Ramification Theory for Arbitrary Valuation Rings in Positive Residue Characteristic

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*May contain some typos or errors, please feel free to contact me with comments.

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I study arithmetic geometry. My current work and future projects aim to develop ramification theory for arbitrary valuation fields, that is compatible with the classical theory of complete discrete valuation fields with perfect residue fields.

We consider arbitrary valuation rings with possibly imperfect residue fields and possibly non-discrete valuations of rank ≥ 1, since many interesting complications arise for such rings. In particular, defect may occur (i.e. we can have a non-trivial extension, such that there is no extension of the residue field or the value group). In this talk, we will focus on degree p Galois extensions of such fields.

In section 1, we review the classical theory involving Swan conductor and Kato's definition of refined Swan conductor, we use Artin-Schreier extensions of valued fields in positive characteristic as examples. Section 2 presents the general results obtained in [Thatte16] and [Thatte18], for the equal characteristic case and mixed characteristic case, respectively. The last section contains some basic examples and computations. Please note that many of the technical details will be skipped during the talk, for convenience, and concepts will be explained mostly via examples.

1 Complete Discrete Valuation Rings

1.1 Classical Case: Perfect Residue Fields

Let K be a complete discrete valued field of residue characteristic $p > 0$ with normalized (additive) valuation v, valuation ring A and perfect residue field k. Consider $L|K$, a finite Galois extension of K with $G = \mathrm{Gal}(L|K)$. Let w be the unique valuation on L that extends v, B the integral closure of A in L and l the residue field of L. Let N denote the norm map $N_{L|K}$. We have the following invariants of ramification theory:

The ramification index $e_{L|K} := (w(L^\times) : v(K^\times))$ and the inertia degree $f_{L|K} := [l : k]$. The Lefschetz number $i(\sigma)$ and the logarithmic Lefschetz number $j(\sigma)$ for $\sigma \in G\backslash\{1\}$ are non-negative integers defined as

$$i(\sigma) = \min\{w(\sigma(a) - a) \mid a \in B\} \tag{1.1}$$

$$j(\sigma) = \min\left\{w\left(\frac{\sigma(a)}{a} - 1\right) \mid a \in L^\times\right\} \tag{1.2}$$

For a finite dimensional representation ρ of G over a field of characteristic zero, the Artin conductor $\mathrm{Art}(\rho)$ and the Swan conductor $\mathrm{Sw}(\rho)$ of ρ are defined by the equations below.

$$\mathrm{Art}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} i(\sigma)(\dim(\rho) - \mathrm{Tr}(\rho(\sigma))) \tag{1.3}$$

$$\mathrm{Sw}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} j(\sigma)(\dim(\rho) - \mathrm{Tr}(\rho(\sigma))) \tag{1.4}$$

As a consequence of the Hasse-Arf Theorem ([Serre]), $\mathrm{Art}(\rho)$ and $\mathrm{Sw}(\rho)$ are non-negative integers. Let us consider cyclic extensions $L|K$ of order p and the wild invariants $j(\sigma)$ and $\mathrm{Sw}(\rho)$.

Definition 1.1 (**Classical Swan Conductor: Degree** p)**.** In this case, the *Swan conductor* Sw *of* $L|K$ is defined by considering the one-dimensional representation $\rho : G \to \mathbb{C}^\times; \rho(\sigma) = \zeta_p =: \zeta$, where $\sigma \in G \setminus \{1\}$ is a fixed generator of G. The definition is independent of the choice of such a generator.

$$\mathrm{Sw} = \frac{j(\sigma)}{e_{L|K}} \left(p - 1 - \sum_{i=1}^{p-1} \zeta^i \right) = \frac{j(\sigma)p}{e_{L|K}} \tag{1.5}$$

1.2 Kato's Swan Conductor and Refined Swan Conductor

Let K be as in 1.1, except possibly with imperfect residue field k. Consider the following two types of non-trivial extensions $L|K$ of degree p. We present Kato's generalization of Swan conductor and his definition of the refined Swan conductor in each case.

1.2.1 Equal Characteristic Case: Artin-Schreier Extensions

In this case, $\mathrm{char}\, K = p$ and $L|K$ is given by the Artin-Schreier polynomial $\alpha^p - \alpha = f$ for some $f \in K^\times$. The Galois group G is cyclic of order p, generated by $\sigma : \alpha \mapsto \alpha + 1$.

Definition 1.2 (**Best** f)**.** Let $\mathfrak{P} : K \to K$ denote the additive homomorphism $x \mapsto x^p - x$. The extension L does not change if f is replaced by any $g \in K$ such that $g \equiv f \bmod \mathfrak{P}(K)$. The *Swan conductor* Sw is defined to be $\min\{-v(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$. This definition is consistent with the classical definition above.

An element f of K which attains this minimum is called *best* f. It is well-defined modulo $\mathfrak{P}(K)$. A concrete description of the Swan conductor is given by the following lemma:

Lemma 1.3 (Kato)**.** *By replacing f with an element of $\{g \in K \mid g \equiv f \mod \mathfrak{P}(K)\}$, we have best f which satisfies exactly one of the following properties (i)-(iii). In the case (i), $L|K$ is unramified and the Swan conductor is 0. In the cases (ii) and (iii), the Swan conductor is n.*

(i) $f \in A^\times$.

(ii) $v(f) = -n$, *where n is a positive integer relatively prime to p.*

(iii) $f = ut^{-n}$, *where $n > 0$, $p|n$, t is a prime element of K and $u \in A^\times$ such that the residue class of u in k does not belong to $k^p = \{x^p \mid x \in k\}$.*

Remark 1.4. Let us compare this with the classical definition in (1.5). Let f be best, consider the corresponding Artin-Schreier generator α given by $\alpha^p - \alpha = f$. Then we have $j(\sigma) = -w(\alpha)$ and hence, by (1.5),

$$\mathrm{Sw} = \frac{j(\sigma)p}{e_{L|K}} = -w(\alpha)\frac{p}{e_{L|K}} = -v(f) = -v(N(\alpha))$$

Thus, the "L-side" definition of Sw involving $j(\sigma)$ is connected to the "K-side" definition involving f via the norm map. We generalize this connection later (theorem 2.5).

Notation 1.5 (Kähler Differentials). Let A be a henselian valuation ring with field of fractions K. Let $L|K$ be an extension of henselian valued fields, B the integral closure of A in L and hence, a valuation ring. We consider the A-module Ω^1_A of differential 1-forms over A and the B-module $\Omega^1_{B|A}$ of relative differential 1-forms over A. The A-module of logarithmic differential 1-forms over A is denoted by ω^1_A. We also consider the B-module $\omega^1_{B|A}$ of logarithmic relative differential 1-forms over A, defined as the cokernel of the map $B \otimes_A \omega^1_A \to \omega^1_B$.

In his papers [Kato87], [Kato89] defines the notion of the refined Swan conductor for complete discrete valued fields with arbitrary residue fields. We rephrase this definition in [Thatte16] as follows.

Definition 1.6 (Refined Swan Conductor). Let K be as described in 1.2.1 and $L = K(\alpha)$ the Artin-Schreier extension given by best f. The *refined Swan conductor (*rsw*)* of this extension is defined to be the A-homomorphism $df : \left(\frac{1}{f}\right) \to \omega^1_A/\mathbb{I}\omega^1_A$ given by $h \mapsto (hf)\ \mathrm{dlog}\, f$; the ideal $\mathbb{I}$ of A is described by $\mathbb{I} := \{x \in K \mid v(x) \geq \left(\frac{p-1}{p}\right) v\left(\frac{1}{f}\right)\}$. For $h \in \left(\frac{1}{f}\right)$, $hf \in A$ and hence, $(hf)\ \mathrm{dlog}\, f$ is indeed an element of ω^1_A.
We show some explicit computations of rsw in example 3.1.

1.2.2 Mixed Characteristic Case: Kummer Extensions

Let K be as in 1.1, except possibly with imperfect residue field k. We assume that $\mathrm{char}\, K = 0$ and that K contains a primitive p^{th} root ζ of 1. Let $L = K(\alpha)$ be the (non-trivial) Kummer extension defined by $\alpha^p = h$ for some $h \in K^\times$. For any $a \in K^\times$, h and ha^p give rise to the same extension L. Let $\mathfrak{A} = \{h \in K \mid \text{the solutions of the equation } \alpha^p = h \text{ generate } L \text{ over } K\}$. The Galois group $\mathrm{Gal}(L|K) = G$ is cyclic of order p, generated by $\sigma : \alpha \mapsto \zeta\alpha$. Let $\mathfrak{z} := \zeta - 1$.

Definition 1.7 (Best h). We define the Swan conductor of this extension by

$$\mathrm{Sw}(L|K) := \min_{h \in \mathfrak{A}} v\left(\frac{\mathfrak{z}^p}{h-1}\right) \tag{1.6}$$

This definition coincides with the classical definition of $\mathrm{Sw}(L|K)$ when k is perfect. Any element h of $\mathfrak{A}$ that achieves this minimum value is called *best* h. It is well-defined upto multiplication by $a^p; a \in K^\times$. Refined Swan conductor for such extensions is defined in a similar fashion as in the Artin-schreier case (see [Thatte18]).

2 Generalization to Arbitrary Valuations

Notation 2.1. Let K be an arbitrarily valued field with henselian valuation ring A, valuation v and residue field k of characteristic p. Let $L|K$ be a non-trivial extension of degree p with Galois group G, Let σ generate G. Let B be the integral closure of A in L, w the unique valuation on L that extends v and let l denote the residue field of L. The value group $\Gamma := v(K^\times)$ of K need not be isomorphic to $\mathbb{Z}$. The residue field k is possibly imperfect.

Definition 2.2 (Defect). Let $L|K$ be as above. The ramification index and the inertia degree of $L|K$ are denoted by $e_{L|K}$ and $f_{L|K}$, respectively. Then there is a positive integer $d_{L|K}$, called the *defect* of the extension, such that $[L : K] = d_{L|K}e_{L|K}f_{L|K}$. The extension $L|K$ is *defectless* if $d_{L|K} = 1$. For a more general discussion on defect, see [Kuhlmann].

2.1 Equal Characteristic

2.1.1 Main Results

Let $L|K$ be a non-trivial Artin-Schreier extension, where char $K = p$.

Notation 2.3. Let $\mathfrak{A} = \{f \in K^\times \mid \text{the solutions of the equation } \alpha^p - \alpha = f \text{ generate } L \text{ over } K\}$. Consider the ideals $\mathcal{J}_\sigma$, $\mathcal{I}_\sigma$ of B and the ideals $\mathcal{H}$, $\mathcal{N}_\sigma$ of A, defined as below:

$$\mathcal{J}_\sigma = \left(\left\{\frac{\sigma(b)}{b} - 1 \mid b \in L^\times\right\}\right), \ \mathcal{I}_\sigma = (\{\sigma(b) - b \mid b \in B\}) \subset B \tag{2.1}$$

$$\mathcal{H} = \left(\left\{\frac{1}{f} \mid f \in \mathfrak{A}\right\}\right), \ \mathcal{N}_\sigma = (N(\mathcal{J}_\sigma)) \subset A \tag{2.2}$$

Remark 2.4. The ideals $\mathcal{J}_\sigma, \mathcal{I}_\sigma, \mathcal{H}$ are analogs of the numbers $j(\sigma), i(\sigma), \mathrm{Sw}$. In particular, in the classical case, the ideal $\mathcal{J}_\sigma$ is a principal ideal generated by an element of valuation $j(\sigma)$ and a similar relation is true for $\mathcal{I}_\sigma$ and $\mathcal{H}$.

It is not apparent from the definition that $\mathcal{H}$ is indeed a subset of A, it is proved by using that K is henselian. The invariants do not depend on the choice of σ, we fix a particular σ purely for convenience. We prove the following results in [Thatte16]. Note that in this paper, we assume that the valuation is of rank 1, but this condition is not necessary and the results are true for valuations of arbitrary rank (see [Thatte18]).

Theorem 2.5 (Thatte)**.** *We have the following equality of ideals of A:*

$$\mathcal{H} = \mathcal{N}_\sigma \tag{2.3}$$

Remark 2.6. Theorem 2.5 generalizes the relation between the classical Swan conductor and the invariant $j(\sigma)$, described in (1.5). The ideal $\mathcal{H}$ is the analog of the Swan conductor.

Theorem 2.7 (Thatte)**.** *We consider the A-module ω^1_A of logarithmic differential 1-forms and the B-module $\omega^1_{B|A}$ of relative logarithmic differential 1-forms. Then*

(i) There exists a unique homomorphism of A-modules $\mathrm{rsw} : \mathcal{H}/\mathcal{H}^2 \to \omega^1_A/(\mathcal{I}_\sigma \cap A)\omega^1_A$ *such that* $\frac{1}{f} \mapsto \mathrm{dlog}\, f$; *for all* $f \in \mathfrak{A}$.

(ii) There is a B- module isomorphism $\varphi_\sigma : \omega^1_{B|A}/\mathcal{J}_\sigma\omega^1_{B|A} \xrightarrow{\cong} \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$ such that for all $x \in L^\times$, $\operatorname{dlog} x \mapsto \dfrac{\sigma(x)}{x} - 1.$

(iii) Furthermore, these maps induce the following commutative diagram:

$$\begin{array}{ccc}
\omega^1_{B|A}/\mathcal{J}_\sigma\omega^1_{B|A} & \xrightarrow[\cong]{\varphi_\sigma} & \mathcal{J}_\sigma/\mathcal{J}_\sigma^2 \\
\Big\downarrow{\scriptstyle \overline{\Delta_N}} & & \Big\downarrow{\scriptstyle \overline{N}} \\
\omega^1_A/(\mathcal{I}_\sigma \cap A)\omega^1_A & \xleftarrow[\operatorname{rsw}]{} & \mathcal{H}/\mathcal{H}^2
\end{array}$$

The map $\overline{N}$ is induced by the norm map N and the map $\overline{\Delta_N}$ is given by $b \operatorname{dlog} x \mapsto N(b) \operatorname{dlog} N(x)$, for all $b \in B, x \in L^\times$.

Remark 2.8. The map rsw in (i) is a further refinement and generalization of Kato's refined Swan conductor. Naïvely speaking, the differential df is replaced by a map that is "multiplication by df".

2.1.2 Key Ideas in the Proofs

In the classical case, there exists $x \in B$ which generates B as an A-algebra. This x can be used to describe the main invariants described in 1.1. However, such element does not exist in the general case.
First we prove Theorem 2.5. Then we prove the following result, which allows us to describe various ideals and modules using a single element, in the absence of defect.

Lemma 2.9. *The following statements are equivalent:*

(a) $\mathcal{J}_\sigma$ is principal.

(b) $L|K$ is defectless.

(c) Best f exists.

(d) $\mathcal{H}$ is principal.

Consequently, in the defectless case, $L|K$ is generated by $\alpha^p - \alpha = f$, where f is best. Then $\mathcal{J}_\sigma = \left(\frac{1}{\alpha}\right), \mathcal{H} = \left(\frac{1}{f}\right)$ are principal ideals of B and A, respectively. The B-module $\omega^1_{B|A}$ is generated by the single element $\operatorname{dlog} \alpha$.

The case with defect is more difficult to deal with, since the objects involved are not singly generated. However, we are able to write B as a "filtered union" of $A[x_\alpha]$ over A, where the elements x_α are chosen very carefully for each α that generates $L|K$ as an Artin-Schreier extension. Now it is enough to consider the commutative diagram for each such α and take appropriate limits. Precise statements of these results are available in [Thatte16] and [Thatte18]

2.2 Mixed Characteristic Case

Let $L|K$ be as in 2.1, $\operatorname{char} K = 0$. First we consider Kummer extensions $L|K$, where K contains a primitive p^{th} root ζ of unity. The general case is then reduced to this case, by using tame extensions and Galois invariance. These results are discussed in [Thatte18].

2.2.1 Kummer Extensions

We assume that K contains a primitive p^{th} root ζ of 1, let $\mathfrak{z} := \zeta - 1$. Consider a (non-trivial) Kummer extension $L = K(\alpha)$ defined by $\alpha^p = h$ for some $h \in K^\times$. Let $\mathfrak{A} = \{h \in K \mid$ the solutions of the equation $\alpha^p = h$ generate L over $K\}$. We define the ideal $\mathcal{H}$ by

$$\mathcal{H} = \left(\left\{\frac{\mathfrak{z}^p}{h-1} \mid h \in \mathfrak{A}\right\}\right) \subset A \tag{2.4}$$

Analogs of Theorem 2.5, Theorem 2.7, Lemma 2.9 in this case are true. The refined Swan conductor rsw is the unique A-module homomorphism $\mathrm{rsw} : \mathcal{H}/\mathcal{H}^2 \to \omega_A^1/(\mathcal{I}_\sigma \cap A)\omega_A^1$ such that for all $h \in \mathfrak{A}$, $\dfrac{\mathfrak{z}^p}{h-1} \mapsto \dfrac{1}{h-1}\,\mathrm{dlog}\, h$.

2.2.2 Non-Kummer Case

Notation 2.10. Let K' be a valued field of characteristic 0 with henselian valuation ring A', valuation v' and residue field k' of characteristic $p > 0$. Consider a non-trivial Galois extension $L'|K'$ of degree p, with Galois group $G' := \mathrm{Gal}(L'|K')$. Let w', B', l' denote the valuation, valuation ring and the residue field of L'. We consider the fields $K := K'(\zeta)$, $L := L'(\zeta)$ and the Kummer extension $L|K$ described by $\alpha^p = h$ for some $h \in K$. We will use the notation of 2.2.1 for the extension $L|K$. The Galois group G is cyclic of order p, generated by $\sigma : \alpha \mapsto \zeta\alpha$. Let $\Lambda := \mathrm{Gal}(K|K') \cong \mathrm{Gal}(L|L')$, the order of Λ is coprime to p.

2.3 Invariants for $L'|K'$

First we define the corresponding invariants for the extension $L'|K'$ as follows.

$$\mathcal{I}' := (\{\sigma(b) - b \mid b \in B'\}) \subset B' \tag{2.5}$$

$$\mathcal{J}' := \left(\left\{\frac{\sigma(b)}{b} - 1 \mid b \in L'^\times\right\}\right) \subset B' \tag{2.6}$$

$$\mathcal{N}' := \left(N_{L'|K'}(\mathcal{J}')\right) \subset A' \tag{2.7}$$

$$\mathcal{H}' := (\mathcal{H})^\Lambda \subset A' \tag{2.8}$$

Proposition 2.11. *The invariants for $L|K$ and the invariants for $L'|K'$ are related as follows.*

1. $\mathcal{J}_\sigma = \mathcal{J}'B$, $(\mathcal{J}_\sigma)^\Lambda = \mathcal{J}'$.
2. $\mathcal{N}_\sigma = \mathcal{N}'A$, $(\mathcal{N}_\sigma)^\Lambda = \mathcal{N}'$.
3. $(\mathcal{I}_\sigma)^\Lambda = \mathcal{I}'$

Observe that $L'|K'$ and $L|K$ have the same defect. We have the analogs of the main results using Proposition 2.11 and Λ-invariance. The map rsw' is the restriction of the map rsw to $\mathcal{H}'/\mathcal{H}^2$. It is worth noting that the statement $\mathcal{I}_\sigma = \mathcal{I}'A$ is not always true.

3 Some Examples of Artin-Schreier Extensions

Example 3.1. CDVR Example (Type (iii) in Lemma 1.3)

Consider $K = k((X))$ and $L|K$ given by $\alpha^p - \alpha = u/X^{mp}$; where $m \in \mathbb{N}$ and u is a unit whose residue class is not a p^{th} power in k. Then u/X^{mp} is our best f. Since $X^m\alpha$ generates B as an A-algebra, we have the following:

$$e_{L|K} = 1, f_{L|K} = p$$

$$j(\sigma) = w(\sigma(X^m\alpha)/X^m\alpha - 1) = w(\sigma(\alpha)/\alpha - 1) = w(1/\alpha) = m$$

$$\mathrm{Sw} = v(1/f) = v(X^{mp}/u) = mp = mp/e_{L|K}$$

$$\mathcal{J}_\sigma = (1/\alpha)B \text{ and } \mathcal{H} = (1/f)A = (N(1/\alpha))A$$

The refined Swan conductor is given by

$$df = du/X^{mp}$$

We can see that rsw captures $\mathrm{Sw} = mp$ here, along with information about u.

Example 3.2. Higher Rank Valuation

$K = k((X))((Y))$ can be given the lexicographic ordering where $v(X) = (1, 0)$ and $v(Y) = (0, 1)$. In this case, the valuation has rank 2.

Example 3.3. Defect and Best f

$$K = \cup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{F}_p((X))(X^{1/p^r})$$

Let $L|K$ be the (non-trivial) extension given by

$$\alpha^p - \alpha = 1/X$$

Observe (details below) that there is no best f and the extension has non-trivial defect in this case. We can replace $f_1 = 1/X$ with $f_2 = 1/X + (-1/X^{1/p})^p - (-1/X^{1/p}) = 1/X^{1/p}$. This does not change the extension, but we obtain a "better" f. We can continue this process to get $\inf_i\{-v(f_i)\} = \inf_i\{1/p^i\} = 0$. But there is no f that achieves this $\inf$.

This example is a rather simple one in that our results are "trivially true" in this case, since this extension is almost unramified. In particular, all the objects in the commutative diagram are 0. A non-trivial example is available in the last section (Appendix) of [Thatte16]. We constructed this example using successive blow-ups. The $\inf$ in this case is non-zero, and we can see the complications caused by the presence of defect rather clearly.

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