On algebraic cycles with modulus

BU-KEIO Workshop 2019

June 25 2019, Boston University

RIKEN iTHEMS

Hiroyasu MIYAZAKI

Key words

Motives

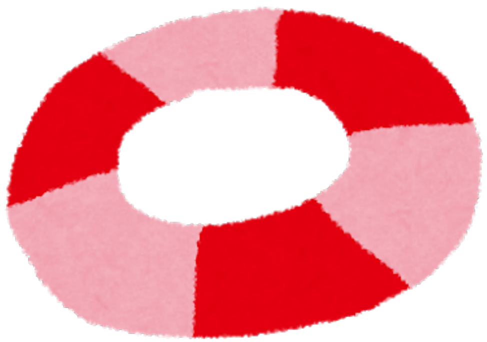
Motives with modulus

Higher Chow group with modulus

Invariants

Geometric Objects

Invariants

X = 

$g(X) = 1$

Number of Holes
(**genus**)

Invariant = essential aspect of shape

Geometric Objects → Invariants

X =

→ $g(X) = 1$

Number of Holes
(**genus**)

https://ja.wikipedia.org/wiki/位相幾何学

Invariant = essential aspect of shape

Geometric Objects → Invariants

X =  → $g(X) = \mathbf{2}$

https://en.wikipedia.org/wiki/Genus-two_surface

Number of Holes
(**genus**)

Invariant = essential aspect of shape

Geometric Objects → **Invariants**

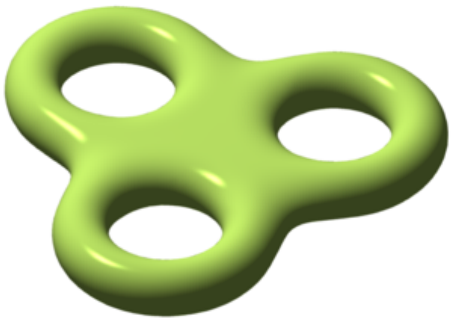
X = 

→ $g(X)$ = **3**

Number of Holes
(**genus**)

Invariant = essential aspect of shape

Geometric Objects → **Invariants**

X =  → $g(X)$ = **3**

https://en.wikipedia.org/wiki/Genus_g_surface#Genus_3

Number of Holes
(**genus**)

Invariant = essential aspect of shape

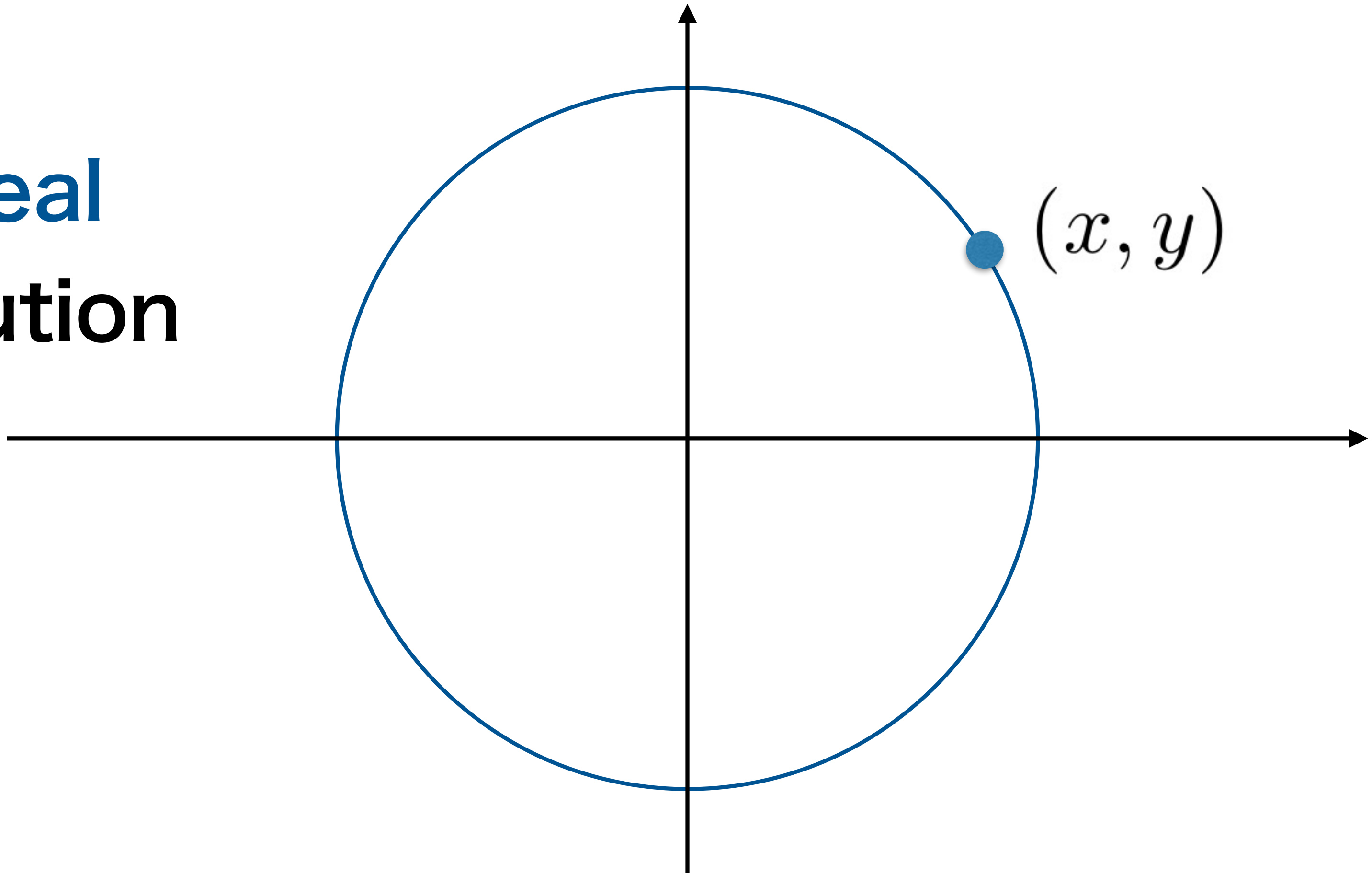
Arithmetic Geometry

Zeros of polynomials

are the main subjects

$$f(x, y) = x^2 + y^2 - 1$$

Real solution

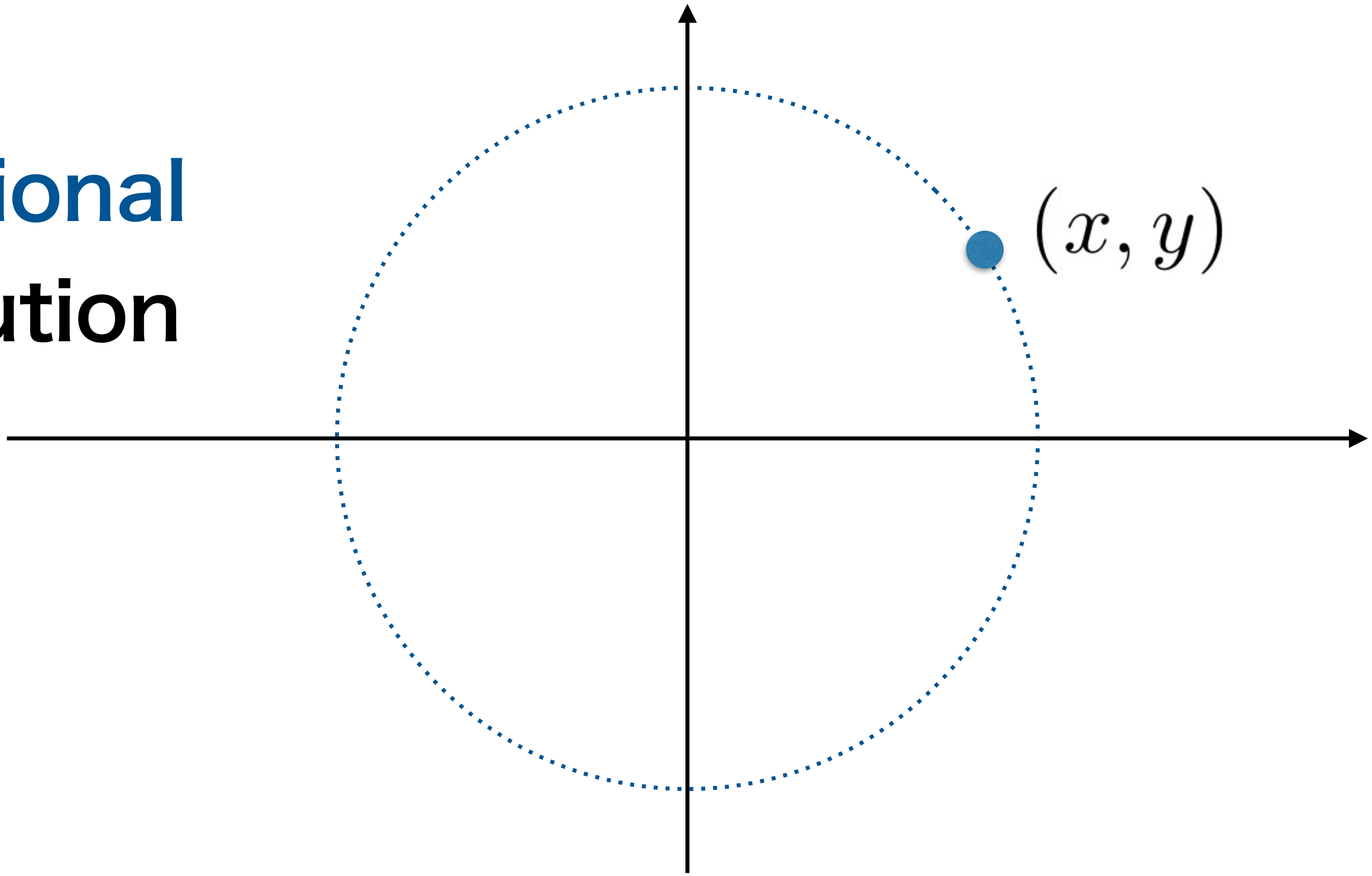


Zeros of polynomials

are the main subjects

$$f(x,y) = x^2 + y^2 - 1$$

Rational
solution

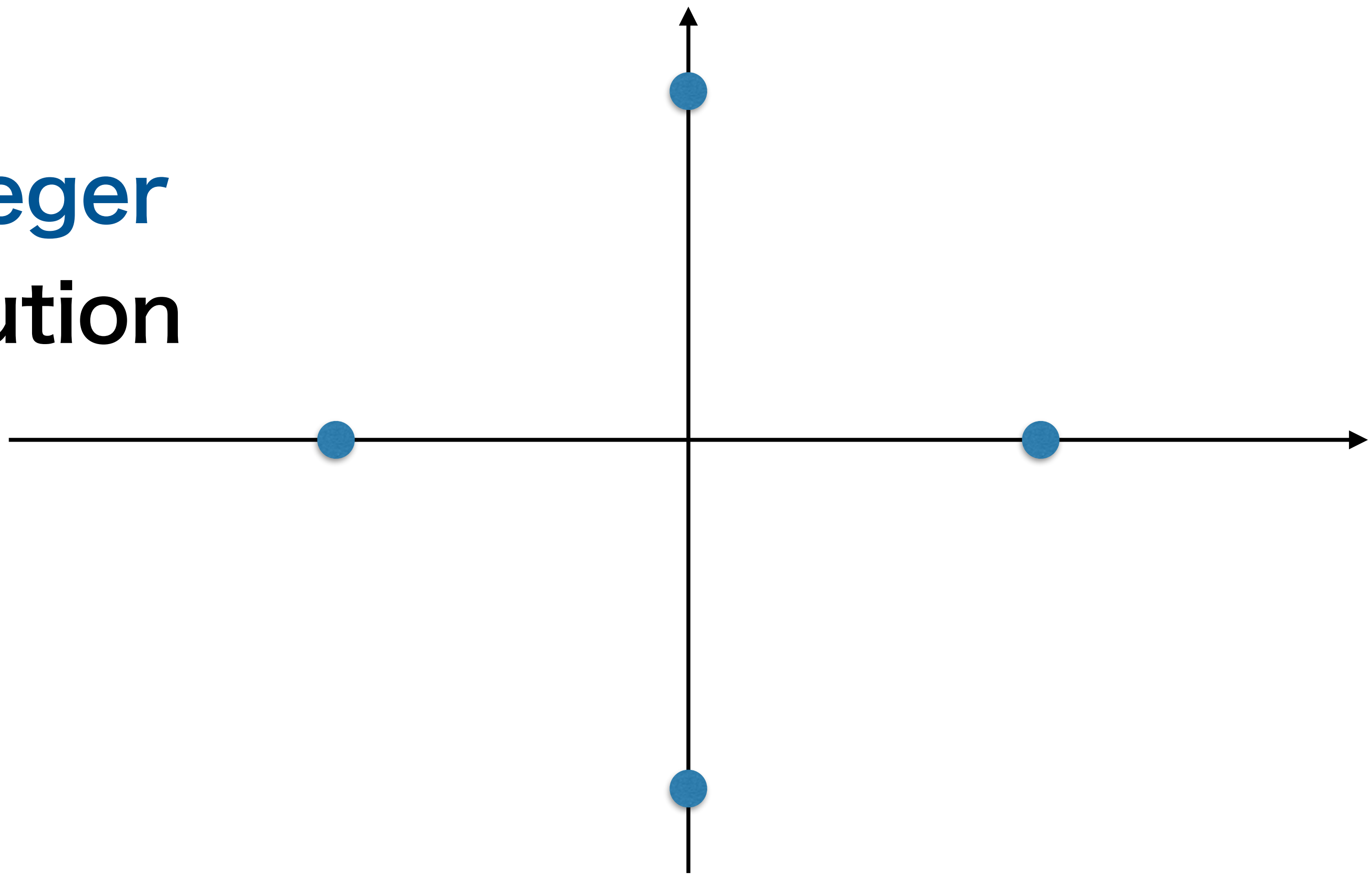


Zeros of polynomials

are the main subjects

$$f(x,y) = x^2 + y^2 - 1$$

Integer
solution



Zeros of polynomials

are the main subjects

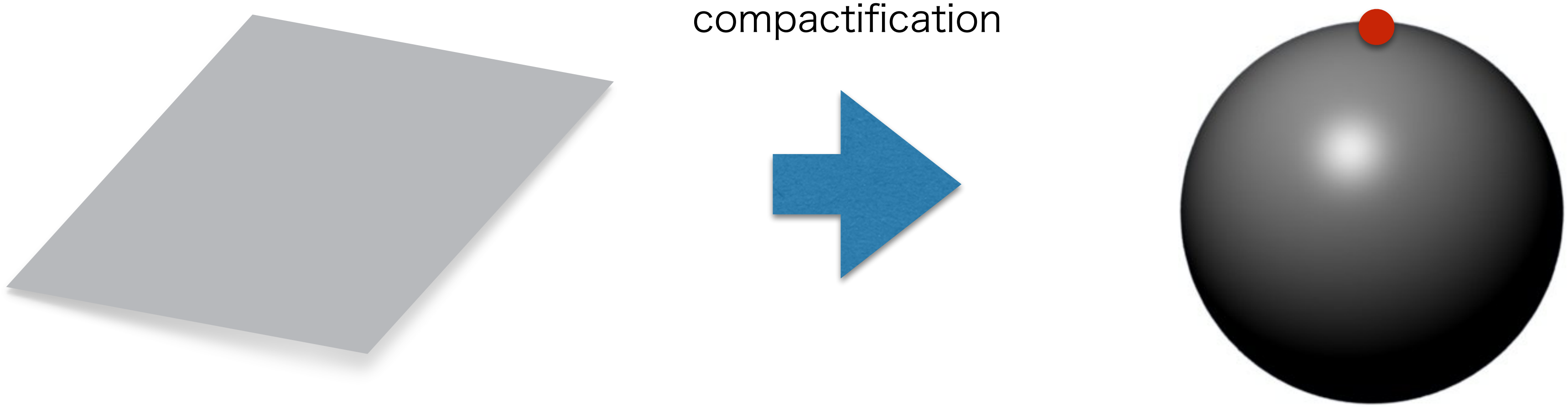
$$f(x, y) = x^2 + y^2 - 1$$

Complex

solution

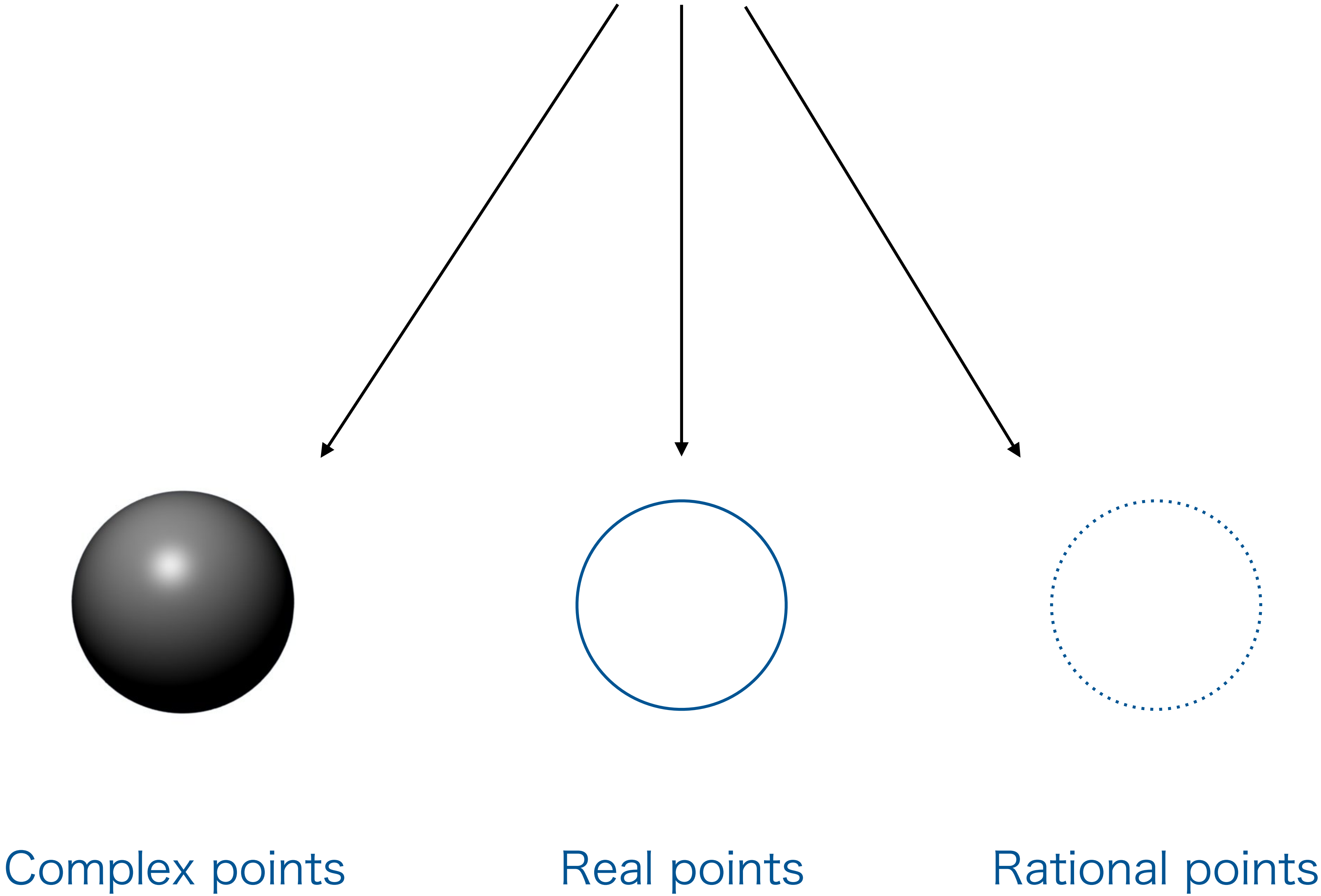
1 point
compactification

∞



$\dim_{\mathbb{R}} = 2$

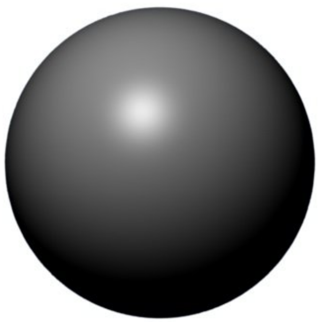
Polynomial $f(x, y) = x^2 + y^2 - 1$



Polynomial $\quad f(x,y) = x^2 + y^2 - 1$

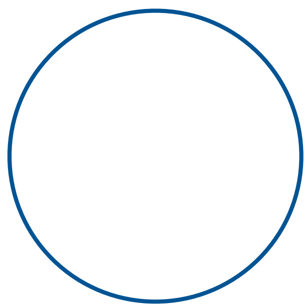
↓ $\quad$ ↓

Algebraic Variety $\quad X \quad$ A space independent of the area of solutions



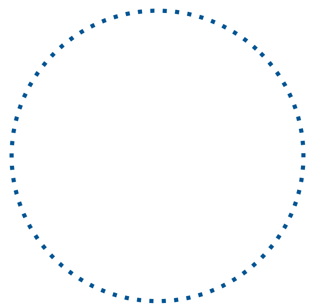
$X(\mathbb{C})$

Complex points



$X(\mathbb{R})$

Real points



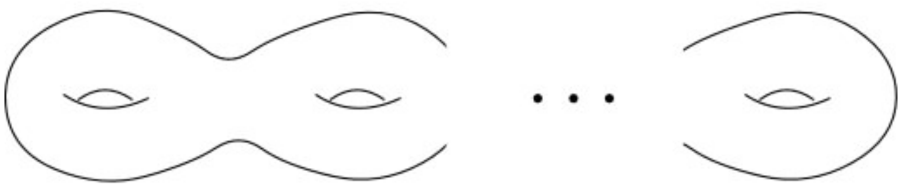
$X(\mathbb{Q})$

Rational points

Polynomial $f(x, y) = x^{n} + y^{n} - 1$

Algebraic Variety X

A space independent of the area of solutions



?

$X(\mathbb{C})$ **Genus** $\dfrac{(n-1)(n-2)}{2}$

Complex points

$\#X(\mathbb{Q})$

# of Rational points

Faltings's theorem

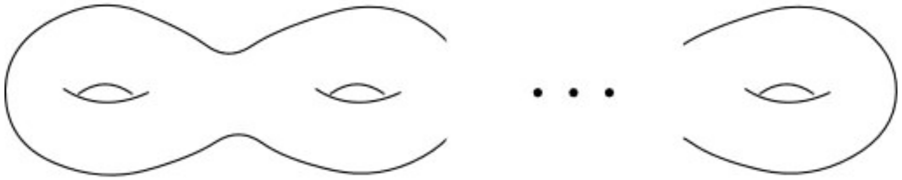
X (smooth projective) algebraic curve $/\mathbb{Q}$

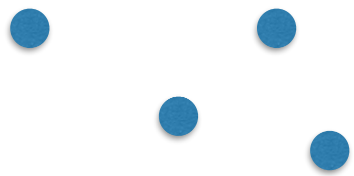
$$g(X(\mathbb{C})) > 1 \Rightarrow \#X(\mathbb{Q}) < \infty$$

If genus (number of holes) is greater than 1
then there are only finitely many rational solutions

(**Also called Modell's conjecture**)

$$X$$





$$g(X(\mathbb{C})) > 1 \qquad \Longleftrightarrow \qquad \#X(\mathbb{Q}) < \infty$$

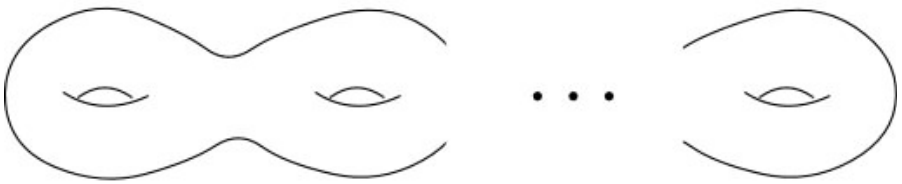
topological data

mysterious relation between Invariants

arithmetic data

$n \geq 3$ $\qquad f(x, y) = x^n + y^n - 1$

$$X$$



$g(X(\mathbb{C})) > 1$

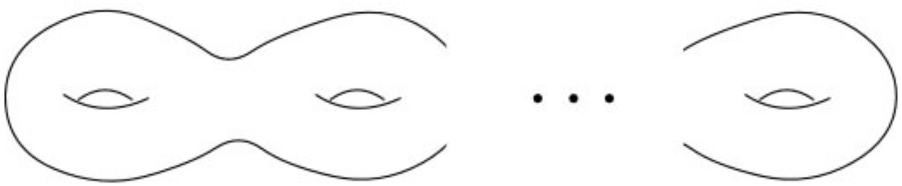
?

$X(\mathbb{Q})$

$n \geq 3$ $\qquad f(x, y) = x^n + y^n - 1$

$$\Downarrow$$

$$X$$



$$g(X(\mathbb{C})) > 1$$

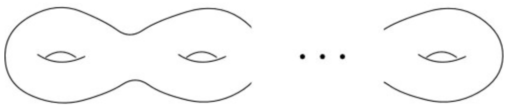
(Fermat's conjecture)

$$X(\mathbb{Q}) = \{\text{trivial solutions}\}$$

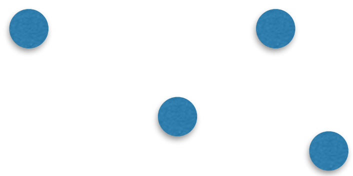
Motif

Alg var X

topological data (genus etc)	analytic data (zeta etc)	arithmetic data (rat. points etc)

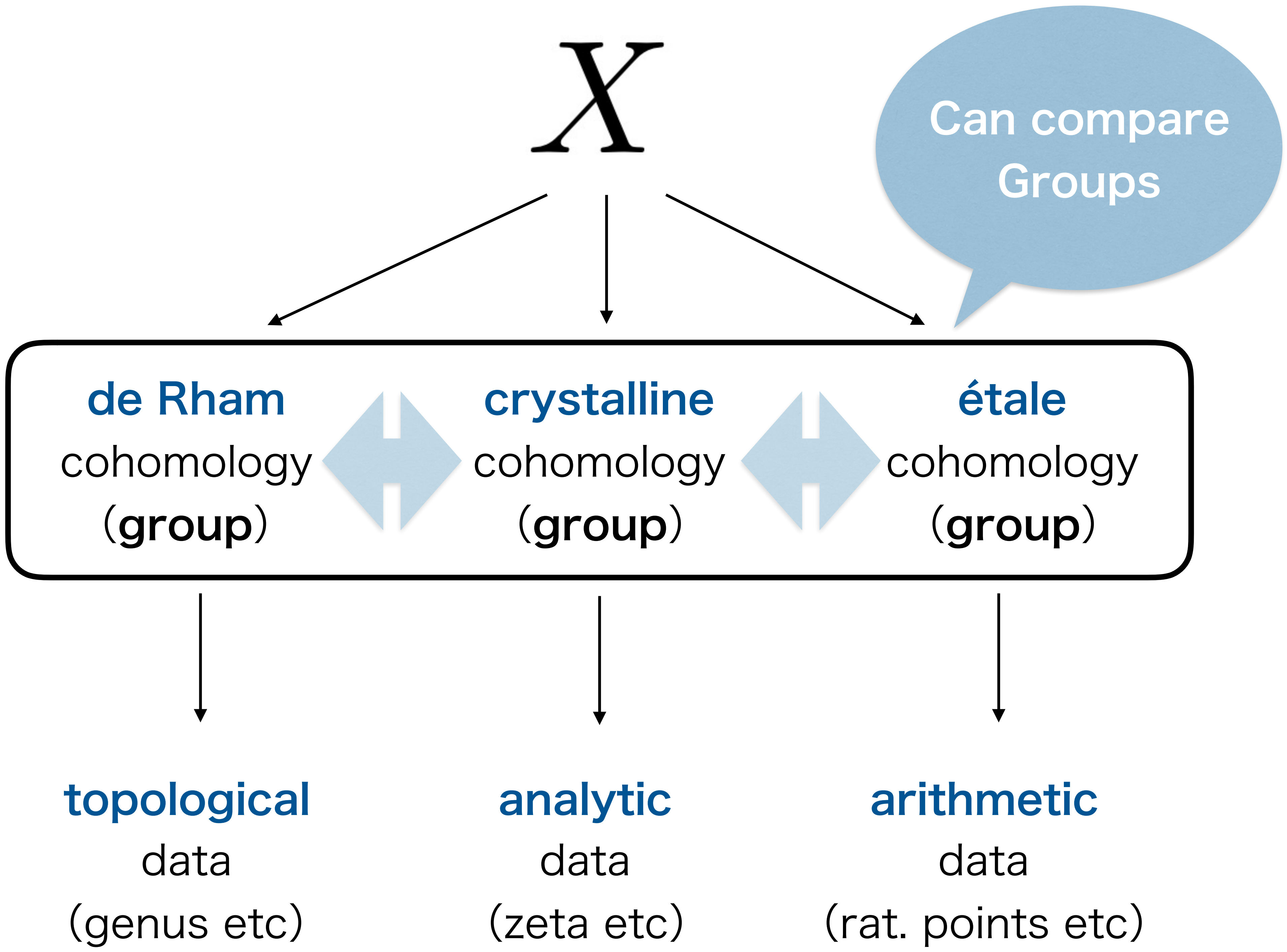


$$\zeta_X(s) = \prod_{x \in X_{(0)}} \frac{1}{1 - N(x)^{-s}}$$



Mysteriously related invariants.

Difficult to compare
because they look too different


X
Can compare
Groups
de Rham
cohomology
(group)
crystalline
cohomology
(group)
étale
cohomology
(group)
topological
data
(genus etc)
analytic
data
(zeta etc)
arithmetic
data
(rat. points etc)

There should be a universal cohomology

X

Motive

A. Grothendieck

https://en.wikipedia.org/wiki/Alexander_Grothendieck

de Rham cohomology (**group**)

crystalline cohomology (**group**)

étale cohomology (**group**)

There should be a universal cohomology

X

We can construct it

Mixed Motive

$\mathscr{M}(X)$

A. Grothendieck

https://en.wikipedia.org/wiki/Alexander_Grothendieck

V. Voevodsky

https://en.wikipedia.org/wiki/Vladimir_Voevodsky

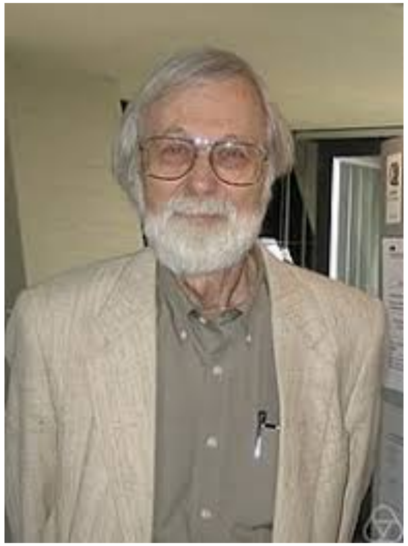
de Rham cohomology (**group**)	**crystalline** cohomology (**group**)	**étale** cohomology (**group**)

X

New connection through motives

Mixed Motive

$\mathscr{M}(X)$



J. W. Milnor

https://en.wikipedia.org/wiki/John_Milnor

V. Voevodsky

https://en.wikipedia.org/wiki/Vladimir_Voevodsky

Milnor's K-group

$$K_n^M(k)/l \cong H_{et}^n(k, \mu_l^{\otimes n})$$

étale cohomology

Milnor's conjecture (l=2)

Bloch-Kato conjecture

Homotopy invariance

- Voevodsky’s construction of motives is **abstract**.
- For concrete applications, we must compute them.

$$X \Rightarrow \mathscr{M}(X)$$

The most important property of motives is

Homotopy invariance (HI)

$$\mathscr{M}(X) \cong \mathscr{M}(X \times \mathbb{A}^1)$$

$\mathbb{A}^1$ is a replacement of $[0,1]$

HI is strong!

- It enables us to catch geometric information well.
- It makes motives **computable**.

Th **(Voevodsky)**

For any smooth X and Y, we have

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

Higher Chow group (concrete group)

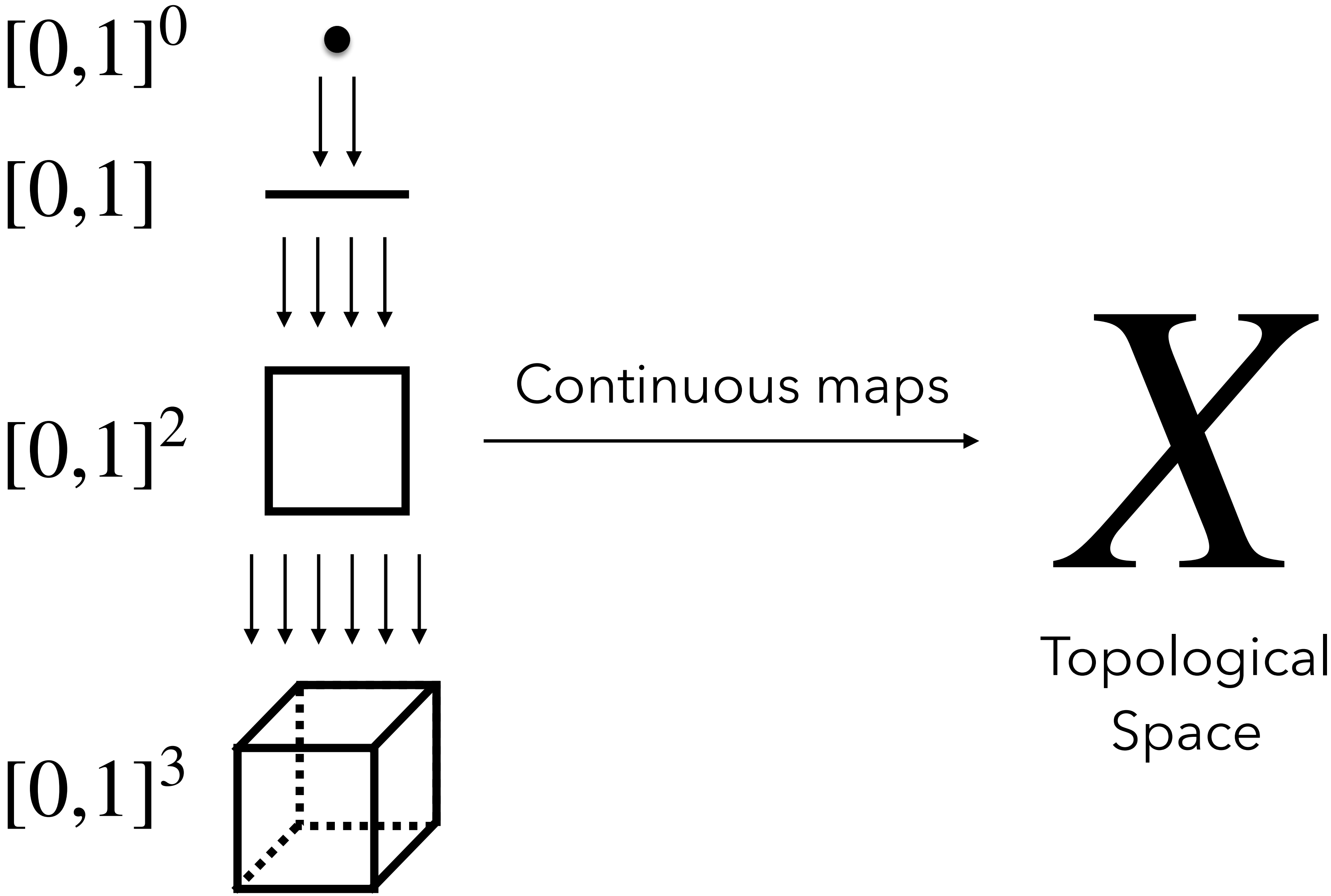
Higher Chow group satisfies **HI**, too (Bloch).

$$\mathrm{CH}^r(X, n) \cong \mathrm{CH}^r(X \times \mathbb{A}^1, n)$$

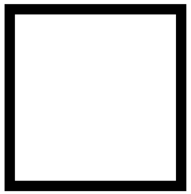
This is the most fundamental property.

What is higher Chow?

Singular (co)homology?

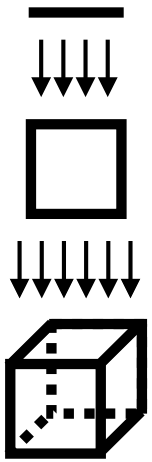


Singular (co)homology?

$[0,1]^n$  $\xrightarrow{\text{Continuous maps}}$ X

$$C_n(X)$$

$$:= \mathbb{Z}\left\{\text{continuous maps from } \mathbb{A}^n \text{ to } X\right\}$$



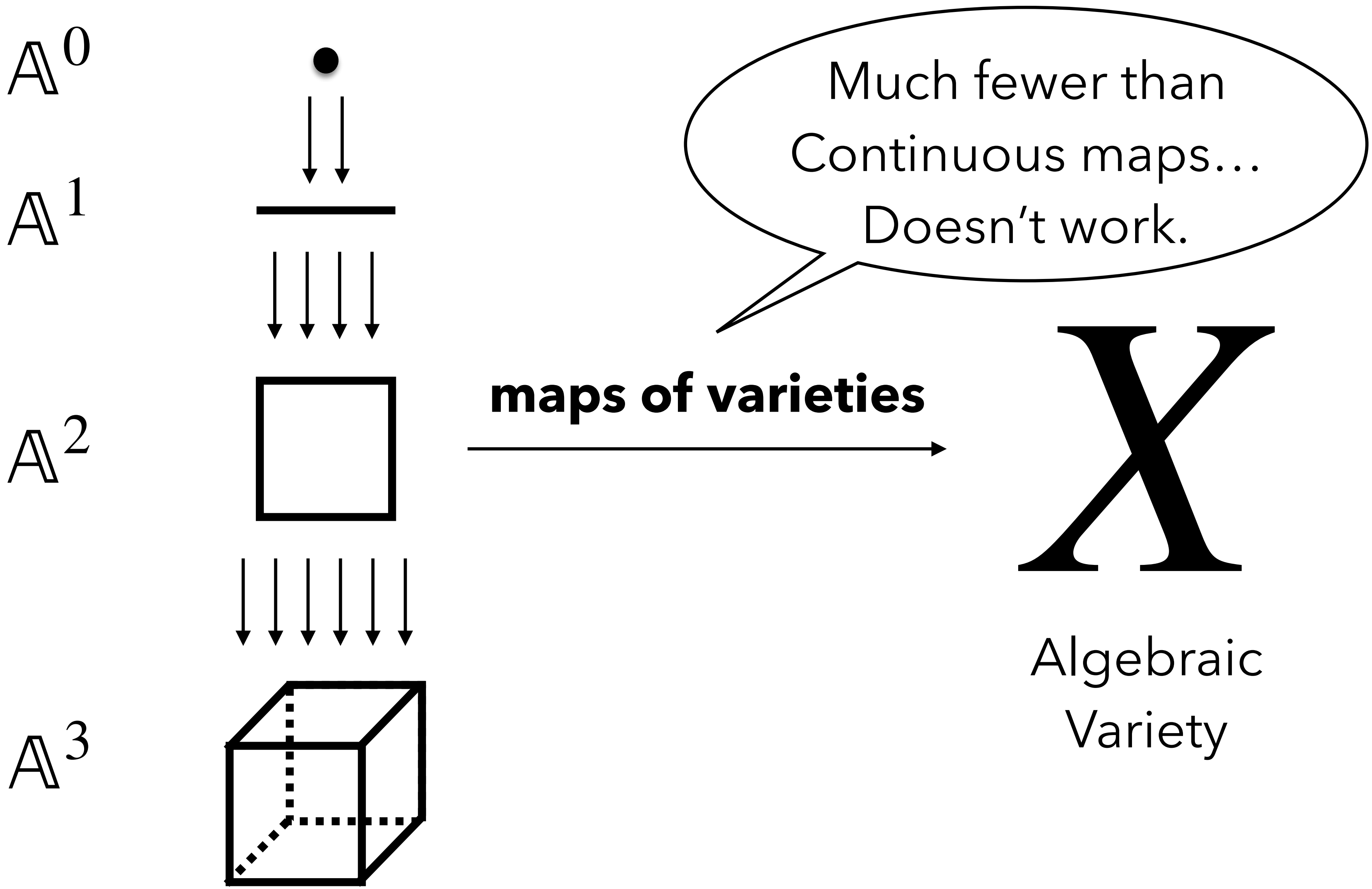
complex

$$\begin{array}{c} C_1(X) \\ \uparrow \\ C_2(X) \\ \uparrow \\ C_3(X) \end{array}$$

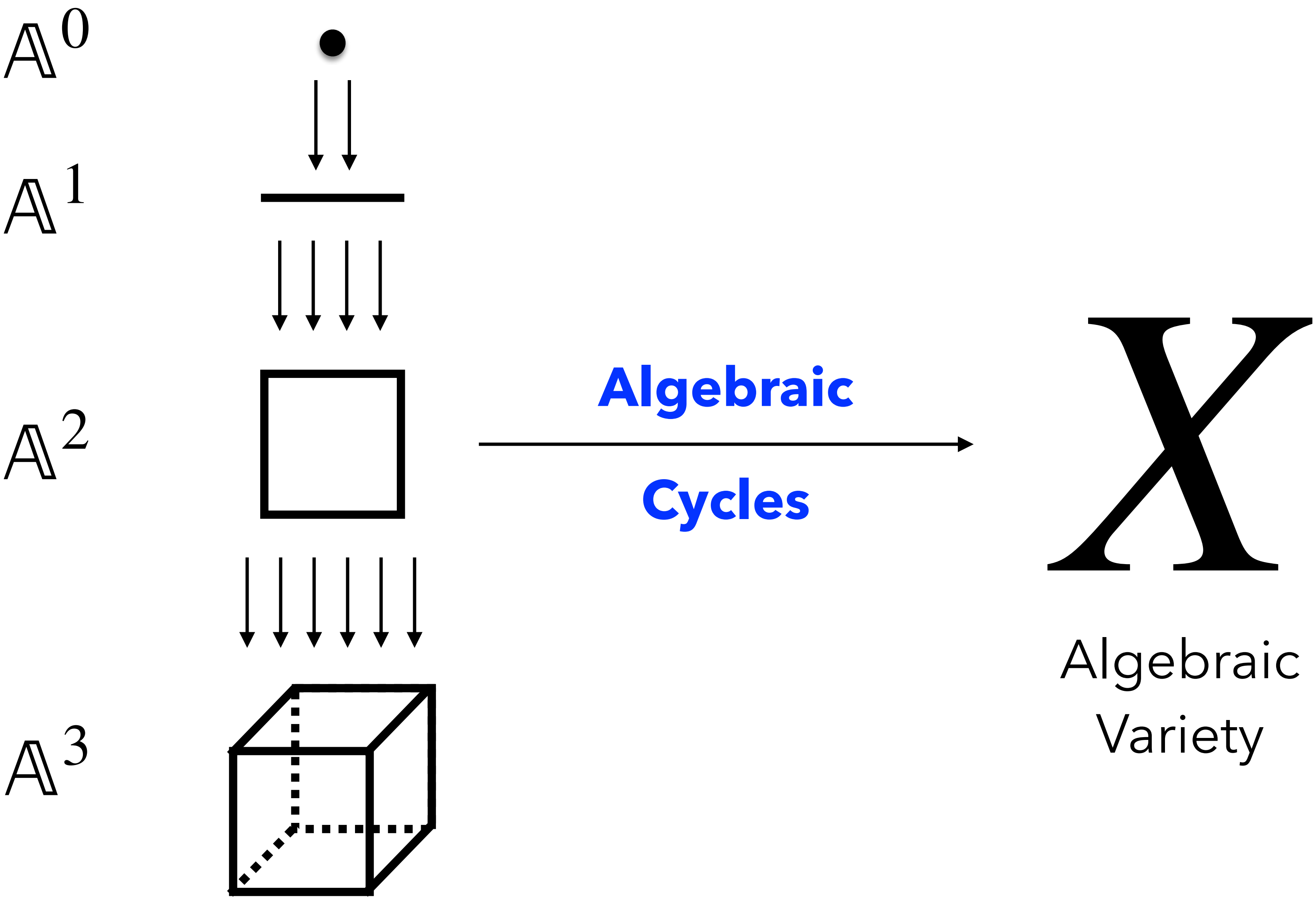
homology

$$H_n(X)$$

Higher Chow group?



Higher Chow group?



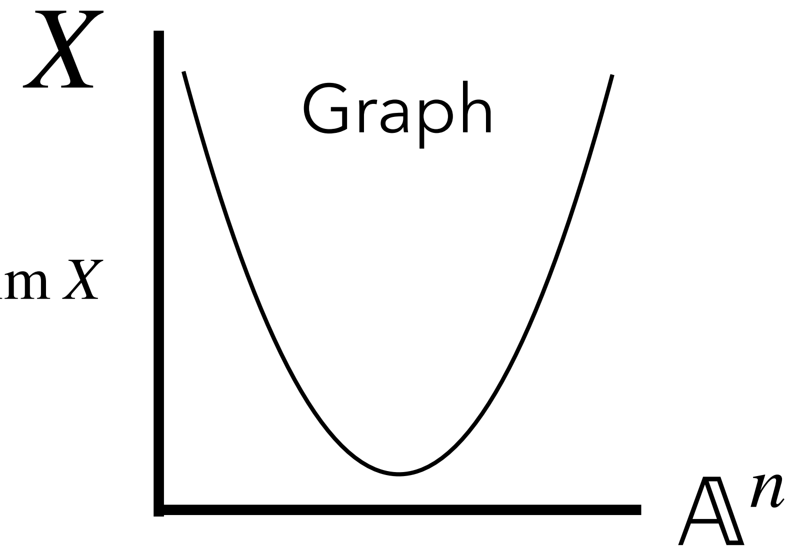
Higher Chow group?

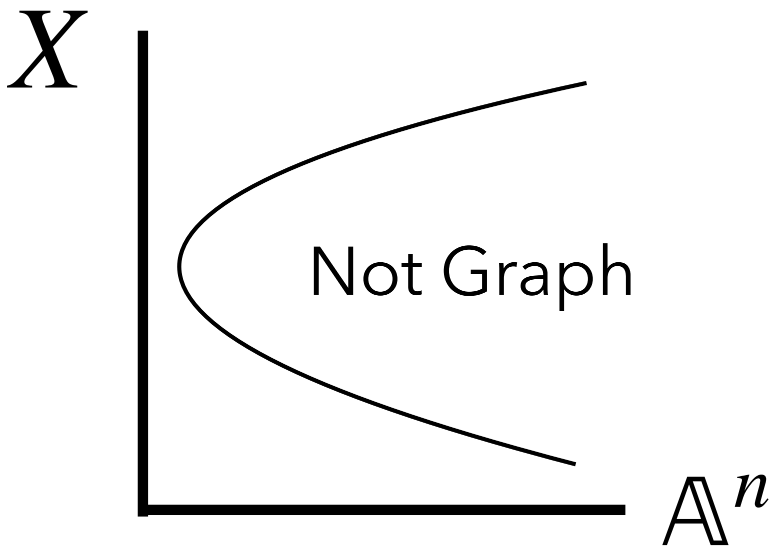
$$\mathbb{A}^n \qquad \square \xrightarrow[\textbf{Cycles}]{\textbf{Algebraic}} X$$

$$z^r(X, n)$$

$$:= \mathbb{Z}\{\text{closed subvarieties of } X \times \mathbb{A}^n \text{ of codim } r \text{ at good position}\}$$

$r = \dim X$



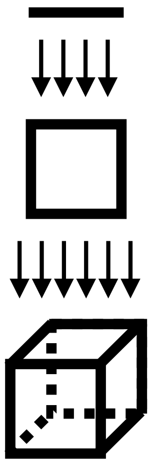


Higher Chow group?

$\mathbb{A}^n \qquad \square \xrightarrow[\textbf{Cycles}]{\textbf{Algebraic}} X$

$$z^r(X, n)$$

$:= \mathbb{Z}\{$closed subvarieties of $X \times \mathbb{A}^n$ of codim r at good position$\}$



complex

$$\begin{array}{c} z^r(X,1) \\ \uparrow \\ z^r(X,2) \\ \uparrow \\ z^r(X,3) \end{array}$$

homology

$$\mathrm{CH}^r(X, n)$$

Back to the story

HI is strong!

- It enables us to catch geometric information well.

- It makes motives **computable**.

<u>Th</u> (Voevodsky)

For any smooth X and Y, we have

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

Higher Chow group is connected to many other invariants.

HI is too strong!

- It disables us from catching **arithmetic** information.

E.g. $$X \Rightarrow \pi_1^{\mathrm{ab}}(X)$$

Arithmetic fundamental group
(knows **ramifications**)

$\pi_1^{\mathrm{ab}}(X)$ does not satisfy homotopy invariance.
It cannot be captured by motives!

We have to generalize motives.

How?

We have to generalize motives.

Motives ('00 Voevodsky) Higher Chow group ('86, Bloch)

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

Abstract side Concrete side

We have to generalize motives.

Motives ('00 Voevodsky) Higher Chow group ('86, Bloch)

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

generalized

Binda-Saito
(2014)

Higher Chow group
with modulus

Abstract side

Concrete side

We have to generalize motives.

Motives | Higher Chow group

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

generalized

Kahn-M.
-Saito-Yamazaki
(upcoming)

generalized

Binda-Saito
(2014)

Motives
with modulus

Higher Chow group
with modulus

Abstract side

Concrete side

We have to generalize motives.

Motives

Higher Chow group

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

generalized

Kahn-M.
-Saito-Yamazaki
(upcoming)

generalized

Binda-Saito
(2014)

Motives
with modulus

Abstract side

Higher Chow group
with modulus

Concrete side

First thing to study!

Higher Chow group

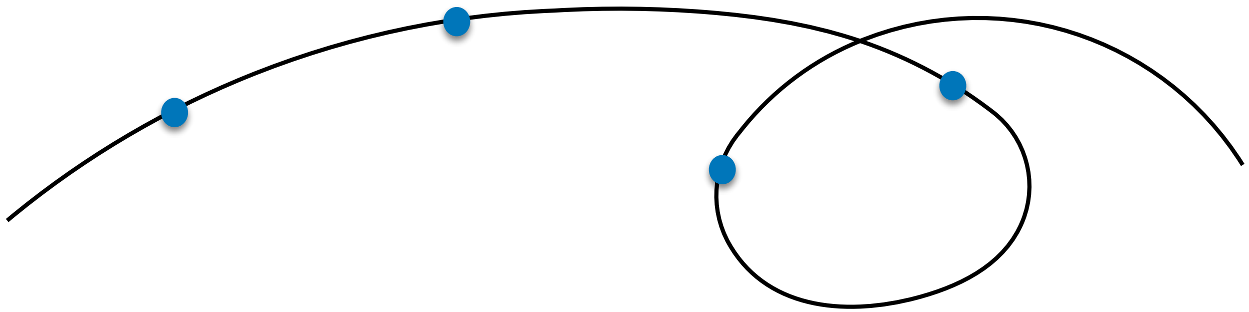
with modulus

Basic idea is to replace X with

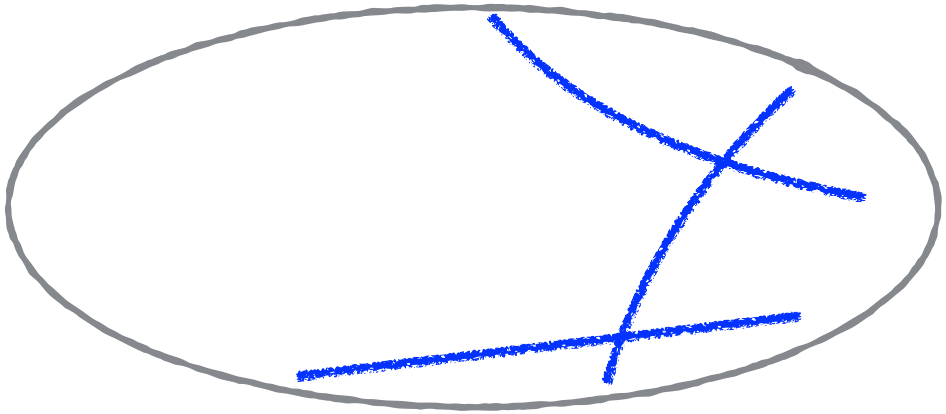
$$\mathscr{X} = (X, D)$$ "Pair of spaces"

Precisely, D is a Cartier divisor on X.

$\dim X = 1$



$\dim X = 2$



Basic idea is to replace X with

$$\mathscr{X} = (X, D)$$ "Pair of spaces"

Precisely, D is a Cartier divisor on X.

$$\mathrm{CH}^r(\mathscr{X}, n) = \mathrm{CH}^r(X, D, n)$$

Higher Chow group with modulus

$$\mathrm{CH}^r(\mathscr{X}, n) = \mathrm{CH}^r(X, D, n)$$

E.g.

$$\mathrm{CH}^r(X, \varnothing, n) = \mathrm{CH}^r(X, n)$$

$$\mathrm{CH}^r(X \times \mathbb{A}^1, m(X \times \{0\}), n) = \mathrm{TCH}^r(X, n; m)$$

Additive higher Chow group (Bloch-Esnault, Park)

Computes de Rham-Witt complex $\mathbb{W}_*\Omega^\bullet$

non-HI

Th (Kerz-Saito)

For any X smooth over a finite field, we have

$$\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \cong \varprojlim_{m\geq 1} \mathrm{CH}^{\dim X}(\overline{X}, mD, 0)^{\deg=0}$$

where $X \subset \overline{X}$ is a compactification s.t. $D = \overline{X} \setminus X$ is Cartier.

Chow with modulus
captures
ramifications!

- Higher Chow group with modulus (**CHM**) is good.
- But it does not satisfy homotopy invariance.

- Higher Chow group with modulus (**CHM**) is good.
- But it does not satisfy homotopy invariance.
- **Q1**. How far is **CHM** from **HI**?
- **Q2**. How does **CHM** depend on multiplicities of D?
- **Q3**. Is there a generalization of **HI** for **CHM**?

Main Results

“Cube invariance of higher Chow groups with modulus”

(J. Algebraic Geometry **28** (2019) 339-390)

How far is **CHM** from **HI** ?

Th (M.)

Obstruction to HI

1) There exists a canonical splitting

$$\mathrm{CH}^r(\mathscr{X} \times \mathbb{A}^1, n) \cong \mathrm{CH}^r(\mathscr{X}, n) \oplus \mathrm{NCH}^r(\mathscr{X}, n)$$

$$\mathscr{X} \times \mathbb{A}^1 := (X \times \mathbb{A}^1, D \times \mathbb{A}^1)$$

2) $\mathrm{NCH}^r(\mathscr{X}, n)$ is a p-group when $\mathrm{ch}(k) = p > 0$

Cor

$$\mathrm{CH}^r(\mathscr{X}, n) \otimes \mathbb{Z}[1/p] \cong \mathrm{CH}^r(\mathscr{X} \times \mathbb{A}^1, n) \otimes \mathbb{Z}[1/p]$$

"Non-HI part" is p-primary torsion

(Known by Binda-Cao-Kai-Sugiyama for $r = \dim X$, $n = 0$, X proper).

How does **CHM** depend on D ?

Th (M.)

If $\mathrm{ch}(k) = p > 0$

$$\mathrm{CH}^r(X, D, n) \otimes \mathbb{Z}[1/p] \cong \mathrm{CH}^r(X, mD, n) \otimes \mathbb{Z}[1/p] \quad \forall m \geq 1$$

Only p-primary torsion part
depends on multiplicity of D

Remark: ∃ similar results in charactetristic 0.

(Known by Binda-Cao-Kai-Sugiyama for $r = \dim X$, $n = 0$, X proper).

Generalization of **HI** ?

Th (M.)

For any $\mathscr{X} = (X, D)$, we have a caonical isomorphsim

$$\mathrm{CH}^r(\mathscr{X}, n) \cong \mathrm{CH}^r(\mathscr{X} \times \overline{\square}^{\vee}, n)$$

where $\overline{\square}^{\vee} = (\mathbb{P}^1, -\infty)$.

If $D = \varnothing$, then this coincides with HI of higher Chow group.

Remark: Motives with modulus satisfy the same property.

This suggest the connection between MwM and CHM.

Future?

Motives | Higher Chow group

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

generalized ⇓ Kahn-M. -Saito-Yamazaki (upcoming)

generalized ⇓ Binda-Saito (2014)

Motives **with modulus**

Higher Chow group **with modulus**

Abstract side

Concrete side

Future?

Motives

Higher Chow group

$$\mathrm{Hom}_{\mathbf{DM}}(\mathscr{M}(X)[n], \mathscr{M}(Y)) \xrightarrow{\simeq} \mathrm{CH}^{\dim Y}(X \times Y, n)$$

generalized

Kahn-M.
-Saito-Yamazaki
(upcoming)

generalized

Binda-Saito
(2014)

Motives
with modulus

$$\simeq$$

Higher Chow group
with modulus

Abstract side

???

Concrete side

→ Better control of π_1, K-group etc.

Thank you very much!