EQUIDISTRIBUTION ESTIMATES FOR THE k-FOLD DIVISOR FUNCTION TO LARGE MODULI

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ABSTRACT. For each positive integer k, let $\tau_k(n)$ denote the k-fold divisor function, the coefficient of n^{-s} in the Dirichlet series for $\zeta(s)^k$. For instance $\tau_2(n)$ is the usual divisor function $\tau(n)$. The distribution of $\tau_k(n)$ in arithmetic progression is closely related to distribution of the primes. Aside from the trivial case $k = 1$ their distributions are only fairly well understood for $k = 2$ (Selberg, Linnik, Hooley) and $k = 3$ (Friedlander and Iwaniec; Heath-Brown; Fouvry, Kowalski, and Michel). In this note I'll survey and present some new distribution estimates for $\tau_k(n)$ in arithmetic progression to special moduli applicable for any $k \geq 4$. This work is a refinement of the recent result of F. Wei, B. Xue, and Y. Zhang in 2016, in particular, leads to some effective estimates with power saving error terms.

1. INTRODUCTION AND SURVEY OF KNOWN RESULTS

Let $n \geq 1$ and $k \geq 1$ be integers. Let $\tau_k(n)$ denote the *k-fold divisor function*

$$\tau_k(n) = \sum_{n_1 n_2 \cdots n_k = n} 1,$$

where the sum runs over ordered k-tuples $(n_1, n_2, \ldots, n_k)$ of positive integers for which $n_1 n_2 \cdots n_k = n$. Thus $\tau_k(n)$ is the coefficient of n^{-s} in the Dirichlet series

$$\zeta(s)^k = \sum_{n=1}^{\infty} \tau_k(n) n^{-s}.$$

This note is concerned with the distribution of $\tau_k(n)$ in arithmetic progressions to moduli d that exceed the square-root of length of the sum, in particular, provides a sharpening of the result in [53].

1.1. **Distribution of arithmetic functions.** C. F. Gauss, towards the end of the 18th century, conjectured the celebrated Prime Number Theorem concerning the sum

$$\sum_{p \leq X} 1$$

as X approaches infinity, where p denotes a prime. It is more convenience to count primes with weight $\log p$ instead of weight 1, c.f. Chebyshev; this leads to consideration of the sum

$$\sum_{p \leq X} \log p.$$

To access the Riemann zeta function more conveniently we also count powers of primes, leading to the sum

$$\sum_{\substack{p^\alpha \leq X \\ \alpha \geq 1}} \log p,$$

which is equal to the unconstrained sum over n

$$\sum_{n \leq X} \Lambda(n)$$

where $\Lambda(n)$ is the von Mangoldt function–the coefficient of n^{-s} in the series $-\zeta'(s)/\zeta(s)$. In 1837, P. G. L. Dirichlet considered the deep question of primes in arithmetic progression, leading him to consider sums of the form

$$\sum_{\substack{n\le X\\ n\equiv a(\mathrm{mod}\ d)}} \Lambda(n)$$

for $(d, a) = 1$. More generally, the function $\Lambda(n)$ is replaced by an arithmetic function $f(n)$, satisfying certain growth conditions, and we arrive at the study of the congruence sum

$$\sum_{\substack{n\le X\\ n\equiv a(\mathrm{mod}\ d)}} f(n). \tag{1.1}$$

This sum (1.1) is our main object of study.

For most f appearing in applications, it is expected that f is distributed equally among the reduced residue classes $a(\mathrm{mod}\ d)$ with $(a, d) = 1$, e.g., that the sum (1.1) is well approximated by the average

$$\frac{1}{\varphi(d)} \sum_{\substack{n\le X\\ (n,d)=1}} f(n) \tag{1.2}$$

since there are $\varphi(d)$ reduced residue classes modulo d, where $\varphi(n)$ is the Euler's totient function. The quantity (1.2) is often thought of as the 'main term'. Different main terms are also considered. Thus, the study of (1.1) is reduced to studying the 'error term'

$$\Delta(f; X, d, a) := \sum_{\substack{n\le X\\ n\equiv a(\mathrm{mod}\ d)}} f(n) - \frac{1}{\varphi(d)} \sum_{\substack{n\le X\\ (n,d)=1}} f(n), \quad \text{for } (a, d) = 1. \tag{1.3}$$

measuring the discrepancy between the the sum (1.1) and the expected value (1.2). If f satisfies

$$f(n) \le C\tau^B(n) \log^B X$$

for some constants $B, C > 0$, which is often the case for most f in applications, then a trivial bound for the discrepancy $\Delta(f; X, d, a)$ is

$$\Delta(f; X, d, a) \le C' \frac{1}{\varphi(d)} X \log^{-B'} X,$$

for some constants $B', C' > 0$. The objective is then to obtain an upper bound for $\Delta(f; X, d, a)$ as small as possible for d as large as possible–the smaller the discrepancy and the larger the range of d, the better the distribution estimates for f is.

We consider the k-fold divisor function $f(n) = \tau_k(n)$, $k \ge 1$. It is well known that the function τ_k is closely related to prime numbers; see Remark 1 below. Let us next survey known results on the distribution of $\tau_k(n)$.

1.2. **Individual estimates for each modulus d.** For $(a, d) = 1$ define

$$T_k(X, d, a) = \sum_{\substack{n\le X\\ n\equiv a(\mathrm{mod}\ d)}} \tau_k(n).$$

For $k = 1$ we have

$$T_1(X, d, a) = \sum_{\substack{n\le X\\ n\equiv a(\mathrm{mod}\ d)}} 1 = \frac{X}{d} + O(1),$$

and this is valid for all $d < X$. We wish to find real numbers $\theta_k > 0$, as large as possible, such that the following statement holds.

(S1) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$T_k(X, d, a) - \frac{X}{\varphi(d)} P_k(\log X) \ll \frac{X^{1-\delta}}{\varphi(d)} \tag{1.4}$$

uniformly for all $d \leq X^{\theta_k - \epsilon}$.

Here $P_k(\log X)$ is a polynomial in $\log X$ of degree $k-1$ given by Cauchy's theorem as

$$P_k(\log X) = \operatorname{Res}(s^{-1} L^k(s, \chi_0) X^{s-1}; s = 1),$$

where χ_0 is the principal character of modulus d; for instance, see [23] and [44]. The number θ_k is called the level of distribution for τ_k. It is widely believed that (S1) is valid for all $\theta_k < 1$ for each $k \geq 1$; though, the only known case is for $k = 1$. For ease of referencing, we record this as

Conjecture 1. *For each $k \geq 2$ statement (S1) holds for any $\theta_k < 1$.*

The Generalized Riemann Hypothesis implies that statement (S1) holds for all $\theta_k < 1/2$ for any k. In Table 1 we summarize known unconditional results towards Conjecture 1. We now give a brief survey of the known results.

Table 1. Known results towards Conjecture 1 for individual estimates and references. Only for $k = 1, 2, 3$ is the exponent of distribution θ_k for τ_k known to hold for a value larger than 1/2.

$\boldsymbol{k}$	$\boldsymbol{\theta_k}$	**References**
$k = 2$	$\theta_2 = 2/3$	Selberg, Linnik, Hooley (independently, unpublished, 1950's); Heath-Brown (1979) [27, Corollary 1, p. 409].
$k = 3$	$\theta_3 = 1/2 + 1/230$	Friedlander and Iwaniec (1985) [24, Theorem 5, p. 338].
	$\theta_3 = 1/2 + 1/82$	Heath-Brown (1986) [28, Theorem 1, p. 31].
	$\theta_3 = 1/2 + 1/46$	Fouvry, Kowalski, and Michel (2015) [21, Theorem 1.1, p. 122], (for prime moduli, polylog saving).
$k = 4$	$\theta_4 = 1/2$	Linnik (1961) [40, Lemma 5, p. 197].
$k \geq 4$	$\theta_k = 8/(3k+4)$	Lavrik (1965) [39, Teopema 1, p. 1232].
$k = 5$	$\theta_5 = 9/20$	Friedlander and Iwaniec (1985) [23, Theorem I, p. 273].
$k = 6$	$\theta_6 = 5/12$	Friedlander and Iwaniec (1985) [23, Theorem II, p. 273].
$k \geq 7$	$\theta_k = 8/3k$	Friedlander and Iwaniec (1985) [23, Theorem II, p. 273].
$k \geq 5$	$\theta_k \geq 1/2$	Open.

The classical result for $k = 2$ giving $\theta_2 = 2/3$ depends crucially on the Weil bound for Kloosterman's sum $S(a, b; d)$:

$$S(a, b; d) := \sum_{\substack{n=1 \\ (n,d)=1}}^{d} e_d\left(an + b\overline{n}\right) \ll \tau(d)(a, b, d)^{1/2} d^{1/2}. \tag{1.5}$$

This important result for θ_2 is an unpublished result of Selberg, Hooley, and Linnik obtained independently in the 1950's, though none of them seem to have formally written it down. They discovered that Weil bound (1.5) for Kloosterman sums implies that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$T_2(X, d, a) = \frac{X}{\varphi(d)} P_d(\log X) + O\left(\frac{X^{1-\delta}}{\varphi(d)}\right),$$

uniformly for all $d < X^{2/3-\epsilon}$, where P_d is the linear polynomial given by

$$P_d(\log X) = \operatorname{Res}\left\{\zeta^2(s)\prod_{p|d}(1-p^{-s}\frac{X^{s-1}}{s}; s=1\right\} \tag{1.6}$$
$$= \frac{\varphi(d)^2}{d^2}(\log X + 2\gamma - 1) - \frac{2\varphi(d)}{d}\sum_{\delta|d}\frac{\mu(\delta)\log\delta}{\delta}.$$

(Note that the main term here is differed from that in (1.3) by an admissible quantity

$$\sum_{\substack{n\le X\\(n,d)=1}}\tau(n) - XP_d(\log X) \ll X^{1/2}d^{\epsilon};$$

see, e.g., [1, Lemma 2.5, p. 7].) As noted in [28], the work [31] of Hooley in 1957 essentially gives this result for θ_2. A formal proof of this result can be found in [27, Corollary 1, p. 409], the work of Heath-Brown in 1979.

The *divisor problem for arithmetic progressions* (c.f. [23]) then amounts to extending θ_2 beyond 2/3. This difficult question has seen no improvement since the 1950's. In one aspect, the problem is asking for a better uniform estimates for Kloosterman sums beyond those that are immediately available from the Weil bound (1.5); see, e.g., the discussion in [1]. The next, and only known, case where $\theta_k > 1/2$ is for $k = 3$.

Friedlander and Iwaniec's spectacular breakthrough work [24] in 1985 yields, in particular, $\theta_3 = 58/115 > 1/2$ for $k = 3$. More precisely, they proved in [24, Theorem 5, p. 338] that, for any $\epsilon > 0$, $X^{92/185} < d < X^{58/115}$, $(a, d) = 1$, we have

$$T_3(X, d, a) = \frac{X}{\varphi(d)}P(\log X) + O(X^{A+\epsilon}d^{-B}),$$

where $A = 271/300$, $B = 97/120$, and P is the quadratic polynomial for which $P(\log X)$ is the residue at $s = 1$ of $(\prod_{p|d}(1-p^{-s}\zeta(s))^3(X^{s-1}/s)$. The proof uses multiple exponential sums which rests on Deligne's deep work on the Riemann Hypothesis over finite fields together with Burgess' bounds [8] on character sums.

Heath-Brown's improvement [28, Theorem 1, p. 31] in 1986 of this exponent of distribution θ_3 to $1/2 + 1/82$ gives a different proof and removes the condition $(a, d) = 1$. Consequently, he replaced the expected main term in (1.4) by

$$M_k(X, d, a) = \frac{X}{\varphi(d/\delta)}\operatorname{Res}\left\{(\sum_{\substack{m=1\\(m,d)=\delta}}^{\infty}\tau_k(m)m^{-s})\frac{X^{s-1}}{s}; s=1\right\}$$

where $\delta = (a, d)$. Heath-Brown showed in [28, Theorem 1, p. 31] that for any $\epsilon > 0$, if $q \le X^{21/41}$, then

$$T_3(X, d, a) = M_3(X, d, a) + O(X^{86/107+\epsilon}d^{-66/107}).$$

His proof uses deep estimates for multiple Kloosterman sums also powered by Deligne's Riemann Hypothesis.

Most recently, in 2015, Fouvry, Kowalski, and Michel's result [21] for $k = 3$ to prime moduli uses the spectral theory of modular forms and estimates for exponential sums using Frobenius trace functions. They showed in [21, Theorem 1.1, p. 122] that for every non-zero integer a, every $\epsilon, A > 0$, every $X \ge 2$, and every prime q, coprime with a, satisfying

$$q \le X^{1/2+1/46+\epsilon},$$

we have

$$\Delta(\tau_3; X, q, a) \ll \frac{X}{q \log^A X},$$

where the implied constant depends on ϵ and A and not on a. The estimates for $k = 1, 2$, and 3 are the only known cases where (S1) holds with $\theta_k > 1/2$.

The exponent of distribution $\theta_4 = 1/2$ for $k = 4$ is explicit in Linnik's work [40] of 1961. Lavrik's result [39] in 1965 for $k \geq 4$ uses Burgess' estimates for character sums and the fourth power moment estimate for $L(s, \chi)$ averaged over characters χ modulo q. Friedlander and Iwaniec's improvement [23] in 1985 of Lavrik's result for $k \geq 5$ uses Burgess' estimates for character sums and Heath-Brown and Iwaniec's work [29] on the difference between consecutive primes.

Remark 1. *The k-fold divisor problem for arithmetic progressions, which asks whether the range of d for which statement (S1) holds can be extended beyond $X^{1/2}$ for $k \geq 4$, has important implications to the distribution of prime numbers. For instance, the estimate* (1.4) *has application to the asymptotic for the divisor problem*

$$\sum_{n \leq X} \tau_k(n) \tau_\ell(n+h).$$

See, for instance, the recent works [42,43], or [56, p. 4] for discussion towards application to power moments of the Riemann zeta function.

More concretely, the distribution of the ternary divisor function $\tau_3(n)$ in arithmetic progression [24] has shown to play an important rôle in the sensational work of Y. Zhang [57] towards the problem of bounded gaps between primes, bringing the gap from infinity to a finite number. The distribution of τ_4 to large moduli will undoubtedly have important consequence to prime numbers. See also [17, Théorème 4] for precise connection between distribution of τ_k and distribution of prime numbers.

1.3. **Average estimates over d.** To obtain further progress on Conjecture 1, a larger range for d can be obtained by averaging over d. This type of result is of Bombieri-Vinogradov type. More precisely, let (S2) be the following statement.

(S2) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{d < X^{\theta_k}} \max_{(a,d)=1} \left| T_k(X, d, a) - \frac{X}{\varphi(d)} P_k(\log X) \right| \ll X^{1-\delta}, \tag{1.7}$$

provided that $d \leq X^{\theta_k - \epsilon}$.

We have an analogous conjecture for θ_k.

Conjecture 2. *For each $k \geq 2$ statement (S2) holds for any $\theta_k < 1$.*

Of course, Conjecture 1 implies Conjecture 2. In Table 2 we list known results towards Conjecture 2.

Averaging over d, Fouvry in 1985 was able to break through the $X^{2/3}$-barrier for θ_2 and showed in [18, Corollaire 5, p. 74] that for any $\epsilon > 0$, there exists a constant $c = c(\epsilon) > 0$ such that

$$\sum_{\substack{(q,a)=1 \\ d \leq X^{1-\epsilon} \\ d \notin [X^{2/3-\epsilon}, X^{2/3+\epsilon}]}} |\Delta(\tau_2; X, d, a)| \ll X \exp(-c \log^{1/2} x)$$

uniformly for all $|a| \leq \exp(c \log^{1/2} x)$. His proof uses estimates for Kloosterman sums together with Poisson's formula and the dispersion method.

TABLE 2. Known results towards Conjecture 2 for average over all moduli d, and references.

k	θ_k	**References**
$k = 2$	$\theta_2 = 1$	Fouvry (1985) [18, Corollaire 5, p. 74] (exponential saving); Fouvry and Iwaniec (1992) [20, Theorem 1, p. 272].
$k = 3$	$\theta_3 = 1/2 + 1/42$	Heath-Brown (1986) [28, Theorem 2, p. 32].
	$\theta_3 = 1/2 + 1/34$	Fouvry, Kowalski, and Michel (2015) [21, Theorem 1.2, p. 123], (for prime moduli, polylog saving).
$k \geq 4$	$\theta_k < 1/2$	Follows from the general version of Bombieri-Vinogradov theorem, see, e.g., [44] or [54], (polylog saving).
$k \geq 4$	$\theta_k \geq 1/2$	Open.

The gap $X^{2/3-\epsilon} < d < X^{2/3+\epsilon}$ is closed by Fouvry and Iwaniec [20] for the case of squarefree moduli d which have a square-free factor r of a certain size. They showed in [20, Theorem 1, p. 272] that if r is square-free with $(a, r) = 1$, $r \leq X^{3/8}$, then we have

$$\sum_{\substack{rs^2 < X^{1-6\epsilon} \\ (s,ar)=1}} |\Delta(\tau_2; X, d, a)| \ll r^{-1} X^{1-\epsilon},$$

where the implied constant depends on ϵ alone. The proof depends on an estimate for sums of Kloosterman sums [20, Theorem 2, p. 272]. In the proof of Theorem 2, some estimates of sums of Laurent polynomials in five variables over a finite field were required; an estimate for these sums is proved in the appendix of [20] by Katz.

A larger range for θ_3 obtained by Heath-Brown [28] is a result of a sharper average bound for

$$\#\{n_1, n_2, n_3, n_4 : 1 \leq n_i \leq N, (n_i, d) = 1, n_1 + n_2 \equiv n_3 + n_4 \pmod{d}\}.$$

He proved in [28, Theorem 2, p. 32] that for any $\epsilon > 0$, we have

$$\sum_{d \leq D} \max_{x \leq X} \max_{a \pmod{d}} |T_3(x, d, a) - M_3(x, d, a)| \ll X^{40/51+\epsilon} D^{7/17},$$

and this is non-trivial for $D \leq X^{11/21-3\epsilon}$.

With a fixed, Fouvry, Kowalski, and Michel's second result in [21] shows that for every non-zero integer a, for every $\epsilon > 0$, and every $A > 0$, we have,

$$\sum_{\substack{q \leq X^{9/17-\epsilon} \\ q \text{ prime}, q \nmid a}} |\Delta(\tau_3; X, d, a)| \ll \frac{X}{\log^A X},$$

where the implied constant depends on ϵ, A, and a. This is the best result on the numerical value of the exponent of distribution θ_k for τ_k presently. Their proof combines estimates for divisor twists of trace functions together with "Kloostermaniac" techniques through the seminal work of Deshouillers and Iwaniec in [16].

Notably, in this type, the statement (S2) holds for any $\theta_k < 1/2$ for all $k \geq 5$. This follows from the general version of the Bombieri-Vinogradov theorem; see, e.g., [44] or [54].

1.4. **Average estimates over a.** Instead of averaging over the modulus d, a different average over the reduced residue classes $a \pmod{d}$ has also been considered by several authors. For instance, in 2005, Banks, Heath-Brown, and Shparlinski showed in [1, Theorem 3.1, p. 9] that for every $\epsilon > 0$, we have the mean value estimate

$$\sum_{\substack{a=1\\(a,d)=1}}^{d}\left|T_2(X,d,a)-\frac{X}{\varphi(d)}P_d(\log X)\right| \ll \begin{cases} d^{1/5}X^{4/5+\epsilon}, & \text{if } d>X^{1/2},\\ d^{2/5}X^{7/10+\epsilon}, & \text{if } X^{1/3}<d\le X^{1/2},\\ d^{1/2}X^{2/3+\epsilon}, & \text{if } X^{1/6}<d\le X^{1/3},\\ X^{3/4+\epsilon}, & \text{if } d\le X^{1/6},\end{cases} \tag{1.8}$$

where $P_d(\log X)$ is given as in (1.6), and the implied constant depends only on ϵ. In particular, their result shows that the left side of the above is $\ll X^{1-\epsilon/6}$ uniformly for all $d<X^{1-\epsilon}$. Their method relies on average bounds for incomplete Kloosterman sums–they showed and crucially used that the Weil-type bound for certain incomplete Kloosterman sums can be sharpened when averaged over all reduced residue classes modulo d.

The upper bounds in (1.8) are substantially sharpened by Blomer [4] in 2008 where the condition $(a,d)=1$ is removed and the summand is squared instead. More precisely, letting

$$\tilde{P}_d(\log X)=\sum_{\delta\mid d}\frac{r_\delta(a)}{\delta}(P(\log X)+2\gamma-1-2\log\delta), \tag{1.9}$$

Blomer showed in [4, Theorem 1.1, p. 277] that for X a large real number and $d\le X$ a positive integer, then for any $\epsilon>0$, we have

$$\sum_{a=1}^{d}\left(T_2(X,d,a)-\frac{X}{d}\tilde{P}_d(\log X)\right)^2 \ll X^{1+\epsilon}, \tag{1.10}$$

where $r_c(h)$ is the Ramanujan's sum. When $(a,d)=1$, the above main term matches that of (1.8). In particular, by Cauchy's inequality, (1.10) gives, for all $d>X^{1/3}$

$$\sum_{a=1}^{d}\left|T_2(X,d,a)-\frac{X}{d}\tilde{P}_d(\log X)\right| \ll (dX)^{1/2+\epsilon}.$$

The proof of (1.10) uses Voronoi summation (the functional equation of twists of the corresponding L-function). Blomer's proof also works for other arithmetic functions such as Fourier coefficients of primitive (eigenform of all Hecke operators) holomorphic cusp forms [4, Theorem 1.2] and the square-free numbers μ^2 [4, Theorem 1.3]. Even though $\tau(n)>0$ for all n while Hecke eigenvalues can be negative, it is well known that their mean values share analogous oscillatory properties; see, e.g., the detailed exposition [34] of Jutila.

Lau and Zhao [38] in 2012 obtained asymptotic results for the variance of $\tau(n)$ for d and X going to infinity at different rates. Fouvry, Ganguly, Kowalski, and Michel [19] in 2014 show, in a restricted range, that the divisor function $\tau(n)$ in residue classes modulo a prime follows a Gaussian distribution, and a similar result for Hecke eigenvalues of classical holomorphic cusp forms.

There is also a conjecture for general k. In a function field variant, the work of Keating, Rodgers, Roditty-Gershon, and Rudnick in [35] leads to the following conjecture over the integers for the variance of τ_k (c.f. [47, Conjecture 1]).

Conjecture 3 (Keating–Rodgers–Roditty-Gershon–Rudnick [35])**.** *For $X,d\to\infty$ such that $\log X/\log d\to c\in(0,k)$, we have*

$$\sum_{\substack{a=1\\(a,d)=1}}^{d}\Delta(\tau_k;X,d,a)^2 \sim a_k(d)\gamma_k(c)X(\log d)^{k^2-1},$$

where $a_k(d)$ is the arithmetic constant

$$a_k(d) = \lim_{s\to 1^+} (s-1)^{k^2} \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} \frac{\tau_k(n)^2}{n^s},$$

and $\gamma_k(c)$ is a piecewise polynomial of degree k^2-1 defined by

$$\gamma_k(c) = \frac{1}{k!G(k+1)^2} \int_{[0,1]^k} \delta_c(w_1 + \cdots w_k)\Delta(w)^2 d^k w,$$

where $\delta_c(x) = \delta(x-c)$ is a Dirac delta function centered at c, $\Delta(w) = \prod_{i<j}(w_i - w_j)$ is a Vandermonde determinant, and G is the Barnes G-function, so that in particular $G(k+1) = (k-1)!(k-2)!\cdots 1!$.

This conjecture is closely related to the problem of moments of Dirichlet L-functions; see, e.g, the works [10–13] of Conrey and Keating on moments of the Riemann zeta function and correlations of divisor sums.

1.5. **Average estimates over both a and d.** The estimate of the form (1.7) in (S2) is of Bombieri-Vinogradov type where an average over moduli d is taken. In addition to this average, taking an additional average over all primitive residue classes a in each modulus d yields better result for the range for d. These results are of Barban-Davenport-Halberstam type (see, e.g., [14, §29] and have a long history, starting from the initial works [2,3] of Barban in 1963 on primes in arithmetic progressions, and of Davenport and Halberstam [15] a few years latter. Hooley has written a series of papers on the Barban-Davenport-Halberstam sums, dating from 1975, which numbers nineteen, as of currently.

In 1976 Motohashi [44] obtained an asymptotic formula for $\tau(n)$ averaged over both a and d. More specifically, there are explicit numerical constants $\mathfrak{S}_j$, $0 \le j \le 3$, such that

$$\begin{aligned}\sum_{d<X} \sum_{a=1}^{d} \Big(T_2(X,d,a) - \tilde{P}_d(\log X)\Big)^2 \\ = X^2(\mathfrak{S}_3 \log^3 X + \mathfrak{S}_2 \log^2 X + \mathfrak{S}_1 \log X + \mathfrak{S}_0) + O(X^{15/8}\log^2 X),\end{aligned}$$

where $\tilde{P}_d(\log X)$ is as in (1.9)

Very recently, Rodgers and Soundararajan [47] in 2018 consider smoothed sums of τ_k averaging over both a and d, and confirm an averaged version of Conjecture 3 for a restricted range. More precisely, for $k \ge 2$, let

$$\Delta_k(D;X) = \sum_d V_k(d;X)\Phi\left(\frac{d}{D}\right),$$

where

$$V_k(d;X) = \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left(\sum_{n\equiv a(d)} \tau_k(n)\Psi\left(\frac{n}{N}\right) - \frac{1}{\varphi(d)} \sum_{(n,d)=1} \tau_k(n)\Psi\left(\frac{n}{N}\right)\right)^2,$$

with Φ and Ψ fixed smooth non-negative functions compactly supported in the positive reals normalized so that $\int \Phi = 1$ and $\int \Psi^2 = 1$. Then letting $c = \log x/\log D$, Rodgers and Soundararajan in [47, Theorem 1] obtained asymptotics for $\Delta_k(D;X)$ as $X, D \to \infty$ uniformly in c for all $\delta \le c \le (k+2)/k - \delta$ for some $\delta > 0$ sufficiently small. Under GRH, a larger range $\delta \le c \le 2-\delta$ independent of k is obtained in [47, Theorem 2]. Their results are closely related to moments of Dirichlet L-functions as discussed above, and their proof relies on the asymptotic large sieve.

The latter work of de la Bretèche and Fiorilli in [7] considers a related variance in the range $1 \le c < 1+O(1/k)$ using an arithmetic approximation motivated by work of Vaughan. Interestingly, the asymptotic for their arithmetic variance in [7, Theorem 1.2] matches those in [47, Theorem 1].

In our first result, we prove a distribution estimate for τ_k averaging over both a and d to moduli d as large as X, similar to that of Barban-Davenport-Halberstam type theorems for the von Mangoldt function.

Theorem 1. *For $k \ge 4$ we have*

$$\sum_{d\le D} \sum_{\substack{a=1\\(a,d)=1}}^{d} \Delta(\tau_k; X, d, a)^2 \ll (D + X^{1-1/6(k+2)})X(\log X)^{k^2-1}.$$

Motivated by the recent work [30] of Heath-Brown and Li in 2017, we also prove analogous estimates for pairs of $\tau_k(n)$'s and $\tau_k(n)\Lambda(n)$ to moduli d that can taken to be almost as large as X^2.

Theorem 2. *For $k \ge 4$ and any $\epsilon > 0$ there holds*

$$\sum_{d\le D} \sum_{\substack{a=1\\(a,d)=1}}^{d} \left(\sum_{\substack{m,n\le X\\ m\equiv an (\mathit{mod}\ d)}} \tau_k(m)\tau_k(n) - \frac{1}{\varphi(d)} \left(\sum_{\substack{n\le X\\(n,d)=1}} \tau_k(n)\right)^2 \right)^2 \ll X^{4-1/3(k+4)} \tag{1.11}$$

for any $D \le X^{2-1/3(k+2)}$.

In particular, the above estimate is valid if one of the τ_k is replaced by the von Mangoldt function Λ. We have

$$\sum_{d\le D} \sum_{\substack{a=1\\(a,d)=1}}^{d} \left(\sum_{\substack{m,n\le X\\ m\equiv an (\mathit{mod}\ d)}} \tau_k(m)\Lambda(n) - \frac{X}{\varphi(d)} \sum_{\substack{n\le X\\(n,d)=1}} \tau_k(n) \right)^2 \ll X^{4-1/3(k+4)} \tag{1.12}$$

for any $D \le X^{2-1/3(k+2)}$.

It might look surprising at first that the moduli in Theorems 2 can be taken almost as large as X^2, but proof is in fact rather simple; the proof of Theorem 2 follows essentially from the multiplicative large sieve inequality.

Remark 2. *Assuming the Generalized Riemann Hypothesis, it might be possible to show that the estimates* (1.11) *and* (1.12) *hold in a larger range for d with right side replaced by*

$$\begin{cases} X^{2-\delta}, & \text{for } 1 \le D \le X^{1+\epsilon}, \\ X^2 D(\log X)^{k^2}, & \text{for } X^{1+\epsilon} < D \le X^2, \end{cases}$$

for some constant $\delta > 0$.

1.6. **Average estimates over smooth moduli.** Recently progress has been made breaking the $X^{2/3}$-barrier towards the divisor problem for arithmetic progression for special moduli. It has been discovered that if, in addition to averaging over moduli d, we also restrict d to those that have good factorization properties, we can obtain results for $\theta_k \ge 1/2$ beyond the $1/2$-barrier. A natural number m is called X^δ-smooth if all proper prime divisors of m are less than or equal to X^δ for some $\delta > 0$. Let (S3) be the following statement.

(S3) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{\substack{d < X^{\theta_k} \\ d \text{ is } X^{\eta}\text{-smooth}}} |T_k(X, d, a) - M_k(X, q)| \ll X^{1-\delta}$$

for some $\eta > 0$, provided that $d \le X^{\theta_k - \epsilon}$.

Conjecture 4. *For each $k \ge 2$ statement (S3) holds for any $\theta_k < 1$.*

Table 3. Known results towards Conjecture 4 for average over smooth moduli d, and references.

k	θ_k	**References**
$k = 2$	$\theta_2 = 2/3 + 1/246$	Irving (2015) [32, Theorem 1.2, p. 6677] (square-free moduli).
	$\theta_2 = 2/3 + \epsilon$	Khan (2016) [36, Theorem 1.1, p. 899] (prime powers moludi).
$k = 3$	$\theta_3 = 1/2 + 1/34$	Xi (2018) [55, Theorem 1.1, p. 702], (square-free moduli, polylog saving).
$k \ge 4$	$\theta_k = 1/2 + 1/584$	Wei, Xue, and Y. Zhang (2016) [53, Theorem 1.1, p. 1664] (square-free moduli, exponential saving).

In 2015, Irving [32] succeeded in broken through the Weil bound for smooth moduli and obtained sharp individual estimate for each d for d almost as large as $X^{55/82}$, thus showing $\theta_2 > 2/3$. More precisely, Irving proved in [32, Theorem 1.2, p. 6677] that for any $\varpi, \eta > 0$ satisfying $246\varpi + 18\eta < 1$, there exists a $\delta > 0$, depending on ϖ and η, such that for any X^{η}-smooth, squarefree $d \le X^{2/3+\varpi}$ and any $(a, d) = 1$, we have

$$\Delta(\tau_2; X, d, a) \ll d^{-1} X^{1-\delta}. \tag{1.13}$$

The proof is based on a q-analog of van der Corput's method and bounds on complete Kloosterman sums of Fouvry, Kowalski, and Michel [22, Corollary 3.3].

Khan [36] in 2016 considered an important case of prime power moduli $d = p^m$ and proved in [36, Theorem 1.1, p. 899] that for a fixed integer $m \ge 7$, there exists some constants $\delta > 0$ and $\rho > 0$, depending only on m, such that (1.13) holds uniformly for $X^{2/3-\rho} < d < X^{2/3+\rho}$. Khan's method is different from that of Irving. The proof uses cancellation of a sum of Kloosterman sums to prime power moduli via the method of Weyl differencing. Khan's result is made uniform in m in [41].

For the ternary divisor function, in an extension of the work of Fouvry, Kowalski, and Michel [21] on θ_3 for prime moduli, Xi [55] in 2018 obtained individual estimates for smooth, square-free d to the same level of distribution $\theta_3 = 9/17$. More precisely, for each $\epsilon > 0$ and $A > 0$, there exists a constant $\eta > 0$ such that if $X^{9/17-\epsilon} \ll d \ll X^{9/17-\epsilon}$, d is X^{η}-smooth and square-free, then

$$\Delta(\tau_3; X, d, a) \ll \frac{X}{d} (\log X)^{-A}$$

holds uniformly for $(a, d) = 1$.

In [53], F. Wei, B. Xue, and Y. Zhang showed that the methods of Zhang in [57] in the problem of bounded gaps between primes applies not only to the von Mangoldt function Λ, but also equally to the k-fold divisor function τ_k. They proved in [53, Theorem 1.1, p. 1664] that for any $k \ge 4$ and $a \ne 0$, we have

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{1/2+1/584}}} \mu(d)^2 \, |\Delta(\tau_k; X, d, a)| \ll X \exp(-\log^{1/2} X), \tag{1.14}$$

where

$$\mathcal{D} = \{d \geq 1 : (d, a) = 1, (d, \prod_{p<X^{1/1168}} p) > X^{71/584}\}, \tag{1.15}$$

and the implied constant depends on k and a. The condition on the moduli d in (1.15) slightly relaxes the constraint on d being smooth, allowing for d to have some, but not too many, prime factors larger than $X^{1/1168}$. The error term and, more importantly, the exponent of distribution $\theta_k = 293/584 = 1/2 + 1/548$ in (1.14) hold uniformly in k. The proof uses the Cauchy-Schwartz inequality, combinatorial arguments, the dispersion method, the Weil bound on Kloosterman sums, together with an estimate of Birch and Bombieri for a variant of a three-variable Kloosterman sum proved in the Appendix to [24] powered by Deligne's Riemann Hypothesis, and, crucially, the factorization

$$d = qr$$

to Weyl shift a certain incomplete Kloosterman sum to the modulus r, thus gaining over Deligne's Riemann Hypothesis by a power of r, hence saving a power of d, since d is a multiple of r.

Our main result improves upon the error term obtained by F. Wei, B. Xue, and Y. Zhang [53] in 2016 for the distribution of family of the k-fold divisor function τ_k for $k \geq 4$, and is the following:

Theorem 3 (Main theorem). *Let*

$$\varpi = \frac{1}{1168}$$

and

$$\theta_k = \min\left\{\frac{1}{12(k+2)}, \varpi^2\right\}.$$

For $a \neq 0$, let

$$\mathcal{D} = \{d \geq 1 : (d, a) = 1,\ |\mu(d)| = 1,\ (d, \prod_{p \leq X^{\varpi^2}} p) < X^{\varpi},\ \text{and } (d, \prod_{p \leq X^{\varpi}} p) > X^{71/584}\},$$

where μ is the Möbius function. Then for each $k \geq 4$ we have

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{293/584}}} \left| \sum_{\substack{n \leq X \\ n \equiv a (mod\ d)}} \tau_k(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right| \ll X^{1-\theta_k}. \tag{1.16}$$

The implied constant is effective, and depends at most on a and k.

Remark 3. *Theorem 3 admits several refinements. The particular choice of $\varpi = 1/1168$ is not optimal and there are certain ways to improve the numerics in Theorem 3, for instance using the extensive work [46] of the Polymath 8 project. Though we do not focus on this aspect here, it is an open problem to replace ϖ and θ_k on the right side of (1.16) by values that are as large as possible.*

Conditionally, if we assume the Generalized Riemann Hypothesis, or the weaker Generalized Lindelöf Hypothesis, for Dirichlet L-functions we can obtain a stronger result.

Theorem 4. *On the Generalized Lindelöf Hypothesis, the estimate* (1.16) *holds with the right side replaced by*

$$X^{1-\varpi^2},$$

where the θ_k power saving is replaced by a positive constant independent of k.

This uniform power saving is the result of sharper estimates of $L(s, \chi)^k$ on the critical line that are independent of k.

2. Sketch of proof

The actual details are quite lengthy. We summarize here the main steps. To prove (1.16) we follow standard practice and split the summation over moduli d in into two sums: one over $d < X^{\frac{1}{2}-\delta}$ which are called small moduli and the other over $X^{\frac{1}{2}-\delta} \le d < X^{\frac{1}{2}+2\varpi}$ which are called large moduli.

For small moduli, we estimate (1.16) directly using the large sieve inequality together with a direct substitute for the Siegel-Walfisz condition. For the von Mangoldt function $\Lambda(n)$, the Möbius function $\mu(n)$ is involved and, hence, the Siegel-Walfisz theorem is needed to handle very small moduli. For us, fortunately, τ_k is simpler than Λ in that μ is absent–this feature of τ_k allows us to get a sharper bound in place of the Siegel-Walfisz theorem. The constant here is effective.

For large moduli, we adapt the methods of Zhang in [57] to bound the error term which goes as follows. After applying suitable combinatorial arguments, we split τ_k into appropriate convolutions as Type I, II, and III, as modeled in [57]. We treat the Type I and II in our Case (b), Type III in our Case (c), and Case (a) corresponds to a trivial case which we treat directly. The main ingredients in Case (b) are the dispersion method and Weil bound on Kloosterman sums. The Case (c) depends crucially on the factorization $d = qr$ of the moduli to Weil shift a certain incomplete Kloosterman sum to the modulus r. The shift modulo this r then induces a Ramanujan sum. This allows for a saving of a power of r, and since d is a multiple of r, this saves a small power of X from the trivial bound.

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