Distribution of zeros of the derivatives of the Riemann zeta function and its relations to zeros of the zeta function itself

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– a joint work with Fan Ge –

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The *Riemann zeta function* $\zeta(s)$ is the analytic function on $\mathbb{C}\backslash\{1\}$ satisfying

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Remarks

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- The equality $\sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1}$ tells us that $\zeta(s)$ has no zeros in $\operatorname{Re}(s) > 1$.
- Subtracting the term $(s-1)^{-1}$ from the Dirichlet series (1) and using its integral representation, we find that $\zeta(s)$ can be analytically continued to $\operatorname{Re}(s) > 0$ $(s \neq 1)$.

Functional equation and trivial zeros of $\zeta(s)$

$\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s). \tag{2}$$

From this we can deduce:

- Since $\zeta(s)$ is analytic on $\mathrm{Re}(s) > 0$ ($s \neq 1$), $\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ is too.

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- At $s = 2, 4, 6, \ldots$, $\sin(\pi s/2) = 0$ cancels out poles of $\Gamma(1-s)$.
- $\zeta(1-s)$ has simple zeros at $s = 3, 5, 7, \ldots$ due to poles of $\Gamma(1-s)$.

Hence $\zeta(s)$ has trivial zeros at $s = -2, -4, -6, -8, -10, \ldots$.

Zeros of $\zeta(s)$

From

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}(s) > 1),$$

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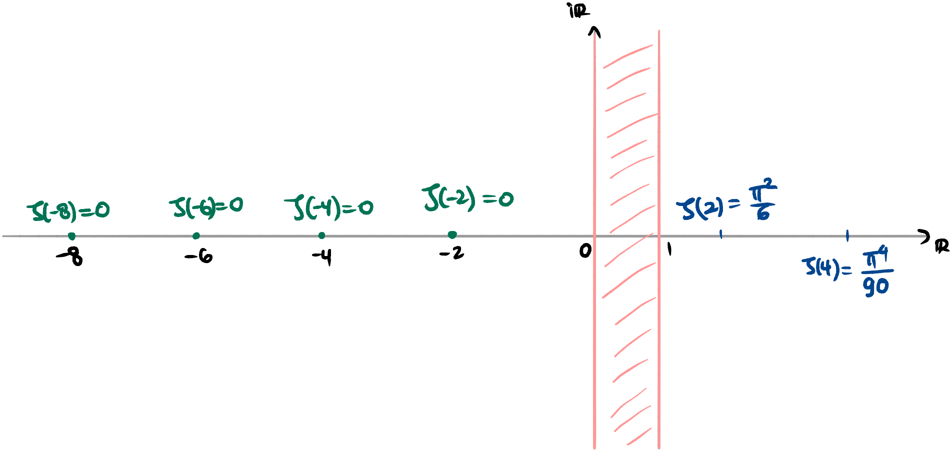
Hence, zeros of $\zeta(s)$ other than $s = -2, -4, -6, -8, -10, \ldots$, if exist, should lie within $0 \leq \mathrm{Re}(s) \leq 1$.



iR
$\zeta(-8)=0$
$\zeta(-6)=0$
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$\zeta(2)=\frac{\pi^2}{6}$
$\zeta(4)=\frac{\pi^4}{90}$
-8
-6
-4
-2
0
1
R

Counting prime numbers (I)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \frac{1}{1-p^{-s}}, \qquad \mathrm{Re}(s) > 1,$$

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots, \quad \mathrm{Re}(s) > 1$$

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$\rightsquigarrow$ There are infinitely many prime numbers.

Counting prime numbers (II)

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, \ldots$$

Let $\pi(x)$ count the number of prime numbers up to x and

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

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$$\zeta(1 + it) \neq 0, \quad t \in \mathbb{R} \quad \Longleftrightarrow \quad \pi(x) \sim \mathrm{Li}(x).$$

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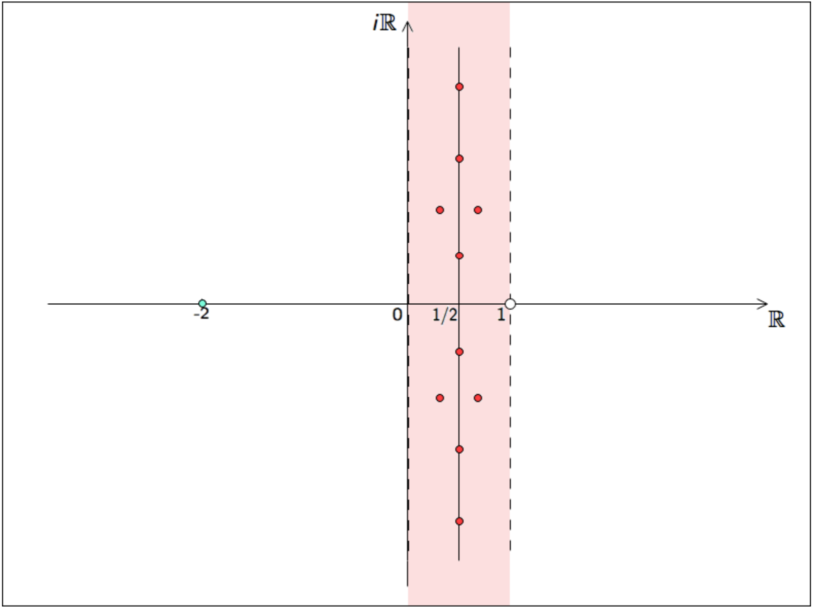
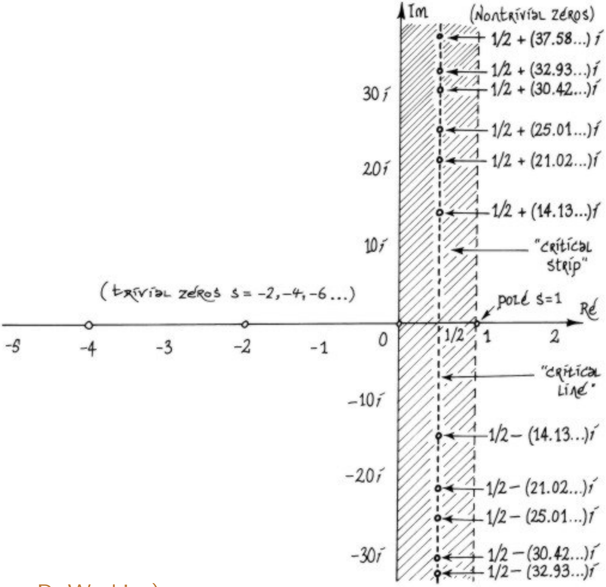
Theorem (H. Koch, 1900)

RH holds $\iff$ $\pi(x) = \mathrm{Li}(x) + O\left(x^{1/2} \log x\right)$ is best possible.

Properties of zeros of $\zeta(s)$



Zeros of $\zeta(s)$



(by Matthew R. Watkins)

Equivalence for RH

Theorem (Speiser, 1935)

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is equivalent to

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Theorem (Levinson and Montgomery, 1974)

$N^-(T)$ (*resp.* $N_1^-(T)$) := the number of zeros of $\zeta(s)$ (*resp.* $\zeta'(s)$) in $\{\sigma + it \mid 0 < \sigma < 1/2,\ 0 < t < T\}$, counted w/ multiplicity.

For $T \geq 2$ we have

$$N^-(T) = N_1^-(T) + O(\log T).$$

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Theorem (Hardy and Littlewood, 1921)

$$N_0(T) \gg T$$

Simple zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$

Theorem (Selberg, 1942)

There exists $c > 0$ (effective) such that $N_0(T) > c\,N(T)$.

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$N_0^*(T) :=$ the number of zeros $\rho_0 = 1/2 + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ where $\zeta'(\rho_0) \neq 0$

Theorem (Levinson, 1974)

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More zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$

Theorem (Conrey, 1989)

$$N_0(T) \geq 0.4088\, N(T)$$

$$N_0^*(T) \geq 0.4013\, N(T)$$

Theorem (Bui, Conrey and Young, 2011)

$$N_0(T) \geq 0.4105\, N(T)$$

$$N_0^*(T) \geq 0.4058\, N(T)$$

Theorem (Feng, 2014(?))

$$N_0(T) \geq 0.4109\, N(T)$$

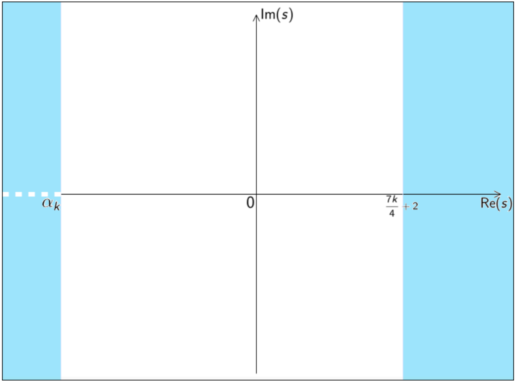
Shifting zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$ by $\zeta'(s)$

1. Study the change of argument of
$$f(s) = \zeta(s) + \frac{\zeta'(s)}{\log t}.$$

2. The number of zeros of $f(s)$ is essentially that of $\zeta(s)$ in $\{\sigma + it \mid 1/2 < \sigma < 1,\ 0 < t < T\}$.

Nontrivial (= non-real) zeros of $\zeta^{(k)}(s)$

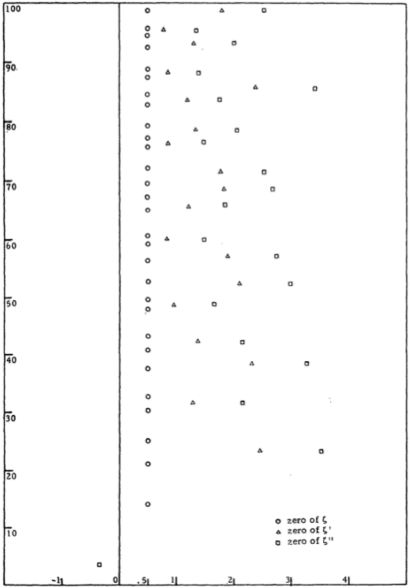
A zero-free region of $\zeta^{(k)}(s)$:



$$\mathcal{Z} = \{\rho \in \mathbb{C} \mid \zeta(\rho) = 0, \ \ \mathrm{Im}(\rho) \neq 0\}$$

$$\begin{aligned}\mathcal{Z}^{(k)} &:= \{\rho \in \mathbb{C} \mid \zeta^{(k)}(\rho) = 0, \ \ \mathrm{Im}(\rho) \neq 0\} \\ &= \text{the set of all } \textit{nontrivial} \text{ zeros of } \zeta^{(k)}(s)\end{aligned}$$

Nontrivial zeros of $\zeta(s)$, $\zeta'(s)$, $\zeta''(s)$



R. Spira, *Zero-free regions of* $\zeta^{(k)}(s)$, J. Lond. Math. Soc. **40** (1965), p. 681

RH and zeros of $\zeta''(s)$ & $\zeta'''(s)$

Theorem (Yıldırım, 1996)

RH implies

$$\zeta''(s) \neq 0 \text{ and } \zeta'''(s) \neq 0 \text{ in } 0 \leq \mathrm{Re}(s) < 1/2.$$

Theorem (Yıldırım, 1996)

$\zeta''(s)$ and $\zeta'''(s)$ have only one pair of non-real zeros in $\mathrm{Re}(s) < 0$.

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RH and non-real zeros of $\zeta^{(k)}(s)$

Theorem (Levinson and Montgomery, 1974)

Let $m \geq 0$.

$\zeta^{(m)}(s)$ has only finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$

$\Rightarrow$

$\zeta^{(m+j)}(s)$ $(j \geq 1)$ also has only finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$.

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$\zeta^{(m+j)}(s)$ $(j \geq 1)$ also has only finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$.

Corollary (Levinson and Montgomery, 1974)

RH $\Rightarrow$

$\zeta^{(k)}(s)$ has at most finitely many non-real zeros in $\mathrm{Re}(s) < 1/2$.

Number of nontrivial zeros of $\zeta^{(k)}(s)$ (under RH)

$N(T)$ (*resp.* $N_k(T)$) := the number of nontrivial zeros ρ of $\zeta(s)$ (*resp.* $\zeta^{(k)}(s)$) with $0 < \mathrm{Im}(\rho) < T$, counted with multiplicity

$$g(T) := \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}, \quad h(T) := \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi}$$

	$N(T)$	$N_k(T)$
unconditional	$g(T) + O(\log T)$ [von Mangoldt, 1905]	$h(T) + O_k(\log T)$ [Berndt, 1970]
under RH	$g(T) + O\left(\frac{\log T}{\log\log T}\right)$ [Littlewood, 1924]	$h(T) + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$ $k = 1$: [Akatsuka, 2012] $k \geq 2$: [A.I.S., 2015]

An improvement by Fan Ge (under RH)

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Theorem 1 (under RH)

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A more general statement (under RH)

Suppose that the error term bound in $N(T)$

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + E_0(T)$$

is $E_0(T) = O(\Phi(T))$ for some increasing function $\log \log T \ll \Phi(T) \ll \log T$.

Theorem 2 (Ge and A.I.S., 2019+)

Assume RH. Then

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\max\left\{\Phi(2T), \sqrt{\log T} \log \log T\right\}\right).$$

A preliminary lemma

Assume RH. Let

$$G_k(s) := (-1)^k \frac{2^s}{(\log 2)^k} \zeta^{(k)}(s)$$

Let $T \geq 2$ satisfy $\zeta(\sigma + iT) \neq 0,\ \zeta^{(k)}(\sigma + iT) \neq 0\ ({}^\forall \sigma \in \mathbb{R})$. Then

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi} \arg G_k\left(\frac{1}{2} + iT\right) + \frac{1}{2\pi} \arg \zeta\left(\frac{1}{2} + iT\right) + O_k(1).$$

The arguments are taken such that $\log \zeta(s)$ and $\log G_k(s)$ tend to 0 as $\sigma \to \infty$, and are holomorphic on $\mathbb{C} \backslash \{\rho + \lambda \mid \zeta(\rho) = 0 \textit{ or } \infty,\ \lambda \leq 0\}$ and $\mathbb{C} \backslash \{\rho + \lambda \mid \zeta^{(k)}(\rho) = 0 \textit{ or } \infty,\ \lambda \leq 0\}$, respectively.

Sketch of proof

Assume RH. Recall the estimate

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O(1).$$

To simplify we only consider the case when

$$\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) = O\left(\frac{\log T}{\log\log T}\right).$$

Hence taking into acccount

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right) + \frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O_k(1),$$

it suffices to show that

$$\arg G_k\left(\frac{1}{2}+iT\right) = O_k\left(\frac{\log T}{\log\log T}\right).$$

Akatsuka's method

$$\arg G_1(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right), \qquad \frac{1}{2} \leq \sigma \leq \frac{3}{4},$$

which gives us

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right).$$

Remark

$$\arg G_1(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right), \quad \frac{1}{2} + \frac{(\log\log T)^2}{\log T} \leq \sigma \leq \frac{3}{4}$$

Fan Ge's method

Write

$$Y := \frac{(\log\log T)^3}{\log T}, \quad U := \frac{Y}{\log\log T} = \frac{(\log\log T)^2}{\log T}$$

and set

$$\Delta_1 := \underset{\infty+iT\to 1/2+U+iT}{\Delta} \arg G_1(\sigma+iT),$$
$$\Delta_2 := \underset{1/2+U+iT\to 1/2+iT}{\Delta} \arg G_1(\sigma+iT).$$

Fan Ge's method

Write

$$Y := \frac{(\log\log T)^3}{\log T}, \quad U := \frac{Y}{\log\log T} = \frac{(\log\log T)^2}{\log T}$$

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$$\Delta_2 := \underset{1/2+U+iT\to 1/2+iT}{\Delta} \arg G_1(\sigma+iT).$$

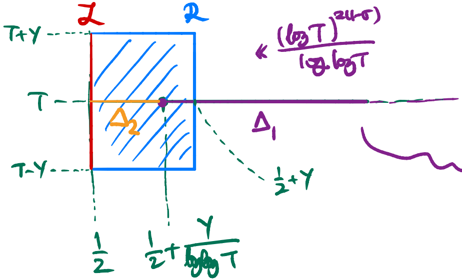
Then from

$$\arg G_1(\sigma+iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right), \quad \frac{1}{2}+U \le \sigma \le \frac{3}{4},$$

we easily deduce

$$\Delta_1 \ll \frac{\log T}{\log\log T}.$$

$$Y := \frac{(\log\log T)^3}{\log T}$$



$$\rightsquigarrow \Delta_1 = O\left(\frac{\log T}{\log\log T}\right)$$

★ $N_{\zeta'}(\mathcal{R}) \ll N_{\zeta}(\mathcal{L})$

★ $$\frac{G_1'}{G_1}(s) = \sum_{\substack{|\mathrm{Im}(\rho_1)-t|<1,\\ \zeta'(\rho_1)=0}} \frac{1}{s-\rho_1} + O(\log t),$$

$s = \sigma + it$, $-1 \leq \sigma \leq 2$.

$$\rightsquigarrow \Delta_2 = O\left(\frac{\log T}{\log\log T}\right)$$

Extending to higher derivatives – Akatsuka's method

$$\arg G_k(\sigma + iT) = O_k\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right), \qquad \frac{1}{2} \le \sigma \le \frac{3}{4}.$$

which again gives us

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right).$$

Remark

$$\arg\frac{G_k}{\zeta}(\sigma + iT) = O_k\left(\frac{\log\log T}{\sigma - \frac{1}{2}}\right), \quad \frac{1}{2} + \frac{(\log\log T)^2}{\log T} < \sigma < 1$$

Extending to higher derivatives – Ge's method

We again need:

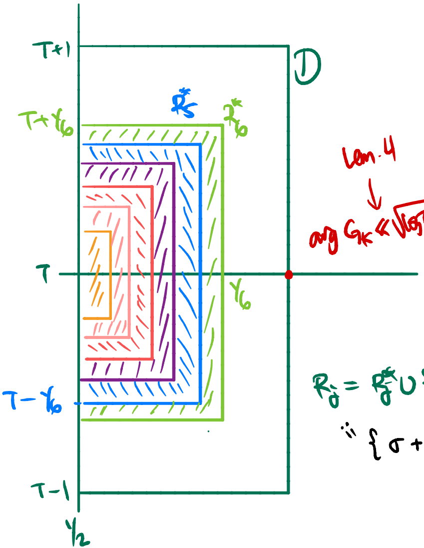
- For sufficiently large t,

$$\mathrm{Re}\,\frac{\zeta^{(k)}}{\zeta^{(k-1)}}(\sigma+it)<0, \qquad 0<\sigma\le 1/2,\ \zeta^{(k-1)}(\sigma+it)\ne 0.$$

- For $s=\sigma+it,\ 1/2\le\sigma\le 1$,

$$\frac{G_k'}{G_k}(s)=\sum_{\substack{|\,\mathrm{Im}(\rho_k)-t|<1,\\ \zeta^{(k)}(\rho_k)=0}}\frac{1}{s-\rho_k}+O_k(\log t).$$

$$Y_j := \frac{2^j}{\sqrt{\log T}}$$



Lem. 4

$\downarrow$

$\arg G_K \ll \sqrt{\log T}\,\log\log T$

$$R_j = R_j^* \cup \{\tfrac{1}{2}+it \mid T-Y_j \le t \le T+Y_j\}$$

$$\because \{\sigma+it \mid \tfrac{1}{2} \le \sigma \le \tfrac{1}{2}+Y_j,\ T-Y_j \le t \le T+Y_j\}.$$

$$N_{\zeta^{(k)}}(R_j) \ll_k Y_j \log T + \frac{\log T}{\log\log T}.$$

keep updating . . .