Arithmetic statistics for knots and knot invariants

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Knots

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A 1-knot $K$ is a embedded submanifold of $S^3$ homeomorphic to $S^1$.

We can also define knots in higher dimensions:

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An $n$-knot $K$ is an embedded submanifold of $S^{n+2}$ homeomorphic to $S^n$. 
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\textbf{Note}

We consider knots to be \textit{oriented}, which means that we keep track of the orientation on $K$ as well as on the ambient $S^{n+2}$. 
Arithmetic Statistics

Studies distribution of the invariants of arithmetic objects, e.g. what is the class group of a random number field? rank of a random elliptic curve?

Cohen-Lenstra Heuristics

- The class group of a random real quadratic field is a 2-group times a finite group drawn from a given distribution (in particular the average size of the odd part is bounded).
- The class group of a random imaginary quadratic field is a 2-group times a cyclic group \( \sim \) 98% of the time.
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Look at distribution of invariants.

Connection to arithmetic statistics
Knot statistics???

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Connection to arithmetic statistics

Families of knots can be parametrized by arithmetic data.
The family of simple $n$-knots

**Definition**

An $n$-knot $K$ is *simple* if $\pi_i(S^{n+2} - K) = \pi_i(S^1)$ for $i \leq (n - 1)/2$. 

All 1-knots are simple. For large $n$, simple $n$-knots have an algebraic classification. Simple $n$-knots have been classified for all $n$ other than 1, 2, 3, 4 and 6. In this talk we’ll focus on the case of $n = 4a + 1$, $a > 1$. (The case of $n = 4a - 1$ is expected to be similar, while even dimensional cases have a different flavor.)
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Classification of simple $n$-knots

Theorem (Kearton-Levine-Trotter)

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The Alexander module $\text{Alex}_K$ is a complicated object, but it turns out that a lot of the information from it is contained in one polynomial, namely the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t]$ of a simple $4a + 1$-knot $K$. 

**Properties of the Alexander Polynomial**

- $\Delta_K(1) = 1$
- $\Delta_K(t^{-1}) = t - \deg \Delta_K$
- $\Delta_K$ has even degree.
- If $\Delta_K(t)$ is a quadratic polynomial, it must have the form $mt^2 + (1 - 2m)t + m$ for some $m \in \mathbb{Z}$. 
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Theorem (Bayer-Michel, Levine)

Let $\Delta$ be a squarefree polynomial. There are only finitely many distinct Alexander modules with Blanchfield pairing having the Alexander polynomial $\Delta$.

Hence, for fixed odd $q \geq 3$, there are only finitely many distinct simple knots with Alexander polynomial $\Delta$.

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Obtain a quantitative form of this finiteness statement.
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Seifert hypersurfaces and Seifert pairings

Our main tool is the theory of Seifert (hyper)surfaces.

Definition

A Seifert hypersurface for an \( n \)-knot \( K \) is a \( n+1 \)-manifold \( V^{n+1} \) embedded in \( S^{2n+2} \) with boundary \( \partial V = K \).

Seifert hypersurfaces exist for all knots, but are not unique. We say that \( V^{n+1} \) is a simple Seifert hypersurface if \( V \) is \( \lfloor n/2 \rfloor \)-connected.

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A simple Seifert hypersurface $V^{4a+2}$ comes with a bilinear pairing $\langle , \rangle$ on $H_{2a+1}(V, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, known as the Seifert pairing.

The Seifert pairing is neither symmetric nor skew-symmetric. However, its skew-symmetric part is equal to the intersection pairing on $H_{2a+1}(V, \mathbb{Z})$. (Any pairing whose skew-symmetric part is unimodular can be realized as a Seifert pairing.)
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Theorem (Kearton, Levine)

If $V^{4a+2}$ is a Seifert surface for $K^{4a+1}$ with nondegenerate Seifert matrix $P$, then the $\mathbb{Z}[t, t^{-1}]$-module $\text{Alex}_K$ is presented by the matrix

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$$\Delta_K(t) = \det(tP - P^t)$$
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We’ve seen that the equivalence class of a simple $4a + 1$-knot, $a \geq 1$, is completely determined by the Seifert matrix. But we have a choice of Seifert matrices! Different choices of Seifert surface for the same knot can have non-isomorphic Seifert pairings. That is, the Seifert matrices that are not equivalent up to change of basis.

\[
\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}
\]
Seifert pairings and orbits

If you pick a basis for $H_q(V, \mathbb{Z})$ in which the intersection pairing has matrix equal to the standard skew-symmetric matrix $J$, the matrix $P$ of the Seifert pairing will satisfy $P - P^t = J$.

**Observation**

Isomorphism classes of Seifert pairings are equivalent to orbits of the group $Sp_{2g}(\mathbb{Z})$ on the set of $2g \times 2g$-matrices $P$ with $P - P^t = J$. (Here $X \in Sp_{2g}(\mathbb{Z})$ acts by $P \mapsto XPX^t$.)
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If we change variables to $Q = P + P^t$, we see that these are also in bijective correspondence with $\text{Sp}_{2g}$-orbits on $2g \times 2g$ symmetric matrices (with a parity condition).
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Conclusion

Isomomorphism classes of Seifert pairings are orbits for the representation $\text{Sym}^2(2g)$ of $\text{Sp}_{2g}(\mathbb{Z})$!
The case $g = 1$

When $g = 1$, $Sp_{2g} = SL_2$, and the arithmetic invariant theory reduces to the question of binary quadratic forms with odd discriminant.
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**Seifert hypersurfaces**

Simple Seifert hypersurfaces with Alexander polynomial $mt^2 + (1 - 2m)t + m$ are in one-to-one correspondence with binary quadratic forms over $\mathbb{Z}$ of discriminant $1 - 4m$, or with narrow ideal classes of the ring $\mathbb{Z}[\gamma_m]$ where $\gamma_m = 1 + \sqrt{1 - 4m^2}$. There is also a generalization to higher degree Alexander polynomials.
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Counting Seifert pairings of genus one is the same as counting $SL_2$-orbits of binary quadratic forms of odd discriminant. We know how to do this, at least for definite forms.

**Theorem (Gauss-Lipschitz-Mertens)**

The number of $SL_2$-orbits of binary quadratic forms of discriminant $1 - 4m$ with $m \in [0, X]$ is $\sim X^{3/2}$.

In the indefinite case, we can only do this counting weighted by the regulator, but we still have the following heuristic for the unweighted count:

**Heuristic (Hooley)**

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So we get the same results for isomorphism classes of Seifert pairings, or of simple Seifert hypersurfaces.
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Asymptotics: Genus 1 Seifert pairings/ hypersurfaces

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Average-case counting simple knots of genus 1, equivalently genus 1 Alexander modules with Blanchfield pairing, is harder because we need to know the average size of the narrow class group of $R_{\Delta} = \mathbb{Z}\left[\frac{1}{m}\right][\frac{1+\sqrt{1-4m}}{2}]$. 

Case: $|m|$ is prime
Then $m$ factors as a product of principal prime ideals $\left(\frac{m}{\gamma}\right) = \left(\gamma\right)^2$, so the map $N\text{Cl}(\mathbb{Z}\left[\frac{1}{m}\right]) \to N\text{Cl}(R_m)$ is an isomorphism.

Case: $|m|$ is composite
For any $p || m$, the ideal $\left(p,\frac{1}{m}\right)^2$ lies in the kernel of the map $N\text{Cl}(\mathbb{Z}\left[\frac{1}{m}\right]) \to N\text{Cl}(R_m)$. 

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Arithmetic knot statistics
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Average-case counting simple knots of genus 1, equivalently genus 1 Alexander modules with Blanchfield pairing, is harder because we need to know the average size of the narrow class group of $R_\Delta = \mathbb{Z}\left[\frac{1}{m}\right]\left[\frac{1+\sqrt{1-4m}}{2}\right]$.

**Case: $|m|$ is prime**

Then $m$ factors as a product of principal prime ideals $(m) = (\gamma)(\overline{\gamma})$, so the map $\text{NCl}(\mathbb{Z}[\gamma_m]) \to \text{NCl}(R_m)$ is an isomorphism.
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**Case: \(|m|\) is composite**

For any \( p \mid |m| \), the ideal \( (p, \gamma_m)^2 \) lies in the kernel of the map \( \text{NCl}(\mathbb{Z}[\gamma_m]) \to \text{NCl}(R_m) \).
Our Heuristic

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There are $\sim X^{3/2}/\log X$ isomorphism classes of simple $2q - 1$ knots whose Alexander polynomial has the form $pt^2 + (1 - 2p)t + p$ for some prime $p$ in the range $[1, X]$. 
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- There are \( \sim X \log X \) isomorphism classes of simple 2q − 1 knots whose Alexander polynomial has the form \( mt^2 + (1 - 2m)t + m \) for some \( m \in [-X, X] \) such \( m \) is not a positive prime.
Bounds on contribution from the prime case

Theorem

The total number of isotopy classes of simple 4a + 1-knots having Alexander polynomial equal to $\Delta_p$ for $0 \leq p \leq X$ is $\gg (X^{3/2} - \epsilon)$ for all $\epsilon > 0$.

Theorem

The total number of isotopy classes of simple 4a + 1-knots having Alexander polynomial equal to $\Delta_p$ for $p$ running over all primes in the range $[1, X]$ is (unconditionally) bounded above by $O(X^{3/2} / \log X)$. 
The lower bound for the contribution of primes also gives us a lower bound for the total:

**Theorem**

*The total number of isotopy classes of simple $4a+1$-knots having Alexander polynomial equal to $\Delta_m$ for $|m| \ll X$ is $\gg \left(\frac{X^{3/2}-\epsilon}{2}\right)$ for all $\epsilon > 0$.***

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Asymptotics: in higher genus?

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$$ht(e_Q) = \max_i \left( c_{2i}^{1/i} \right)$$

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$$e_Q(x) = \det(xJ - Q) = x^{2g} + c_2 x^{2g-2} + \ldots + c_{2g}$$

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Work in progress

The total number of \( \text{Sp}_{2g} \)-orbits of symmetric matrices \( Q \) such that \( \text{ht}(e_Q) < X \) is asymptotic to \( X^{g(g+\frac{1}{2})} \).

As in quadratic case, most orbits come from the cases when the stabilizer of \( Q \) finite, when happens when \( \mathbb{Q}[x]/e_Q(x) \) is a CM field. Can also count the other orbits weighted by regulator.
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For the knot question, expect again largest contribution when \( c_{2g} \) is prime.
Thank you!