Motivation

Faltings' Theorem

Let X be a nice (smooth, proper, geometrically integral) curve of genus $g \geq 2$ over a number field K.

$$\# X(K) < \infty.$$

Theorem (Siegel, Mahler, Lang)

Let S be a finite set of places of K.

Let $\mathcal{O}_{K,S}$ be the ring of S-integers of K.

Then, $\# \{ (x,y) \in (\mathcal{O}_{K,S}^\times)^2 : x+y=1 \} < \infty$

More geometrically,

$$\# \left(\mathbb{P}^1_{\mathcal{O}_{K,S}} \setminus \{0,1,\infty\} \right)(\mathcal{O}_{K,S}) < \infty.$$

$\boxed{\text{Best general upper bounds due to Evertse} \sim 7^{d+2\cdot S_\infty + 2S_f}}$

- Proofs of Faltings' Theorem are ineffective.
  - $\rightsquigarrow$ don't help to <u>compute</u> $X(K)$
  - $\rightsquigarrow$ can get upper bounds, but they are very large.

- Siegel's Theorem is effective (Győry 1974) via Baker's theory of linear forms in logarithms.
- Good testing ground for new approaches to Faltings' thm.

"Want something better (effective)

Chabauty's Method

"Suppose $X(K)$ nonempty"

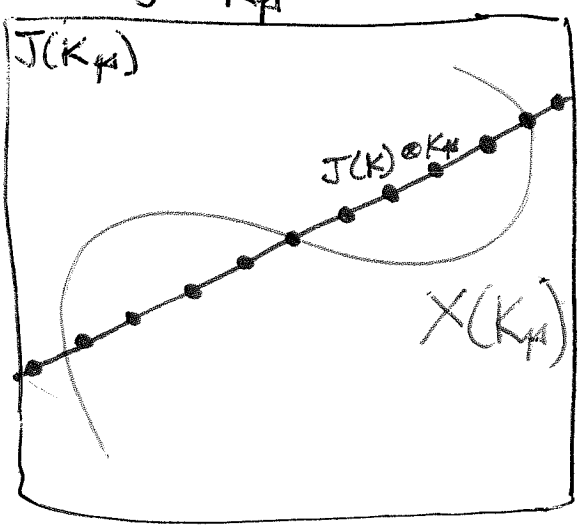
Fix $P \in X(K)$, set $J := \mathrm{Jac}(X)$, $d = [K:\mathbb{Q}]$

Fix a place $\mathfrak{p}$ of K (of good reduction for X).

"Locally $J(K_\mathfrak{p}) \xrightarrow{\cong} \mathrm{Lie}\, J(K_\mathfrak{p}) \cong K_\mathfrak{p}^g$"

via log map

locally $\cong K_\mathfrak{p}^g$



$$X(K) \subseteq \overline{J(K)} \cap X(K_\mathfrak{p})$$
$$\cap\!\!\mid$$
$$(J(K)\otimes K_\mathfrak{p}) \cap X(K_\mathfrak{p}) \;*$$

inside $J(K_\mathfrak{p})$.

mension counts	$X(K_\mathfrak{p})$	$J(K)\otimes K_\mathfrak{p}$	$J(K_\mathfrak{p})$
$K_\mathfrak{p}$ "dim"	1	$\leq \mathrm{rank}\, J(K)$	g

If $\mathrm{rank}\, J(K) \geq g-1$, say C is bad for classical Chabauty (expect $*$ is infinite).

"If C is not bad for classical Chabauty, might expect that $*$ is finite."

Theorem (Coleman)

Let $X/\mathbb{Q}$ be a nice genus g curve w/ good reduction at $p > 2g$. Suppose rank $J(\mathbb{Q}) \leq g-1$ (X is not bad for classical Chabauty)

Then, one can compute an explicit subset

$$\Sigma \subset X(\mathbb{Q}_p) \quad \text{s.t.}$$

$$X(\mathbb{Q}) \subseteq \Sigma \quad \text{and} \quad \#\Sigma \leq \#X(\mathbb{F}_p) + (2g-2).$$

"Coleman covered the number field case as well."

"Can improve bounds when rank $J(\mathbb{Q})$ is smaller."

- Stöll
- uniform bounds of Katz-Rabinoff-Zureck-Brown.

"Requirement that rank $J(K) \leq g-1$ is restrictive!"

How to relax rank restriction

1) a) Descent – Replace X w/ finite set of covering curves.

$$\cancel{\text{E.g. } (\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathcal{O}_{K,S}) = \bigcup_{\alpha \in \mathcal{O}_{K,S}^\times / \mathcal{O}_{K,S}^{\times d}} (\mathbb{P}^1 \setminus \{0,\infty, x : x^d = \alpha\})(\mathcal{O}_{K,S})}$$

e.g. For $\alpha \in \mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^d$

$$f_\alpha : C_\alpha := \mathbb{P}^1 - \{0, \infty, x : \alpha x^d = 1\} \longrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$$

$$x \longmapsto \alpha x^d$$

$$(\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathcal{O}_{K,S}) = \bigcup_\alpha f_\alpha(C_\alpha(\mathcal{O}_{K,S})).$$

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b) Nonabelian Chabauty – Kim

Replace $\overline{J(K)}$ and $J(K_\mu)$ with p-adic Selmer varieties constructed from $\pi_1^{et}(X)$.

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its Kim 2004 – new proof that $\#(\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathcal{O}_{\mathbb{Q},S}) < \infty$.

Balakrishnan-Dogra 2016 – effective bounds for hyperelliptic curves w/ rank $J(\mathbb{Q}) \leq g + NS(J) - 2$.

B.-D.-Muller-Tuitman-Vonk 2017 – implementation of quadratic Chabauty + more improvements "$X_{ns}(13)$"

Dogra 19 many results / number field.

§ Chabauty for Restriction of Scalars.

<u>Idea</u>: (Wetherell '00, and Siksek '15)

Replace X & J w/ $\mathrm{Res}_{K/\mathbb{Q}} X$ and $\mathrm{Res}_{K/\mathbb{Q}} J$.

$$X(K) = (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}) \subseteq \underbrace{(\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p)}_{\text{inside } (\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q}_p)} \cap \overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})} \quad *$$

	$(\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p)$	$\overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})}$	$(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q}_p)$
dim	d	$\leq \operatorname{rank} J(K)$	$d \cdot g$

Might expect $*$ finite if $\operatorname{rank} J(K) \leq d \cdot (g-1)$.

<u>Def</u> X is bad for RoS Chabauty if any of the following hold:

(1) $\operatorname{rank} J(K) > d \cdot (g-1)$

Leave up & fill in as you go.

(2) $\exists\, k \subset K$ subfield and Y/k bad for RoS Chabauty s.t. $Y_K \cong X$. ← Verbal Faltings's sketch.

(3) $\exists\, X \to Y/K$ s.t. Y is bad for R.o.S. Chabauty and

$$\left(\operatorname{rank} J_X(K) - \operatorname{rank} J_Y(K)\right) \geq d \cdot (g_X - g_Y). \quad **$$

Base Change

* Failed sketch of Faltings' Thm. Given X/K, $J = \mathrm{Jac}(X)$
Find L/K s.t. rank $J(L)$ is not too big.
Use R o S. Chabauty to bound $X(L) \supseteq X(K)$.

If $Y_K \cong X$

$$\begin{array}{ccccc} Y(k) & \subseteq & (\mathrm{Res}_{k/\mathbb{Q}}\, Y)(\mathbb{Q}_p) & \cap & \overline{(\mathrm{Res}_{k/\mathbb{Q}}\, J_Y)(\mathbb{Q})} \\ \cap\!\!\mid & & & \cap\!\!\mid & \\ X(K) & \subseteq & (\mathrm{Res}_{K/\mathbb{Q}}\, X)(\mathbb{Q}_p) & \cap & \overline{(\mathrm{Res}_{K/\mathbb{Q}}\, J_X)(\mathbb{Q})} \end{array}$$

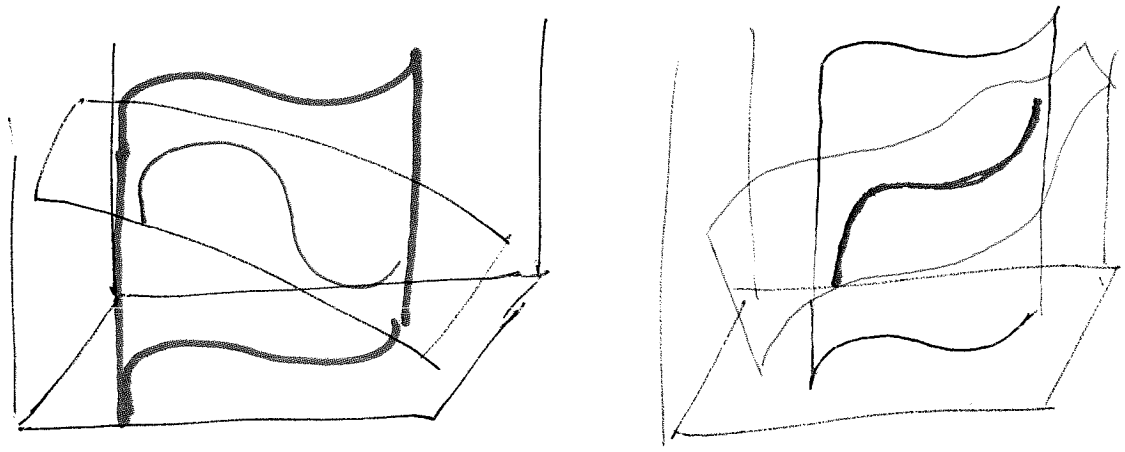
<u>Full Prym</u> If $\exists\, X \longrightarrow Y$, then $J_X \underset{\text{isogenous}}{\sim} J_Y \times \underset{\mathrm{Prym}(X/Y)}{P}$

** is <u>necessary</u> for $\overline{(\mathrm{Res}_{K/\mathbb{Q}}\, P)(\mathbb{Q})} = (\mathrm{Res}_{K/\mathbb{Q}}\, P)(\mathbb{Q}_p)$.

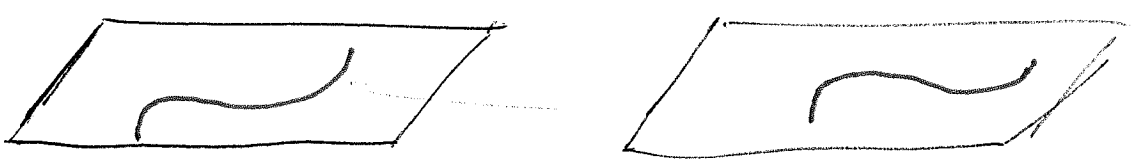
$\mathrm{Res}\, J_X$

$\sim$

$\mathrm{Res}\, J_Y \times \mathrm{Res}\, P$



$\mathrm{Res}\, J_Y$



"Would really like to prove a theorem like Coleman's, but seems tied to hard questions about unlikely intersections inside of abelian varieties."

<u>Theorem</u> (T.)

Suppose that K does not contain a CM subfield.

Then, for ~~$1 \neq \alpha \in \mathcal{O}_{K,S}^{\times}/\mathcal{O}_{K,S}^{\times q}$~~,

q sufficiently large and $1 \neq \alpha \in \mathcal{O}_{K,S}^{\times}/\mathcal{O}_{K,S}^{\times q}$,

$\mathbb{P}^1_{\mathcal{O}_{K,S}} \setminus \{x : \alpha x^q = 1\}$ is <u>not</u> bad for R.o.S. Chabauty.

<u>"Corollary"</u> R.o.S. Chabauty + descent by genus 0 covers is likely to be enough to prove $\#(\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathcal{O}_{K,S}) < \infty$.

<u>Theorem</u> (T.)

Suppose 3 splits completely in K and $[K:\mathbb{Q}]$ is prime to 3.

Then $(\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathcal{O}_K) = \phi$.