\mathrm{RO}^\kappa = \mathrm{SRO}$ is the stable orbital integral.

$$\mathrm{RO}(X, f) = c\left(\mathrm{SRO}(X, f) + \sum_{\kappa \neq 1} \mathrm{RO}^\kappa(X, f)\right)$$

Problem of geometric stabilization

- For global purposes, need to express κ-orbital integrals in terms of stable orbital integrals.

Goal

Find groups of smaller dimension H_κ acting on varieties X_κ so that

$$\kappa\text{-orbital integrals of } (U(V) \times U(V), \mathrm{End}(V))$$

may be expressed in terms of

$$\text{stable orbital integrals of } (H_\kappa, X_\kappa).$$

Adjoint case: $(H_\kappa, X_\kappa) = (H, \mathrm{Lie}(H))$ is an **endoscopic group** acting on its Lie algebra (Langlands,Shelstad, Kottwitz, Waldspurger, Laumon, Ngô...)

Proposed endoscopic spaces

- Decompose $V = V_1 \oplus V_2$ into (split) Hermitian subspaces,
- Then $U(V_1) \times U(V_1)$ and $U(V_2) \times U(V_2)$ acts on

$$\mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \subset \mathrm{End}(V).$$

- Call the pair

$$(U(V_1) \times U(V_1), \mathrm{End}(V_1)) \oplus (U(V_2) \times U(V_2), \mathrm{End}(V_2))$$

an **endoscopic space** for $(U(V) \times U(V), \mathrm{End}(V))$.

Hope

Should satisfy a fundamental lemma and smooth transfer

Conjectural fundamental lemma

Recall that $\Lambda \subset V$ is our self-dual lattice.

Conjecture: Relative endoscopic fundamental lemma (L)

For a decomposition $V = V_1 \oplus V_2$, we may associate a character κ and a transfer factor Δ such that, if

$$X \in \mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \subset \mathrm{End}(V),$$

then

$$\mathrm{SRO}(X, \mathbf{1}_{\mathrm{End}(\Lambda_1)} \oplus \mathbf{1}_{\mathrm{End}(\Lambda_2)}) = \Delta(X)\mathrm{RO}^{\kappa}(X, \mathbf{1}_{\mathrm{End}(\Lambda)}),$$

where $\Lambda = \Lambda_1 \oplus \Lambda_2$.

The case of $U(2) \times U(2)$

In low rank, we can compute both sides.

Theorem (L)

The fundamental lemma is true for $\dim(V) = 2$.

- First example of an endoscopic fundamental lemma for relative orbital integrals in the literature.
- **Key step:** Make precise by developing the appropriate transfer factor for arbitrary $\dim(V)$.

"Regular" smooth transfer

We are also interested in transferring general test functions f. We can do this for many functions.

Theorem (L)

For any $\dim V$, there exists a transfer factor Δ such that

$$\Delta(X)\mathrm{RO}^{\kappa}(X, f)$$

is the stable orbital integral for some $f' \in C_c^{\infty}(\mathrm{End}(V_1) \oplus \mathrm{End}(V_2))$ whenever $\mathrm{supp}(f) \subset \mathrm{GL}(V)$.

This is overly simplistic: we must include terms associated to non-split Hermitian spaces as well.

Transfer factors: Try to reduce to adjoint case

Lemma

The invariant map $\chi : \mathrm{End}(V) \to F^d$ may by factored

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[rr, "X \mapsto XX^*"] \arrow[dr, "\chi"'] & & \mathfrak{u}(V) \arrow[dl, "\text{charpoly}"] \\
& F^d &
\end{tikzcd}$$

realizing $\mathfrak{u}(V)$ as the categorical quotient $\mathrm{End}(V)//U(V)$.

- Reduces the problem of defining matching and the transfer factors to the Langlands-Shelstad-Kottwitz case,
- but **not** the FL or smooth transfer.

Where does this come from?

Periods of automorphic forms!

Periods of automorphic forms

- Fix a reductive group G over $\mathbb{Q}$, and let $H \subset G$ be a closed algebraic subgroup.
- An automorphic representation π (always irred cuspidal) of $G(\mathbb{A})$ is H-**distinguished** if the period integral

$$\mathcal{P}_H(\varphi) := \int_{[H]} \varphi(h)dh \neq 0$$

  for some $\varphi \in \pi$. Here $[H] := H(\mathbb{Q})A_{G,H}(\mathbb{A})\backslash H(\mathbb{A})$.
- Closely related to special values/poles of L-functions and Functorial lifting from smaller groups.

Example: Linear periods

- Let V be a d-dimensional $\mathbb{Q}$-vector space, set $W = V \oplus V$,
- $\mathrm{GL}(V) \times \mathrm{GL}(V) \subset \mathrm{GL}(W)$

Theorem (Friedberg-Jacquet, ′95)

Let π be a cuspidal automorphic representation of $\mathrm{GL}(W)_{\mathbb{A}}$. The following are equivalent:

1. π is $\mathrm{GL}(V) \times \mathrm{GL}(V)$-distinguished
2. $L(s, \pi, \wedge^2)$ has a pole at $s = 1$

This implies that $\pi \cong \pi^\vee$.

Our case of interest: Unitary linear periods

- $E/\mathbb{Q}$ a quadratic extension
- (V_1, Φ_1) and (V_2, Φ_2) d dim'l Hermitian spaces over E
- $W = V_1 \oplus V_2$, $\Phi = \Phi_1 \oplus \Phi_2$
- $U(V_1) \times U(V_2) \subset U(W)$

Study representations distinguished by $U(V_1) \times U(V_2)$

For any data $\{W, \Phi, W = V_1 \oplus V_2\}$, consider the distribution

$$J(f) := \int_{[U(V_1)\times U(V_2)]} \int_{[U(V_1)\times U(V_2)]} \left(\sum_{\gamma \in U(W)(\mathbb{Q})} f(h^{-1}\gamma g) \right) dgdh.$$

Relative trace formula

RTF

$$\sum_{\pi} c(\pi) \sum_{\varphi} |\mathcal{P}_H(\varphi)|^2 \approx J(f) \approx \sum_{\gamma} a(\gamma)\mathrm{RO}(\gamma, f),$$

where π sums over $U(V_1) \times U(V_2)$-distiguished reps and γ sums over orbits.

Similar stability issue as before:

$$J(f) = \underbrace{SJ(f)}_{\text{Stable part of RTF}} + \underbrace{\sum_{\kappa} J^{\kappa}(f)}_{\text{endoscopic pieces}}$$

Need to express as sum of stable distibutions on endoscopic spaces.

Motivating global result

Let σ be an automorphic representation (some local constraints...) of $U(W)$ and let $\Sigma = BC(\sigma)$ be the base change to $\mathrm{GL}(W)$.

Theorem: Pollack-Wan-Zydor ('19)

Assume $U(W)$ is quasi-split. If σ is $U(V_1) \times U(V_2)$-distinguished, then Σ is $\mathrm{GL}(V) \times \mathrm{GL}(V)$- distinguished.

Goal: prove the converse

Show that if Σ is $\mathrm{GL}(V) \times \mathrm{GL}(V)$-distinguished, then σ is $U(V_1) \times U(V_2)$-distinguished.

Method: Comparison of relative trace formulas

Lemma

If Σ is both $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$- and $U(W, \Phi)$-distinguished (for some form Φ), then it is a base change $\Sigma = BC(\sigma)$ from the quasi-split unitary group $U(W)$.

This suggests the following comparison: For $f \in C_c^\infty(\mathrm{GL}(W)_{\mathbb{A}})$,

$$I(f) := \int_{[\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)]} \int_{[U(W)]} \left(\sum_{x \in \mathrm{GL}(W)_{\mathbb{Q}}} f(g^{-1}xh) \right) dg dh$$

Also satisfies a (simple) relative trace formula.

Naïve comparison

Possible comparison of RTFs

Find a matching of functions $f' \leftrightarrow f$ such that

$$I(f') = J(f),$$

by comparing geometric sides of RTFs.

- Way too simplistic, but something "spiritually" related to this makes sense.
- With this, standard techniques would imply the converse to Pollack-Wan-Zydor.

Problem: Can only compare to **the stable part**

$$SJ(f) = J(f) - \sum_{\kappa} J^{\kappa}(f).$$

Problem of geometric stabilization redux

Reduces to the local problem:

- Study κ-orbital integrals of $U(V_1) \times U(V_2)$-action on

$$\mathcal{Q} = U(W)/U(V_1) \times U(V_2)$$

- Standard technique: reduce to the Lie algebra $\mathrm{Lie}(\mathcal{Q})$.

Lemma

There is a natural identification

$$\mathrm{Lie}(\mathcal{Q}) \cong Hom_E(V_1, V_2).$$

When $V_1 \cong V_2$ are both split, we arrive at our objects of primary interest from earlier.

Conclusion

Two main steps toward geometric stabilization:

1. Establish the relative endoscopic fundamental lemma (Done for $(U(4), U(2) \times U(2))$),
2. Establish existence of transfer for all test functions (Known for "regularly supported functions").

For other global applications: still need to regularize spectral/geometric decompositions in the RTFs, relative character identities....

THANK YOU!!