RIGID HYODO–KATO THEORY AND p-ADIC POLYLOGARITHMS
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This talk consists of a rigid analytic reconstruction of Hyodo–Kato theory and its application to the explicit computation of p-adic polylogarithms of Tate curves. The former is a joint work with Veronika Ertl of Regensburg University, and the preprint [EY] is available at https://arxiv.org/abs/1907.10964.

1. Introduction

Let V be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field k. Let $\mathfrak{m}$ be the maximal ideal of V. Let K and F be the fraction fields of V and the ring of Witt vectors $W=W(k)$ of k, respectively.

Original construction. Let X be a proper semistable scheme over V with the log structure induced from the special fiber. (More generally we may consider a fine proper log scheme which is log smooth and of Cartier type in a certain sense.)

The crystalline Hyodo–Kato cohomology $R\Gamma^{\mathrm{cris}}_{\mathrm{HK}}(X)$ and the crystalline Hyodo–Kato map

$$\Psi^{\mathrm{cris}}_{\pi}\colon R\Gamma^{\mathrm{cris}}_{\mathrm{HK}}(X)\to R\Gamma_{\mathrm{dR}}(X_K)$$

were constructed by Hyodo and Kato [HK], depending on the choice of a uniformizer $\pi\in V$.

They play important roles in number theory, for example Bloch–Kato's Tamagawa number conjecture, via the semistable conjecture which is now a theorem ([Ts], [Fa], [Ni], [Bei]). More precisely,

- the rational cohomology groups $H^{\mathrm{cris},i}_{\mathrm{HK}}(X)_{\mathbb{Q}}$ are finite dimensional F-vector space endowed with a σ-semilinear automorphism φ and an F-linear endomorphism N satisfying $N\varphi=p\varphi N$.
- The de Rham cohomology groups $H^i_{\mathrm{dR}}(X_K)$ are finite dimensional K-vector spaces endowed with the Hodge filtration $F^\bullet$.
- The map $\Psi^{\mathrm{cris}}_{\pi}\otimes 1\colon H^{\mathrm{cris},i}_{\mathrm{HK}}(X)_{\mathbb{Q}}\otimes_F K\to H^i_{\mathrm{dR}}(X_K)$ is an isomorphism.

The semistable conjecture states that φ, N, and $F^\bullet$ together restore the Galois action on the étale cohomology groups $H^i_{\mathrm{et}}(X_{\overline{K}},\mathbb{Q}_p)$.

Note that, $R\Gamma^{\mathrm{cris}}_{\mathrm{HK}}(X)$ was defined to be the log crystalline cohomology over W^0 (W with the hollow log structure) of the reduction of X, and $\Psi^{\mathrm{cris}}_{\pi}$ was defined as a zig-zag of technical quasi-isomorphisms.

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Other known constructions. There are at least two other constructions of Hyodo–Kato cohomology and Hyodo–Kato map.

- Beilinson [Bei] represented the crystalline Hyodo–Kato cohomology in another way, and a Hyodo–Kato map which is independent of the choice of π. His theory is very sophisticated, but based on highly abstract considerations.
- Große-Klönne [GK2] proposed a rigid analytic version of the Hyodo–Kato map. Since he used log rigid cohomology instead of log crystalline cohomology, his Hyodo–Kato map was defined in terms of p-adic differential forms on certain dagger spaces, and we may remove the assumption for properness. However it depends on the choice of a uniformizer of V and passes through several zig-zags of quasi-isomorphisms whose intermediate objects are quite complicated. For example, even if X is affine, we have to pass through the cohomology of simplicial log schemes with boundary which are no longer affine.

Motivation and advantages of the new construction. Our motivation is to give a new construction of Hyodo–Kato cohomology and Hyodo–Kato map which are intuitive and computable. For this, we develop Kim–Hain's construction [KH] using log rigid cohomology. Precisely, we construct rigid Hyodo–Kato cohomology $R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X})$ and rigid Hyodo–Kato map

$$\Psi^{\mathrm{rig}}_{\pi,q}\colon R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X}) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})$$

for a strictly semistable weak formal log scheme $\mathcal{X}$, depending on the choice of uniformizer $\pi \in V$ and a branch of p-adic logarithm $\log_q$ defined from $q \in \mathfrak{m} \setminus 0$. These are defined in a very simple way, and in fact independent of π. Then we have the following advantages.

- Our construction gives a natural and intuitive interpretation of Hyodo–Kato theory. In particular, by fixing a branch of p-adic logarithm we obtain rigid Hyodo–Kato map which is compatible with extension of base field.
- Moreover our construction is more consistent with the theory of p-adic Galois representations. Namely, the functor D_{st} associating filtered (φ, N)-modules to p-adic Galois representations also depends only on a branch of p-adic logarithm, as well as our rigid Hyodo–Kato map.
- For given $\mathcal{X}$, one can directly compute its rigid Hyodo–Kato cohomology and rigid Hyodo–Kato map in terms of explicit Čech cocycles.
- Our constrcution is probably useful to make variants. (e.g. cohomology with coefficients in log overconvergent F-isocrystals, cohomoloy with compact support, etc.)

2. Construction

Weak formal schemes. One of technical difficulties of (log) rigid cohomology is that we have to consider overconvergent differential forms. In the original construction of Berthelot, he needed to take a compactification to define the notion of overconvergence. In order to simplify arguments, we use weak formal schemes and dagger spaces, which play the same roles as formal schemes and rigid analytic spaces, but whose structure sheaves are already "overconvergent". Refer to [Me], [GK1], and [LM] for them.

Note that all weak formal schemes in [Me] are adic over the base ring. See [EY] for the definition and properties of weak formal schemes which are not necessary adic over the base.

Log structure. The notion of log schemes was introduced by Kato [Ka], as a good framework including semistable schemes. Working in the category of log schemes, one can naturally extend nice cohomology theories to semistable schemes and open schemes.

To define log rigid cohomology, we extend it to weak formal schemes in the obvious way. Namely, a weak formal log scheme is a weak formal scheme $\mathcal{Z}$ with a sheaf of monoids $\mathcal{N}$ on $\mathcal{Z}$ and a morphism $\alpha\colon \mathcal{N} \to \mathcal{O}_{\mathcal{Z}}$ where we regard $\mathcal{O}_{\mathcal{Z}}$ as a sheaf of monoid by the multiplication, such that the morphism $\alpha^{-1}(\mathcal{O}_{\mathcal{Z}}^{\times}) \to \mathcal{O}_{\mathcal{Z}}^{\times}$ induced by α is an isomorphism.

Note that, if a monoid P and a homomorphism $\alpha_0\colon P_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}}$, where $P_{\mathcal{Z}}$ is the constant sheaf defined by P, are given, one can associate a log structure by setting $\mathcal{N} := P_{\mathcal{Z}} \oplus_{\alpha_0^{-1}(\mathcal{O}_{\mathcal{Z}}^{\times})} \mathcal{O}_{\mathcal{Z}}$. In this note we denote by $(\mathcal{Z}, P \to \mathcal{O}_{\mathcal{Z}})$ the induced weak formal log scheme.

Now we consider the following weak formal log schemes as base:

$$\begin{aligned}
\mathcal{S} &:= (\operatorname{Spwf} W[\![s]\!], \mathbb{N} \to W[\![s]\!]; 1 \mapsto s),\\
W^0 &:= (\operatorname{Spwf} W, \mathbb{N} \to W; 1 \mapsto 0),\\
k^0 &:= (\operatorname{Spec} k, \mathbb{N} \to k; 1 \mapsto 0),\\
V^\sharp &:= (\operatorname{Spwf} V, \mathbb{N} \to V; 1 \mapsto \pi),
\end{aligned}$$

where π is a uniformizer of V. By identifying all $\mathbb{N}$ in the above definitions, we obtain the following commutative diagram of exact closed immersions

$$\begin{tikzcd}
 & \mathcal{S} & \\
W^0 \arrow[ur, "j_0"'] & & V^\sharp \arrow[ul, "j_\pi"] \\
 & k^0 \arrow[ul, "i_0"] \arrow[ur, "i_\pi"'] &
\end{tikzcd}.$$

Note that the log scheme $V^\sharp$ is independent of the choice of π, but i_π and j_π depend on the choice.

Rigid Hyodo–Kato cohomology. Let π be a uniformizer of V. Let $\mathcal{X}$ be a strictly semistable weak formal log scheme over $V^\sharp$. Namely, Zariski locally on $\mathcal{X}$ there exists a smooth morphism

$$\mathcal{X} \to \operatorname{Spwf} V[x_1, \dots, x_n]^\dagger/(x_1 \cdots x_n - \pi),$$

and the log structure on $\mathcal{X}$ is associated to the map $\mathbb{N}^n \to \mathcal{O}_{\mathcal{X}}; 1_i \mapsto x_i$. Note that this condition is independent of the choice of π.

For simplicity, we assume that $\mathcal{X}$ can be extended to a rigid analytic family over the open unit disk. More precisely, assume that there exists a weak formal log scheme $\mathcal{Z}$

which is log smooth over $\mathcal{S}$, such that $\mathcal{Z} \times_{\mathcal{S},j_\pi} V^\sharp \cong \mathcal{X}$. Further we assume that there is a lift ϕ on $\mathcal{Z}$ of the p-th power Frobenius on $\mathcal{Z}$ modulo p.

Let $\mathfrak{X}$ and $\mathfrak{Z}$ be the dagger spaces associated to $\mathcal{X}$ and $\mathcal{Z}$ respectively. Let $\omega^\bullet_{\mathcal{Z}/W^\varnothing}$ be the log de Rham complex of $\mathcal{Z}$ over $W^\varnothing$ (not $\mathcal{S}$). Let $\omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}$ be the complex on $\mathfrak{Z}$ induced from $\omega^\bullet_{\mathcal{Z}/W^\varnothing}$ by tensoring with $\mathbb{Q}$.

Definition 2.1. We define a complex $\omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}[u]$ on $\mathfrak{Z}$ to be the CDGA generated by $\omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}$ and degree zero elements $u^{[i]}$ for $i \geq 0$ with relations

$$du^{[i+1]} = -d\log s \cdot u^{[i]}, \qquad\qquad u^{[0]} = 1,$$

and the multiplication

$$u^{[i]} \wedge u^{[j]} = \frac{(i+j)!}{i!j!} u^{[i+j]}.$$

We define the Frobenius operator φ on $\omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}[u]$ by the action of ϕ to $\omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}$ and $\varphi(u^{[i]}) := p^i u^{[i]}$. We define the monodromy operator $\mathbf{N}$ to be the $\mathcal{O}_{\mathfrak{Z}}$-linear map with $\mathbf{N}(u^{[i]}) := e^{-1}u^{[i-1]}$, where e is the ramification index of K over $\mathbb{Q}_p$.

We define the rigid Hyodo–Kato cohomology to be

$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X}) := R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}[u]).$$

Remark 2.2.
- The relations defining the differentials and products $u^{[i]}$ suggests that $u^{[i]}$ imitates $\frac{(-\log s)^i}{i!}$.
- Kim and Hain [KH] worked in de Rham-Witt setting. The direct analogue of their complex is on the fiber of $\mathfrak{Z}$ at $s = 0$, and we slightly enhanced it to a complex on $\mathfrak{Z}$. This small difference is important for our simple definition of rigid Hyodo–Kato map.

Then one can prove that $R\Gamma_{\mathrm{HK}}(\mathcal{X})$ is independent of the choice of π and $\mathcal{Z}$, up to canonical isomorphisms. Multiplying by e^{-1} makes the monodromy operator compatible with base change.

Proposition 2.3. *Let $Y := \mathcal{X} \times_{V^\sharp, i_\pi} k^0$ be the special fiber of $\mathcal{X}$. There is a commutative diagram of natural morphisms*

$$\begin{tikzcd} R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) \arrow[r] \arrow[d, "\sim"] & R\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) \arrow[dl, "j_0^*"'] \\ R\Gamma_{\mathrm{rig}}(Y/W^0) & \end{tikzcd}$$

compatible with Frobenius operators, such that the left vertical map is a quasi-isomorphism.

The upper horizontal and left vertical maps in the proposition are defined by the natural surjection between the log de Rham complexes and by sending $u^{[i]} \mapsto 0$ for $i > 0$.

p-adic logarithm. Let $\boldsymbol{\mu}$ be the image of $k^\times$ under the Teichmüller map, that is the set of $|k^\times|^{\text{th}}$ roots of unity in K. Then there is a decomposition $V^\times = \boldsymbol{\mu}(1+\mathfrak{m})$.

Definition 2.4. Let $\log\colon V^\times \to V$ be the p-adic logarithm function defined by

$$\log(v) := -\sum_{n\geq 1} \frac{(1-v)^n}{n} \qquad \text{for } v \in (1+\mathfrak{m}),$$
$$\log(u) := 0 \qquad \text{for } u \in \boldsymbol{\mu}.$$

A *branch of the p-adic logarithm* on K is a group homomorphism from $K^\times$ to (the additive group of) K whose restriction to $V^\times$ coincides with log as above.

For $q \in \mathfrak{m} \setminus \{0\}$, let $\log_q\colon K^\times \to K$ be the unique branch of the p-adic logarithm which satisfies $\log_q(q) = 0$. Note that any branch of p-adic logarithm can be written in the form $\log_q$ for some q.

Rigid Hyodo–Kato map. Let $\mathcal{X}$, $\mathfrak{X}$, $\mathcal{Z}$, and $\mathfrak{Z}$ be as above.

Definition 2.5. We define the rigid Hyodo–Kato map with respect to π and q to be the map

$$\Psi^{\mathrm{rig}}_{\pi,q}\colon R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X}) = R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}}[u]) \to R\Gamma(\mathfrak{X}, \omega^\bullet_{\mathcal{X}/V^\sharp,\mathbb{Q}}) \cong R\Gamma_{\mathrm{dR}}(\mathfrak{X}/K)$$

defined by the natural surjection $j^*_\pi\colon \omega^\bullet_{\mathcal{Z}/W^\varnothing,\mathbb{Q}} \to \omega^\bullet_{\mathcal{X}/V^\sharp,\mathbb{Q}}$ and by sending $u^{[i]} \mapsto \frac{(-\log_q(\pi))^i}{i!}$.

We often write

$$\Psi^{\mathrm{rig}}_{\pi,q,K} := \Psi^{\mathrm{rig}}_{\pi,q} \otimes 1\colon R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X}) \otimes_F K \to R\Gamma_{\mathrm{dR}}(\mathfrak{X}/K).$$

Theorem 2.6. (1) *For uniformizers $\pi, \pi' \in V$ and elements $q, q' \in \mathfrak{m} \setminus \{0\}$, we have*

$$\Psi^{\mathrm{rig}}_{\pi,q,K} = \Psi^{\mathrm{rig}}_{\pi',q',K} \circ \exp\left(-\frac{\log_q(q')}{\mathrm{ord}_p(q')} \cdot \mathbf{N}\right).$$

In particular, $\Psi^{\mathrm{rig}}_{\pi,q}$ is independent of the choice of π.

(2) *For any π and q, the map $\Psi^{\mathrm{rig}}_{\pi,q,K}$ is a quasi-isomorphism.*

(3) *If $\mathcal{X}$ is proper, $\Psi^{\mathrm{rig}}_{\pi,\pi}$ is compatible with the classical crystalline Hyodo–Kato map Ψ^{cris}_π.*

3. p-adic polylogarithms of a Tate curve

A family of Tate curves. As an application, we compute the p-adic elliptic polylogarithms of a Tate curve. For any $n \in \mathbb{Z}$, let $\mathcal{Z}_n := \mathrm{Spwf}\, W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s)$ and endow it with the log structure associated to the map

$$\mathbb{N}^2 \to W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s);\ (1,0) \mapsto v_n, (0,1) \mapsto w_n.$$

If we set $t := s^{n-1} v_n = \frac{s^n}{w_n}$, we may write

$$\mathcal{Z}_n = \mathrm{Spwf}\, W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s) = \mathrm{Spwf}\, W[\![s]\!][\frac{t}{s^{n-1}}, \frac{s^n}{t}]^\dagger.$$

Let

$$\mathcal{V}_n := \operatorname{Spwf} W[\![s]\!][v_n, v_n^{-1}]^\dagger = \operatorname{Spwf} W[\![s]\!][\frac{t}{s^{n-1}}, \frac{s^{n-1}}{t}]^\dagger,$$
$$\mathcal{W}_n := \operatorname{Spwf} W[\![s]\!][w_n, w_n^{-1}]^\dagger = \operatorname{Spwf} W[\![s]\!][\frac{s^n}{t}, \frac{t}{s^n}]^\dagger$$

be open subsets of $\mathcal{Z}_n$.

For $r \geq 1$, we glue $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ via the natural isomorphisms $\mathcal{V}_{n+1} \cong \mathcal{W}_n$ and the isomorphism $\mathcal{V}_1 \cong \mathcal{W}_r$, $v_1 \mapsto w_r^{-1}$. Denote by $\mathcal{Z}^{(r)}$ the resulting weak formal log scheme over $\mathcal{S}$. Let $\phi\colon \mathcal{Z}^{(r)} \to \mathcal{Z}^{(r)}$ be a Frobenius lift defined by the Frobenius action on W and $v_n \mapsto v_n^p$, $w_n \mapsto w_n^p$ on each $\mathcal{Z}_n$.

Fix a uniformizer $\pi \in V$ and $r \geq 2$. Let $\mathcal{X} := \mathcal{Z}^{(r)} \times_{\mathcal{S}, j_\pi} V^\sharp$ be the fiber at $s = \pi$. This is a weak formal model of the Tate curve over K of period π^r. Namely, the dagger space $\mathfrak{X}$ associated to $\mathcal{X}$ is isomorphic to $K^\times/\pi^{r\mathbb{Z}}$. ($v_1$ is identified with the canonical parameter t of $K^\times/\pi^{r\mathbb{Z}}$.) Note that $\mathcal{X}^{(r)}$ is covered by $\mathcal{X}_n := \mathcal{Z}_n \times_{\mathcal{S}, j_\pi} V^\sharp$ for $n = 1, \ldots, r$.

Computation of cohomology. By using the (ordered) Čech complex

$$\check{C}^\bullet_{\mathrm{HK}} = \check{C}^\bullet(\{\mathfrak{Z}_n\}_{n=1}^r, \omega^\bullet_{\mathcal{Z}_n/W^\varnothing, \mathbb{Q}}[u]) \qquad \text{and} \qquad \check{C}^\bullet_{\mathrm{dR}} = \check{C}(\{\mathfrak{X}_n\}_{n=1}^r, \Omega^\bullet_{\mathfrak{X}_n/K}),$$

we may compute the rigid Hyodo–Kato cohomology and the rigid Hyodo–Kato map directly. For example H^1 is computed as follows:

Proposition 3.1. (1) *We have $H^1_{\mathrm{HK}}(\mathcal{X}) \cong F e_1^{\mathrm{HK}} \oplus F e_2^{\mathrm{HK}}$ with $\varphi(e_1^{\mathrm{HK}}) = e_1^{\mathrm{HK}}$, $\mathbf{N}(e_1^{\mathrm{HK}}) = 0$, $\varphi(e_2^{\mathrm{HK}}) = p e_2^{\mathrm{HK}}$, and $\mathbf{N}(e_2^{\mathrm{HK}}) = \frac{r}{e} e_1^{\mathrm{HK}}$, where the classes e_1^{HK} and e_2^{HK} are represented by the cocycles*

$$(0, \ldots, 0, 1) \in \prod_{n=1}^r \Gamma(\mathfrak{W}_n, \mathcal{O}_{\mathfrak{W}_n}) \subset \check{C}^1_{\mathrm{HK}},$$
$$(d\log w_1, \ldots, d\log w_r) + (-u^{[1]}, \ldots, -u^{[1]}, u^{[1]}) \in \prod_{n=1}^r \Gamma(\mathfrak{Z}_n, \omega^1_{\mathcal{Z}_n/W^\varnothing, \mathbb{Q}}) \oplus \prod_{n=1}^r \Gamma(\mathfrak{W}_n, \mathcal{O}_{\mathfrak{W}_n} u^{[1]}) \subset \check{C}^1_{\mathrm{HK}}$$

respectively.

(2) *We have $H^1_{\mathrm{dR}}(\mathcal{X}) \cong K e_1^{\mathrm{dR}} \oplus K e_2^{\mathrm{dR}}$ with the Hodge filtration*

$$F^p H^1_{\mathrm{dR}}(\mathcal{X}) = \begin{cases} K e_1^{\mathrm{dR}} \oplus K e_2^{\mathrm{dR}} & \text{if } p \leq 0, \\ K e_2^{\mathrm{dR}} & \text{if } p = 1, \\ 0 & \text{if } p \geq 2, \end{cases}$$

where the classes e_1^{dR} and e_2^{dR} are represented by the cocycles

$$(0, \ldots, 0, 1) \in \prod_{n=1}^r \Gamma(\mathfrak{X}_n \cap \mathfrak{W}_n, \mathcal{O}_{\mathfrak{X}}) \subset \check{C}^1_{\mathrm{dR}},$$
$$(d\log w_1, \ldots, d\log w_r) \in \prod_{n=1}^r \Gamma(\mathfrak{X}_n, \Omega^1_{\mathfrak{X}}) \subset \check{C}^1_{\mathrm{dR}}$$

respectively.

(3) *For $q \in \mathfrak{m} \setminus \{0\}$, the rigid Hyodo–Kato map $\Psi_q = \Psi_{\pi,q} \colon H^1_{\mathrm{HK}}(\mathcal{X}) \to H^1_{\mathrm{dR}}(\mathcal{X})$ is given by*

$$\Psi_q(e_1^{\mathrm{HK}}) = e_1^{\mathrm{dR}} \qquad \text{and} \qquad \Psi_q(e_2^{\mathrm{HK}}) = e_2^{\mathrm{dR}} - r\log_q(\pi)e_1^{\mathrm{dR}}.$$

Isocrystals. We may extend rigid Hyodo–Kato theory to (unipotent) filtered log overconvergent F-isocrystals. More precisely, let $\mathcal{F}$ be a locally free sheaf of finite rank over $\mathfrak{Z}^{(r)}$ with integrable connection $\nabla \colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{Z}^{(r)}}} \omega^1_{\mathcal{Z}^{(r)}/W^{\varnothing},\mathbb{Q}}$, the Frobenius structure $\Phi \colon \phi^*\mathcal{F} \xrightarrow{\cong} \mathcal{F}$, and the Hodge filtration $F^\bullet$ on $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{Z}^{(r)}}} \mathcal{O}_{\mathfrak{X}}$ with certain conditions. Then we may define a map

$$\Psi^{\mathrm{rig}}_{\pi,q} \colon R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\mathcal{F}) := R\Gamma(\mathfrak{Z}^{(r)}, \mathcal{F}\otimes\mathcal{F}\otimes\omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]) \to R\Gamma(\mathfrak{X}, \mathcal{F}\otimes_{\mathcal{O}_{\mathfrak{Z}^{(r)}}}\omega^\bullet_{\mathcal{X}/V^\sharp,\mathbb{Q}}) \cong R\Gamma_{\mathrm{dR}}(\mathfrak{X}, \mathcal{F}|_{\mathfrak{X}}).$$

Here we use $q = \pi$. We define the syntomic cohomology $R\Gamma_{\mathrm{syn}}(\mathcal{X},\mathcal{F})$ to be the homotopy limit of

$$\begin{array}{rcl} R\Gamma(\mathfrak{Z}^{(r)}, \mathcal{F}\otimes\omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}) \oplus F^0 R\Gamma_{\mathrm{dR}}(\mathfrak{X},\mathcal{F}|_{\mathfrak{X}}) & \to & R\Gamma(\mathfrak{Z}^{(r)}, \mathcal{F}\otimes\omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}) \oplus R\Gamma_{\mathrm{dR}}(\mathfrak{X},\mathcal{F}|_{\mathfrak{X}}) \\ (x,y) & \mapsto & ((1-\varphi)(x), \Psi^{\mathrm{rig}}_{\pi,\pi}(x) - y). \end{array}$$

Then there is a canonical isomorphism $H^1_{\mathrm{syn}}(\mathcal{X},\mathcal{F}) \cong \mathrm{Ext}^1(\mathcal{O},\mathcal{F})$.

Log sheaf. Let H be the filtered (φ,N)-module given by identifying $H^1_{\mathrm{HK}}(\mathcal{X}) \otimes_F K$ and $H^1_{\mathrm{dR}}(\mathfrak{X})$ via $\Psi^{\mathrm{rig}}_{\pi,\pi}$. Let $\mathcal{H}$ be the "constant" F-isocrystal defined by $H^\vee$, where $\vee$ denotes the dual. (Note that $\mathcal{H}$ is not constant as a module with connection because the monodromy operator on $H^\vee$ contribute to the log connection of $\mathcal{H}$.) Then the Leray spectral sequence of syntomic cohomology induces a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(K,H^\vee) \arrow[r] & H^1_{\mathrm{syn}}(\mathcal{X},\mathcal{H}) \arrow[r] & \mathrm{Ext}^0(K, H\otimes H^\vee) \arrow[r] & 0, \\ & & \mathrm{Ext}^1(\mathcal{O},\mathcal{H}) \arrow[u, "\cong"] & (e_1^{\mathrm{HK}}\otimes e_1^{\mathrm{HK},\vee} + e_2^{\mathrm{HK}}\otimes e_2^{\mathrm{HK},\vee})\mathbb{Q}_p \arrow[u, "\cong"] & \end{tikzcd}$$

which splits by a retraction $H^1_{\mathrm{syn}}(\mathcal{X},\mathcal{H}) \to H^1(V^\sharp, \iota^*\mathcal{H}) \cong \mathrm{Ext}^1(K,H^\vee)$ given by the pull-back to origin. Here $\iota \colon \mathcal{S} \hookrightarrow \mathcal{Z}^{(r)}$ is the closed immersion defined by $v_1 = 1$.

Then the log sheaf $\mathcal{L}og$ is defined to be an element $\mathrm{Ext}^1(\mathcal{O},\mathcal{H})$ corresponding to $e_1^{\mathrm{HK}} \otimes e_1^{\mathrm{HK},\vee} + e_2^{\mathrm{HK}} \otimes e_2^{\mathrm{HK},\vee}$ via this splitting. Using the explicit computation of rigid Hyodo–Kato map, one can compute the connection, Frobenius structure, and Hodge filtration of $\mathcal{L}og$.

Polylogarithms. Let $\mathcal{U} := \mathcal{X} \setminus \iota(V^\sharp)$ be the Tate curve minus origin. The elliptic polylogarthm element is defined to be the element of $\mathrm{pol} \in H^1_{\mathrm{syn}}(\mathcal{U}, \mathcal{H}^\vee \otimes \mathcal{L}og(1))$ which maps to 1 by the residue map

$$H^1_{\mathrm{syn}}(\mathcal{U}, \mathcal{H}^\vee \otimes \mathcal{L}og(1)) \xrightarrow{\cong} H^0_{\mathrm{syn}}(V^\sharp, \iota^*(\mathcal{H}^\vee \otimes \mathcal{L}og)) \cong \mathbb{Q}_p.$$

One can describe a representative of pol by an explicit Čech cocycle. I omit the detail, but assert that a function

$$\mathbb{L}_m(s,t) := \sum_{k\in\mathbb{Z}} \mathrm{Li}_m^{(p)}(s^{rk}t)$$

appears. Here $\mathrm{Li}_m^{(p)}(t)$ is the classical p-adic polylogarithm function defined for $|t-1| \le p^{-1/(p-1)}$, whose restriction to $|t| < 1$ is given by

$$\mathrm{Li}_m^{(p)}(t) = \sum_{\substack{n\ge 1\\(p,n)=1}} \frac{t^n}{n^m}.$$

This function should be called a kind of p-adic elliptic polylogarithm function. The case $m = 2$ was treated by Coleman [Co, §IX].

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