On algebraic cycles with modulus

BU-KEIO Workshop 2019
June 25 2019, Boston University

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Key words

Motives

Motives with modulus

Higher Chow group with modulus
Invariants
Geometric Objects $X = \text{Lifebuoy}$

$g(X) = 1$

Number of Holes (genus)

Invariant = essential aspect of shape
Geometric Objects \rightarrow Invariants

\[ X = \begin{array}{c}
\text{图:咖啡杯}
\end{array} \rightarrow g(X) = 1 \]

Number of Holes (genus)

Invariant = essential aspect of shape

https://ja.wikipedia.org/wiki/位相幾何学
Geometric Objects

\[ X = \]

\[ g(X) = 2 \]

Number of Holes (genus)

Invariant = essential aspect of shape

https://en.wikipedia.org/wiki/Genus-two_surface
Geometric Objects \rightarrow \text{Invariants}

\[ X = \text{fidget spinner} \rightarrow g(X) = 3 \]

Number of Holes (genus)

Invariant = essential aspect of shape
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Geometric Objects

$X = \quad$ 🔄

Invariants

$g(X) = 3$

Number of Holes

(genus)

https://en.wikipedia.org/wiki/Genus_g_surface#Genus_3
Arithmetic

Geometry
Zeros of polynomials are the main subjects

\[ f(x, y) = x^2 + y^2 - 1 \]
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\[ f(x, y) = x^2 + y^2 - 1 \]

Complex solution

\[ \dim_\mathbb{R} = 2 \]

1 point compactification
Polynomial \[ f(x, y) = x^2 + y^2 - 1 \]
Polynomial: \[ f(x, y) = x^2 + y^2 - 1 \]

Algebraic Variety: \( X \)

A space independent of the area of solutions

\( X(\mathbb{C}) \) - Complex points

\( X(\mathbb{R}) \) - Real points

\( X(\mathbb{Q}) \) - Rational points
Polynomial \[ f(x, y) = x^n + y^n - 1 \]

Algebraic Variety

A space independent of the area of solutions

\[ X(\mathbb{C}) \quad \text{Genus} \quad \frac{(n-1)(n-2)}{2} \quad \# X(\mathbb{Q}) \]

Complex points

# of Rational points
Faltings’s theorem

\[ X \text{ (smooth projective) algebraic curve } \mathbb{Q} \]

\[ g(X(\mathbb{C})) > 1 \implies \#X(\mathbb{Q}) < \infty \]

If genus (number of holes) is greater than 1
then there are only finitely many rational solutions

(Also called Modell’s conjecture)
\[ g(X(\mathbb{C})) > 1 \]

topological data

mysterious relation between Invariants

\[ \#X(\mathbb{Q}) < \infty \]

arithmetic data
$n \geq 3$

$$f(x, y) = x^n + y^n - 1$$

$X$

$g(X(\mathbb{C})) > 1$

$X(\mathbb{Q})$
\[ n \geq 3 \quad f(x, y) = x^n + y^n - 1 \]

\[ g(X(\mathbb{C})) > 1 \]

\[ X(\mathbb{Q}) = \{ \text{trivial solutions} \} \]
Motif
Alg var $X$

- topological data (genus etc)
- analytic data (zeta etc)
- arithmetic data (rat. points etc)

Mysteriously related invariants.

Difficult to compare because they look too different.
Can compare Groups

de Rham cohomology (group)

crystalline cohomology (group)

étale cohomology (group)

topological data (genus etc)

analytic data (zeta etc)

arithmetic data (rat. points etc)
There should be a universal cohomology

A. Grothendieck

There should be a universal cohomology

We can construct it

Mixed Motive $\mathcal{M}(X)$

A. Grothendieck

V. Voevodsky


https://en.wikipedia.org/wiki/Vladimir_Voevodsky

de Rham cohomology (group)

crystalline cohomology (group)

étale cohomology (group)
New connection through motives

Mixed Motive $\mathcal{M}(X)$

Milnor’s K-group

$K_n^M(k)/l \cong H_{et}^n(k, \mu_l \otimes n)$

Milnor’s conjecture ($l=2$)

Bloch-Kato conjecture

J. W. Milnor

V. Voevodsky


https://en.wikipedia.org/wiki/Vladimir_Voevodsky
Homotopy invariance
• Voevodsky’s construction of motives is **abstract**.

• For concrete applications, we must compute them.
The most important property of motives is

**Homotopy invariance (HI)**

\[ \mathcal{M}(X) \cong \mathcal{M}(X \times \mathbb{A}^1) \]

\( \mathbb{A}^1 \) is a replacement of \([0,1]\)
HI is strong!

- It enables us to catch geometric information well.
- It makes motives computable.

**Th (Voevodsky)**

For any smooth $X$ and $Y$, we have

$$\text{Hom}_{DM}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\text{dim}Y}(X \times Y, n)$$

Higher Chow group (concrete group)
Higher Chow group satisfies $\text{HI}$, too (Bloch).

$$\text{CH}^r(X, n) \cong \text{CH}^r(X \times \mathbb{A}^1, n)$$

This is the most fundamental property.
What is higher Chow?
Singular (co)homology?

Continuous maps

Topological Space
Singular (co)homology?

\[
[0,1]^n \xrightarrow{\text{Continuous maps}} X
\]

\[
C_n(X) = \mathbb{Z} \{\text{continuous maps from } \mathbb{A}^n \text{ to } X\}
\]

\[
\begin{align*}
C_1(X) & \quad \xrightarrow{\text{complex}} \quad C_2(X) \\
C_2(X) & \quad \xrightarrow{\text{homology}} \quad C_3(X) \\
& \quad \quad H_n(X)
\end{align*}
\]
Higher Chow group?

Maps of varieties

\[ \mathbb{A}^0 \rightarrow \mathbb{A}^1 \rightarrow \mathbb{A}^2 \rightarrow \mathbb{A}^3 \]

Much fewer than Continuous maps…

Doesn’t work.

Algebraic Variety
Higher Chow group?

$\mathbb{A}^0$ $\rightarrow$ $\mathbb{A}^1$ $\rightarrow$ $\mathbb{A}^2$ $\rightarrow$ $\mathbb{A}^3$

$\rightarrow$ Algebraic Cycles $\rightarrow$ Algebraic Variety
Higher Chow group?

\[ \mathbb{A}^n \xrightarrow{\text{Algebraic Cycles}} X \]

\[ z^r(X, n) = \mathbb{Z} \{ \text{closed subvarieties of } X \times \mathbb{A}^n \text{ of codim } r \text{ at good position} \} \]
Higher Chow group?

$\mathbb{A}^n \xrightarrow{\text{Algebraic Cycles}} X$

$z^r(X, n) = \mathbb{Z} \{ \text{closed subvarieties of } X \times \mathbb{A}^n \text{ of codim } r \text{ at good position} \}$

$z^r(X, 1) \xrightarrow{\text{complex}} \xrightarrow{\text{homology}} \text{CH}^r(X, n)$

$z^r(X, 2)$

$z^r(X, 3)$
Back to the story
HI is strong!

- It enables us to catch geometric information well.
- It makes motives computable.

**Th (Voevodsky)**

For any smooth $X$ and $Y$, we have

$$\text{Hom}_{\text{DM}}(M(X)[n], M(Y)) \overset{\sim}{\to} \text{CH}^{\dim Y}(X \times Y, n)$$

Higher Chow group is connected to many other invariants.
HI is too strong!

- It disables us from catching arithmetic information.

E.g. 

\[ X \rightarrow \pi_1^{\text{ab}}(X) \]

Arithmetic fundamental group (knows ramifications)

\[ \pi_1^{\text{ab}}(X) \] does not satisfy homotopy invariance. It cannot be captured by motives!
We have to generalize motives.

How?
We have to generalize motives.

Motives ('00 Voevodsky) \quad \text{Higher Chow group ('86, Bloch)}

\[ \text{Hom}_{\mathcal{DM}}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\text{dim}Y}(X \times Y, n) \]
We have to generalize motives.

Motives (’00 Voevodsky) \quad \text{Higher Chow group (’86, Bloch)}

\[ \text{Hom}_{\text{DM}}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\text{dim}Y}(X \times Y, n) \]

generalized

Binda-Saito (2014)

Higher Chow group with modulus

Abstract side

Concrete side
We have to generalize motives.

\[ \text{Hom}_{\text{DM}}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \overset{\sim}{\rightarrow} \text{CH}^{\text{dim} Y}(X \times Y, n) \]

Motives \quad \text{Higher Chow group}

Kahn-M. -Saito-Yamazaki (upcoming) \quad \text{generalized}

Motives with modulus \quad \text{Higher Chow group with modulus}

Abstract side \quad \text{Concrete side}

Binda-Saito (2014) \quad \text{generalized}
We have to generalize motives.

\( \text{Hom}_{DM}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\dim Y}(X \times Y, n) \)

Motives

Higher Chow group

**Kahn-M. -Saito-Yamazaki** (upcoming)

**Binda-Saito** (2014)

Motives with modulus

Concrete side

First thing to study!
Higher Chow group with modulus
Basic idea is to replace $X$ with

$$\mathcal{X} = (X, D) \quad \text{"Pair of spaces"}$$

Precisely, $D$ is a Cartier divisor on $X$.

$\dim X = 1$

$\dim X = 2$
Basic idea is to replace $X$ with

$$\mathcal{X} = (X, D) \quad \text{"Pair of spaces"}$$

Precisely, $D$ is a Cartier divisor on $X$.

$$\text{CH}^r(\mathcal{X}, n) = \text{CH}^r(X, D, n)$$

Higher Chow group with modulus
\[ \text{CH}^r(\mathcal{X}, n) = \text{CH}^r(X, D, n) \]

E.g.

\[ \text{CH}^r(X, \emptyset, n) = \text{CH}^r(X, n) \]

\[ \text{CH}^r(X \times \mathbb{A}^1, m(X \times \{0\}), n) = \text{TCH}^r(X, n; m) \]

Additive higher Chow group (Bloch-Esnault, Park)

Computes de Rham-Witt complex \( \mathbb{W}_* \Omega^\bullet \)

non-HI
\textbf{Th (Kerz-Saito)}

For any $X$ smooth over a finite field, we have

$$\pi_1^{ab}(X)^{\text{geo}} \cong \lim_{\substack{\leftarrow \cr m \geq 1}} \text{CH}^{\dim X}(\bar{X}, mD, 0)^{\deg=0}$$

where $X \subset \bar{X}$ is a compactification s.t. $D = \bar{X} \setminus X$ is Cartier.

\textbf{Chow with modulus captures ramifications!}
• Higher Chow group with modulus (CHM) is good.

• But it does not satisfy homotopy invariance.
• Higher Chow group with modulus (CHM) is good.

• But it does not satisfy homotopy invariance.

• **Q1.** How far is CHM from HI?

• **Q2.** How does CHM depend on multiplicities of D?

• **Q3.** Is there a generalization of HI for CHM?
Main Results

“Cube invariance of higher Chow groups with modulus”

(J. Algebraic Geometry 28 (2019) 339-390)
How far is $\text{CHM}$ from $\text{HI}$?

**Th (M.)**

1) There exists a canonical splitting

$$\text{CH}^r(\mathcal{X} \times \mathbb{A}^1, n) \cong \text{CH}^r(\mathcal{X}, n) \oplus \text{NCH}^r(\mathcal{X}, n)$$

$$\mathcal{X} \times \mathbb{A}^1 := (X \times \mathbb{A}^1, D \times \mathbb{A}^1)$$

2) $\text{NCH}^r(\mathcal{X}, n)$ is a $p$-group when $\text{ch}(k) = p > 0$

**Cor**

$$\text{CH}^r(\mathcal{X}, n) \otimes \mathbb{Z}[1/p] \cong \text{CH}^r(\mathcal{X} \times \mathbb{A}^1, n) \otimes \mathbb{Z}[1/p]$$

"Non-HI part" is $p$-primary torsion

(Known by Binda-Cao-Kai-Sugiyama for $r = \dim X, n = 0, X$ proper).
How does \textbf{CHM} depend on D?

\textbf{Th (M.)}

If \( \text{ch}(k) = p > 0 \)

\[ \text{CH}^r(X, D, n) \otimes \mathbb{Z}[1/p] \cong \text{CH}^r(X, mD, n) \otimes \mathbb{Z}[1/p] \quad \forall m \geq 1 \]

Only p-primary torsion part depends on multiplicity of D

Remark: \( \exists \) similar results in characteristic 0.

(Known by Binda-Cao-Kai-Sugiyama for \( r = \dim X, n = 0, X \) proper).
Generalization of $\text{HI}$?

**Th (M.)**

For any $\mathcal{X} = (X, D)$, we have a canonical isomorphism

\[ \text{CH}^r(\mathcal{X}, n) \cong \text{CH}^r(\mathcal{X} \times \square^\vee, n) \]

where $\square^\vee = (\mathbb{P}^1, -\infty)$.

If $D = \emptyset$, then this coincides with HI of higher Chow group.

**Remark:** Motives with modulus satisfy the same property.

This suggest the connection between MwM and CHM.
Future?

\[
\text{Motives} \quad \text{Hom}_{DM}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\dim Y}(X \times Y, n)
\]

Motives \textbf{with modulus}

Abstract side

Higher Chow group \textbf{with modulus}

Concrete side

Kahn-M.-Saito-Yamazaki (upcoming)

Binda-Saito (2014)
Future?

\[ \text{Hom}_{\text{DM}}(\mathcal{M}(X)[n], \mathcal{M}(Y)) \xrightarrow{\sim} \text{CH}^{\dim Y}(X \times Y, n) \]

Motives with modulus

Abstract side

\[ \Rightarrow \text{Better control of } \pi_1, K\text{-group etc.} \]

Higher Chow group with modulus

Concrete side

Kahn-M. -Saito-Yamazaki (upcoming)

Binda-Saito (2014)
Thank you very much!