KAWASHIMA FUNCTIONS AND MULTIPLE ZETA VALUES

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Introduction

The Kawashima functions provide a type of generalizations ('multiplification') of the digamma function (and polygamma functions), in the same way as the multiple zeta values do of the Riemann zeta values.

$$\begin{tikzcd}
\text{Riemann zeta values} \arrow[r, "\text{multiplify}"] \arrow[d, leftrightarrow, "\text{related (classical)}"] & \text{Multiple zeta values} \arrow[d, leftrightarrow, dotted, "\text{related?}"] \\
\text{Digamma function} \arrow[r, "\text{multiplify}"] & \text{Kawashima functions}
\end{tikzcd}$$

There are classically known relationships between the digamma function and the Riemann zeta values at positive and non-positive integers (see §1 below). In these notes, we consider multiplifications of these relationships, and applications to the study of multiple zeta values. For more detail (except for the content of §5), we refer the reader to [12].

1. Reviews on the digamma function and MZVs

First we define the digamma function $\psi(z)$ as the logarithmic derivative of the gamma function:

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

Many formulas are known for this classical function. Let us note four of them:

$$\psi(N+1) = -\gamma + \sum_{n=1}^{N} \frac{1}{n}, \tag{1.1}$$

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \binom{z}{n} \tag{1.2}$$

$$= -\gamma + \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(1+m) z^m \tag{1.3}$$

$$\sim \log z + \sum_{m=1}^{\infty} (-1)^m \zeta(1-m) z^{-m} \quad \text{as } z \to +\infty. \tag{1.4}$$

Here $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \log N\right)$ denotes the Euler constant, and N in (1.1) is any non-negative integer. We want to obtain multiple versions of these formulas.

Next, we introduce the multiple zeta(-star) values.

2010 *Mathematics Subject Classification.* 11M32, 33B15.

Key words and phrases. Kawashima function; digamma function; polygamma function; multiple zeta value.

Definition 1.1. A tuple $\boldsymbol{k} = (k_1, \dots, k_r)$ of positive integers is called an index. In particular, the index of length 0 (the empty index) is denoted by $\varnothing$. An index $\boldsymbol{k}$ is called admissible if it is empty or its last entry k_r is greater than 1.

For an index $\boldsymbol{k} = (k_1, \dots, k_r)$ and a non-negative integer N, we define

$$s(\boldsymbol{k}; N) \coloneqq \sum_{0<n_1<\cdots<n_r=N} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \qquad S(\boldsymbol{k}; N) \coloneqq \sum_{0<n_1<\cdots<n_r\le N} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$
$$s^\star(\boldsymbol{k}; N) \coloneqq \sum_{0<n_1\le\cdots\le n_r=N} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \qquad S^\star(\boldsymbol{k}; N) \coloneqq \sum_{0<n_1\le\cdots\le n_r\le N} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

For $\boldsymbol{k} = \varnothing$, we set

$$s(\varnothing; N) = s^\star(\varnothing; N) \coloneqq \delta_{N0}, \quad S(\varnothing; N) = S^\star(\varnothing; N) \coloneqq 1.$$

If $\boldsymbol{k}$ is admissible, the limits

$$\zeta(\boldsymbol{k}) \coloneqq \lim_{N\to\infty} S(\boldsymbol{k}; N), \qquad \zeta^\star(\boldsymbol{k}) \coloneqq \lim_{N\to\infty} S^\star(\boldsymbol{k}; N)$$

exist. We call them the multiple zeta and zeta-star values respectively.

In these decades, the multiple zeta(-star) values have been investigated from various points of view. Here we only mention that there are many relations among these values, linear or algebraic over $\mathbb{Q}$, e.g.

$$\zeta(3) = \zeta(1,2), \quad \zeta(4) = 4\zeta(1,3) = \zeta(1,1,2), \quad \zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5).$$

We are interested in how to generate such relations systematically, and the theory of Kawashima functions provide a large class of relations.

To describe some relations, it is convenient to consider a $\mathbb{Q}$-linear map constructed from MZVs. Let $\mathcal{R}$ be the $\mathbb{Q}$-vector space of formal linear combinations of indices, and $\mathcal{R}^0$ its subspace generated by admissible ones:

$$\mathcal{R} \coloneqq \bigoplus_{\boldsymbol{k} \text{ index}} \mathbb{Q} \cdot \boldsymbol{k} \supset \mathcal{R}^0 \coloneqq \bigoplus_{\boldsymbol{k} \text{ adm. index}} \mathbb{Q} \cdot \boldsymbol{k}.$$

Then we define $\mathbb{Q}$-linear maps $S(-; N)$ and $S^\star(-; N)$ from $\mathcal{R}$ to $\mathbb{Q}$ by sending indices $\boldsymbol{k}$ to $S(\boldsymbol{k}; N)$ and $S^\star(\boldsymbol{k}; N)$, respectively. The $\mathbb{Q}$-linear maps $\zeta, \zeta^\star \colon \mathcal{R}^0 \to \mathbb{R}$ are similarly defined. Thus the study of linear relations among MZVs is the study of $\operatorname{Ker} \zeta$.

2. Definition of Kawashima functions

By combining (1.1) and (1.2), we see that

$$\sum_{n=1}^{N} \frac{1}{N} = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} \binom{N}{n} \quad \text{for any integer } N \ge 0.$$

This identity is generalized to multi-index as follows:

Theorem 2.1 (Hoffman [6],Kawashima [7])**.**

$$S^\star(\boldsymbol{k}; N) = \sum_{n=1}^{N} (-1)^{n-1} s^\star(\boldsymbol{k}^\vee; n) \binom{N}{n}.$$

Here $\boldsymbol{k}^\vee$ denotes the Hoffman dual index of $\boldsymbol{k}$. For example, if $\boldsymbol{k} = (2,3,1) = (1+1, 1+1+1, 1)$, its Hoffman dual is $\boldsymbol{k}^\vee = (1, 1+1, 1, 1+1) = (1,2,1,2)$, obtained by interchanging plus and comma.

Definition 2.2. For $\boldsymbol{k} \in \mathbb{Z}_{>0}^r$, we define the Kawashima function $F(\boldsymbol{k}; z)$ of index $\boldsymbol{k}$ by

$$F(\boldsymbol{k}; z) \coloneqq \sum_{n=1}^{\infty} (-1)^{n-1} s^\star(\boldsymbol{k}^\vee; n) \binom{z}{n}.$$

This series converges (at least) if $\Re(z) > -1$.

This definition itself is a generalization of (1.2), while the interpolation property

$$F(\boldsymbol{k}; N) = S^\star(\boldsymbol{k}; N) \quad \text{for any integer } N \geq 0,$$

which generalizes (1.1), is a direct consequence of Theorem 2.1.

3. Stuffle relation

We begin with an example. For integers $k, l \geq 1$, we have

$$\begin{aligned} S(k; N)S(l; N) &= \sum_{0<m,n\leq N} \frac{1}{m^k n^l} \\ &= \left(\sum_{0<m<n\leq N} + \sum_{0<n<m\leq N} + \sum_{0<m=n\leq N} \right) \frac{1}{m^k n^l} \\ &= S(k, l; N) + S(l, k; N) + S(k + l; N). \end{aligned}$$

If we define an element $(k) * (l)$ of $\mathcal{R}$ by

$$(k) * (l) \coloneqq (k, l) + (l, k) + (k + l),$$

we can write the above identity as

$$S(k; N)S(l; N) = S\big((k) * (l); N\big).$$

By generalizing the above computation, for any indices $\boldsymbol{k}$ and $\boldsymbol{l}$, we can find an element $\boldsymbol{k} * \boldsymbol{l} \in \mathcal{R}$ for which the identity (the stuffle relation)

$$S(\boldsymbol{k}; N)S(\boldsymbol{l}; N) = S(\boldsymbol{k} * \boldsymbol{l}; N)$$

holds. The operation $*$, called the stuffle (or harmonic, quasi-shuffle) product, is extended to a bilinear, associative and commutative product on $\mathcal{R}$. Thus $S(-; N)\colon \mathcal{R} \to \mathbb{Q}$ is a $\mathbb{Q}$-algebra homomorphism with respect to the stuffle product. The same is true for $\zeta\colon \mathcal{R}^0 \to \mathbb{R}$.

In fact, we need the variant of the above story for $S^\star$. The first example is modified as

$$\begin{aligned} S^\star(k; N)S^\star(l; N) &= \sum_{0<m,n\leq N} \frac{1}{m^k n^l} \\ &= \left(\sum_{0<m\leq n\leq N} + \sum_{0<n\leq m\leq N} - \sum_{0<m=n\leq N} \right) \frac{1}{m^k n^l} \\ &= S^\star(k, l; N) + S^\star(l, k; N) - S^\star(k + l; N). \end{aligned}$$

Hence we have to set

$$(k) \mathbin{\bar{*}} (l) \coloneqq (k, l) + (l, k) - (k + l)$$

to obtain

$$S^\star(k; N)S^\star(l; N) = S^\star\big((k) \mathbin{\bar{*}} (l); N\big).$$

In general, we have a bilinear, associative and commutative product $\bar{*}$ on $\mathcal{R}$ with respect to which $S^\star(-; N)\colon \mathcal{R} \to \mathbb{Q}$ and $\zeta^\star\colon \mathcal{R}^0 \to \mathbb{R}$ are $\mathbb{Q}$-algebra homomorphisms.

The following result by Kawashima gives the interpolation of the stuffle relation for $S^\star$.

Theorem 3.1 (Kawashima [7])**.** For any indices $\boldsymbol{k}, \boldsymbol{l}$, we have

$$F(\boldsymbol{k};z)F(\boldsymbol{l};z) = F(\boldsymbol{k} \mathbin{\bar{*}} \boldsymbol{l};z). \tag{3.1}$$

Here $F(-;z)$ is defined on $\mathcal{R}$ by linearity.

4. Taylor expansion

By expanding the binomial coefficients $\binom{z}{n}$ as polynomials of z, Kawashima proved the following:

Theorem 4.1 (Kawashima [7])**.**

$$F(\boldsymbol{k};z) = \sum_{m=1}^{\infty} (-1)^{m-1} A(\boldsymbol{k};m) z^m, \tag{4.1}$$

where

$$A(\boldsymbol{k};m) := \sum_{n=1}^{\infty} s(\overbrace{1,\dots,1}^{m};n) s^\star(\boldsymbol{k}^\vee;n).$$

Note that, for each index $\boldsymbol{k}$ and integer $m > 0$, the number $A(\boldsymbol{k};m)$ can be written as a finite sum of MZVs. For example,

$$\begin{aligned}
&A(1,1,2;2) \\
&= \sum_{n=1}^{\infty} s(1,1;n) s^\star(3,1;n) = \sum_{\substack{0 < a_1 < a_2 \\ \| \\ 0 < b_1 \le b_2}} \frac{1}{a_1 a_2} \frac{1}{b_1^3 b_2} \\
&= \left(\sum_{0<a_1<b_1<a_2=b_2} + \sum_{0<a_1=b_1<a_2=b_2} + \sum_{0<b_1<a_1<a_2=b_2} + \sum_{0<a_1<b_1=a_2=b_2} \right) \frac{1}{a_1 a_2} \frac{1}{b_1^3 b_2} \\
&= \zeta(1,3,2) + \zeta(4,2) + \zeta(3,1,2) + \zeta(1,5).
\end{aligned}$$

Remark 4.2. The formula (1.3) in §1 says that

$$F(1;z) = \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(m+1) z^m,$$

while Theorem 4.1 implies that

$$F(1;z) = \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(\overbrace{1,\dots,1}^{m-1},2) z^m.$$

The identity $\zeta(m+1) = \zeta(\overbrace{1,\dots,1}^{m-1},2)$ is a common example of two famous classes of linear relations of MZVs, the duality and the sum formula.

By combining the stuffle relation (3.1) and the Taylor expansion (4.1) of Kawashima functions, we obtain the following:

Corollary 4.3 (Kawashima [7])**.** For any indices $\boldsymbol{k}, \boldsymbol{l}$ and an integer $m > 0$, we have

$$\sum_{\substack{p,q\geq 1\\ p+q=m}} A(\boldsymbol{k};p)\, A(\boldsymbol{l};q) = -A(\boldsymbol{k} \mathbin{\bar{*}} \boldsymbol{l}; m).$$

In the right hand side, we consider the linearly extended map $A(-;m)$ on $\mathcal{R}$.

Since each $A(\boldsymbol{k};m)$ is a linear combination of MZVs, the above equation gives an algebraic relation among them, called Kawashima's relation. In fact, it is conjectured that these relations together with the "shuffle relations" (which we don't recall here) imply all $\mathbb{Q}$-linear relations among MZVs. As supporting evidences, there are several works that deduce some other classes of relations from Kawashima's relations (see Kawashima [**?**], Tanaka [10] and Tanaka–Wakabayashi [11]).

5. Asymptotic expansion

Recall the asymptotic expansion (1.4) for the digamma function. In terms of the Kawashima function $F(1;z) = \psi(z+1) + \gamma$, it is written as

$$F(1;z) \sim \log z + \gamma + \sum_{m=1}^{\infty} (-1)^m \zeta(1-m) z^{-m} \quad \text{as } z \to +\infty.$$

This suggests that the asymptotic expansion for general Kawashima functions $F(\boldsymbol{k};z)$ can be described in terms of the values of the multiple zeta functions

$$\zeta(s_1,\dots,s_r) \coloneqq \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}}$$

at possibly non-positive integer points. On the analytic continuation of these functions, the following result is known (see also Zhao [13] and Matsumoto [9]):

Theorem 5.1 (Akiyama–Egami–Tanigawa [1])**.** The function $\zeta(s_1,\dots,s_r)$ continues meromorphically to $\mathbb{C}^r$, and has singularities on

$$s_r = 1, \quad s_{r-1} + s_r = 2, 1, 0, -2, -4, -6, \dots$$

and

$$\sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j} \quad (j = 3, \dots, r).$$

Therefore, a non-positive integer point often lies on the singular locus of the multiple zeta function. This makes it difficult to define the 'values' at these points. Nevertheless, there are some attempts to define such values properly, e.g. Akiyama–Tanigawa [2], Guo–Zhang [5], Ebrahimi-Fard–Manchon–Singer [3] and Furusho–Komori–Matsumoto–Tsumura [4].

On the other hand, the computation of the asymptotic expansion for Kawashima functions is a problem independent to the difficulty in interpretation of multiple zeta values of non-positive indices. As for this problem, we present a partial result:

For an index $\boldsymbol{k} = (k_1,\dots,k_r) \in \mathbb{Z}_{>0}^r$ and an integer $l \geq 0$, set

$$B(\boldsymbol{k};l) \coloneqq \sum_{\substack{l_1+\cdots+l_r=l\\ l_j\geq k_j-1}} \prod_{j=1}^{r} \left[\frac{B_{l_j-k_j+1}}{(l_j-k_j+1)!} \frac{(l_j+l_{j+1}+\cdots+l_r-1)!}{(k_j+l_{j+1}+\cdots+l_r-1)!} \right],$$

where B_m denotes the Bernoulli number defined by $\frac{te^t}{e^t-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m$ (hence we have $\zeta(1-m) = -\frac{B_m}{m}$).

Theorem 5.2 (Y.)**.** If $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 2}^r$ (all entries are greater than 1), we have

$$F(\boldsymbol{k}; z) \sim \sum_{l=0}^{\infty} (-1)^l \left[\sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta^\star(k_1, \dots, k_i)\, B(k_r, \dots, k_{i+1}; l) \right] z^{-l}$$

as $z \to +\infty$.

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