Motivation

Faltings' Theorem

Let $X$ be a nice (smooth, proper, geometrically integral) curve of genus $g \geq 2$ over a number field $K.

$\# X(K) < \infty$.

Theorem (Siegel, Mahler, Lang)

Let $S$ be a finite set of places of $K$.
Let $\mathcal{O}_K$, $s$ be the ring of $S$-integers of $K$.

Then, $\# \xi (x, y) \in (\mathcal{O}_K, s)^2 : x + y = 1^s < \infty$

More geometrically,

$\# \left( \mathbb{P}^1_{\mathcal{O}_K, s} \setminus \{0, 1, \infty^s \} \right) (\mathcal{O}_K, s) < \infty$.

Best general upper bounds due to Evertse $\sim 7^{d+2} 8^n 2^s$.

Proofs of Faltings' Theorem are ineffective.

- don't help to compute $X(K)$
- can get upper bounds, but they are very large.

- Siegel's Theorem is effective (Györy 1974) via Baker's theory of linear forms in logarithms.
- Good testing ground for new approaches to Faltings' thm.
"Want something better (effective)"

Chabauty's Method

"Suppose $X(K)$ nonempty."

Fix $P \in X(K)$, set $J = \text{Jac}(X)$, $d = [k : \mathbb{Q}]$.

Fix a place $p$ of $k$ (of good reduction for $X$).

"Locally, $\text{Lie } J(K_p) \cong K_p^g$" via log map.

locally $\cong K_p^g$

$\text{dim } X(K) \leq \frac{\text{dim } J(K)}{\text{rank } J(k)} \leq \text{rank } J(k)$

If $\text{rank } J(k) \geq g - 1$, say $C$ is bad for Chabauty.

If $C$ is not bad for Chabauty, might expect that $*$ is finite.
Theorem (Coleman)

Let $X/Q$ be a nice genus $g$ curve w/ good reduction at $p > 2g$. Suppose rank $J(C) = g-1$ (X is not bad for classical Chabauty).

Then, one can compute an explicit subset

$$\Sigma \subseteq X(Q_p) \quad \text{s.t.}$$

$$X(Q) \subseteq \Sigma \quad \text{and} \quad \# \Sigma \leq \# X(F_p) + (2g-2).$$

"Coleman covered the number field case as well."

"Can improve bounds when rank $J(C)$ is smaller."

- Stöll
- Uniform bounds of Katz-Rabinoff-Zureick-Brown.
Requirement that \( \text{rank } J(K) \leq g-1 \) is restrictive.

How to relax rank restriction

1) a) Descent - Replace \( X \) w/ finite set of covering curves.

For \( \alpha \in \mathbb{C}_{K,S} / (\mathbb{C}_{K,S})^d \)

\[
f_{\alpha}: C_{\alpha} \cong \mathbb{P}^{1} \setminus \{0, \infty, x: \alpha x^3 = 1\} \quad \rightarrow \quad \mathbb{P}^{1} \setminus \{0, 1, \infty\}
\]

\[
x \rightarrow \alpha x^3
\]

\[
(\mathbb{P}^{1} \setminus \{0, 1, \infty\})(\mathbb{C}_{K,S}) = \bigcup_{\alpha} f_{\alpha}(C_{\alpha}(\mathbb{C}_{K,S})).
\]

b) Non-abelian Chabauty - Kim

Replace \( J(K) \) and \( J(K_\alpha) \) with \( p \)-adic Selmer varieties constructed from \( \pi^1_t(X) \).

Kim 2004 - new proof that \( \# C_{\alpha, \mathbb{Q}}(\mathbb{Q}) < \infty \).

Balakrishnan-Dogra 2016 - effective bounds for hyperelliptic curves w/ rank \( J(K) \leq g + \text{NS}(K) - 2 \).

B. D. Mueller - Truitman - Vank 2017 - implementation of quadratic Chabauty + more improvements \( \mathbb{X}_{ns}(13) \).

Many more results in number field.
Chabauty for Restriction of Scalars

Idea: (Wetherell '00, and Siksek '15)

Replace $X$ & $J$ by $\text{Res}_{k/\mathbb{Q}} X$ and $\text{Res}_{k/\mathbb{Q}} J$.

$X(K) = (\text{Res}_{k/\mathbb{Q}} X)(\mathbb{Q}) \subseteq (\text{Res}_{k/\mathbb{Q}} X)(\mathbb{Q}_p) \cap (\text{Res}_{k/\mathbb{Q}} J)(\mathbb{Q}_p)$

inside $(\text{Res}_{k/\mathbb{Q}} J)(\mathbb{Q}_p)$


\[
\begin{array}{c|c|c}
\text{dim} & d & d \cdot g \\
\hline
(\text{Res}_{k/\mathbb{Q}} X)(\mathbb{Q}_p) & (\text{Res}_{k/\mathbb{Q}} J)(\mathbb{Q}_p) & (\text{Res}_{k/\mathbb{Q}} J)(\mathbb{Q}_p)
\end{array}
\]

** Might expect finite if $\text{rank} J(K) \leq d \cdot (g-1)$.**

Def $X$ is bad for RoS Chabauty if any of the following hold:

1. $\text{rank} J(K) > d \cdot (g-1)$
2. \exists $k \subset K$ subfield and $Y/k$ bad for RoS Chabauty s.t. $Y_k \cong X$. \hspace{1cm} \text{Leave up & fill in as you go.}
3. $\exists X \rightarrow Y/k$ s.t. $Y$ is bad for RoS Chabauty and $(\text{rank } J_x(K) - \text{rank } J_y(K)) \geq d \cdot (g_x - g_y)$. **
Base Change

Failed sketch of Faltings' Thm. Given $X/K$, $J = J_c(x)$

- Find $L/k$ s.t. rank $J(L)$ is not too big.
- Use RoS. Chabauty to bound $X(L) \supseteq X(K)$.

If $Y_k \sim X$

$$Y(k) \subseteq (\text{Res}_{k/Q} Y)(Q_p) \cap (\text{Res}_{k/Q} J_y)(Q)$$

$$X(k) \subseteq (\text{Res}_{k/Q} X)(Q_p) \cap (\text{Res}_{k/Q} J_y)(Q)$$

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If $\exists X \rightarrow Y$, then $J_X \sim J_y \times P$

** is necessary for $(\text{Res}_{k/Q} P)(Q) = (\text{Res}_{k/Q} P)(Q_p)$

$\text{Res } J_x 
\sim 
\text{Res } J_y \times \text{Res } P$

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\[ \text{Diagram} \]
"Would really like to prove a theorem like Coleman's, but seems tied to hard questions about unlikely intersections inside of abelian varieties."

**Theorem (T.)**

Suppose that $K$ does not contain a CM subfield.

Then, for $\mathcal{M} \neq 0 \in \mathcal{O}_{K,s} / \mathcal{O}_{K,s}^x$, a sufficiently large and $1 \neq \alpha \in \mathcal{O}_{K,s}^x / \mathcal{O}_{K,s}^x$,

$$(P', \exists x : \alpha x^q = 1)$$

is not bad for R.o.S. Chabauty.

"Corollary" R.o.S. Chabauty + descent by genus 0 covers is likely to be enough to prove $(P' \setminus \{0, 1, \infty\}) (\mathcal{O}_{K,s}) < \infty$.

**Theorem (T.)**

Suppose 3 splits completely in $K$ and $[K:Q]$ is prime to 3.

Then $(P' \setminus \{0, 1, \infty\})(\mathcal{O}_K) = \emptyset$. 