Holomorphic Continuation of Adjoint L-functions via Trace Formula

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Basic zeta/L-functions over $\mathbb{Q}$

- Riemann ζ-function (Euler, 1740):

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \Re(s) > 1.$$

- $f = \sum_{n \geq 0} a_n q^n \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ holomorphic cusp form, gives an automorphic L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) \gg 1.$$

- Rankin-Selberg convolution:

$$L(s, f \times \bar{f}) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s} = \langle E(s, \cdot), |f|^2 \rangle, \quad \Re(s) \gg 1.$$

Basic zeta/L-functions over K

- Dedekind ζ-function (Dedekind, 1863): K a number field,

$$\zeta_K(s) = \prod_{\mathfrak{P}} (1 - N(\mathfrak{P})^{-s})^{-1} = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}, \ \Re(s) > 1$$

- $f \rightsquigarrow \pi_f$ a cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$: $L(s, \pi_f) = L(s, f)$
- Consider π : cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_K)$.

$$L(s, \pi) = \int_{\mathrm{GL}_2(\mathbb{A}_K)} \Phi(g) \langle g.f, \bar{f} \rangle |\det g|^s dg, \quad \text{(Godement-Jacquet)}$$

- Rankin-Selberg convolution: $L(s, \pi \times \widetilde{\pi}) = \langle E(s, \cdot), |f|^2 \rangle$, $f \in \pi$.

Algebraic v.s. Analytical

	Algebraic	Analytic
1	$\mathbb{Q}$, $\deg_{\mathbb{Q}/\mathbb{Q}} = 1$	$\mathrm{GL}(1)$, $\mathrm{rk} = 1$
zeta-function	$\zeta(s)$	$\zeta(s) = L(s, \mathbf{1})$
2	K quadratic, i.e. $\deg_{K/\mathbb{Q}} = 2$	$\mathrm{GL}(2)$, $\mathrm{rk} = 2$
L-function	$\zeta_K(s)$	$L(s, f \times \bar{f})$
Ratio	$\zeta_K(s)/\zeta(s)$	$L(s, f \times \bar{f})/\zeta(s)$
Analytic Properties	?	?

Question.

What do we know about these ratios? Why are we interested in them?

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Ratio	$\zeta_K(s)/\zeta(s)$	$L(s, f \times \bar{f})/\zeta(s)$
Upshot	Class Field Theory: $\zeta_K(s)/\zeta(s) = L(s, \eta_K)$ holomorphic	Shimura & Zagier ('70s): $L(s, f \times \bar{f})/\zeta(s)$ holomorphic

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Because I was interested in it!

—— R. Dedekind

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Conjecture. (Dedekind, $\approx$ 1873)

K : number field. Then $\zeta_K(s)/\zeta(s)$ admits an analytic continuation to $\mathbb{C}$.

- $K = Q(\sqrt{-D})$. $\zeta_K(s)/\zeta(s) = L(s, \chi_D)$.
- Dedekind (1900): Holds for K pure cubic field.
- Holds for $K/\mathbb{Q}$ Galois.
- Uchida & van der Waall (1975): K is contained in a solvable extension of $\mathbb{Q}$. (Holds also for general extension K/F.)
- Murty (2000): Twist case, i.e. $L(s, \chi \circ N_{K/F})/L(s, \chi)$.

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E. Artin: Me! Should consider the ratio in terms of Galois representation:

$$\text{Artin-Takagi}: \quad \zeta_K(s)/\zeta_F(s) = \prod_{1 \neq \chi \in \widehat{\mathrm{Gal}(K/F)}} L(s, \chi, K/F)^{\chi(1)}.$$

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Conjecture. (Artin, 1923)

Let $\rho \neq \mathbf{1}$ be a finite-dimensional complex representation of $\mathrm{Gal}(K/F)$. *Then the Artin L-function $L(s, \rho) = L(s, \chi_\rho, K/F)$ is entire.*

- Known for monomial $\mathrm{Gal}(K/F)$, and some 2-dimensional case, e.g. tetrahedral, octahedral.
- Wide open.

Is there any strategy for Artin conjecture?

- Yes! We have **Langlands philosophy**.

Langlands. (1970)

Artin conjecture follows from strong enough results of the **Langlands philosophy**. *In particular, $L(s, \rho)$ should be an automorphic L-function, which is "typically" analytic.*

- Will see:

$$\boxed{\text{Langlands philosophy also indicates entireness of } L(s, f \times \tilde{f})/\zeta(s)!}$$

Summary

$$\begin{tikzcd}
\boxed{\text{Artin Conjecture}} \arrow[d, Rightarrow] & \boxed{\text{Langlands Philosophy}} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\underbrace{\boxed{\text{DC: } \zeta_K(s)/\zeta(s) \text{ entire}}}_{\text{Algebraic Side}} & \underbrace{\boxed{L(s, f \times \tilde{f})/\zeta(s) \text{ entire}}}_{\text{Analytic Side}}
\end{tikzcd}$$

- Langlands implies everything!

Langlands Correspondence

- G : reductive group, e.g., $G = \mathrm{GL}_2$
- ${}^L G$: the L-group, e.g. ${}^L\mathrm{GL}_2 = \mathrm{GL}_2(\mathbb{C})$
- $\mathcal{A}(G)$: set of automorphic representations on $G(\mathbb{A}_F)$, can be thought as a subspace of $L^2(G(F)\backslash G(\mathbb{A}_F))$
- $\pi \in \mathcal{A}(G)$, standard L-function $L(s, \pi)$
- W_F' : conjectural "Weil-Deligne" group over global field F

Conjecture. (Langlands)

There is a map such that

$$\mathcal{A}(G) \longrightarrow \{\rho : \ W_F' \longrightarrow {}^L G\}$$

Some remarks:

- Makes sense locally; for $G = \mathrm{GL}_n$, the correspondence is 1-to-1.
- $L(s, \rho) = L(s, \pi)$.

$$
\begin{tikzcd}
 & {}^L G \arrow[rr, "r"] & & \mathrm{GL}_m(\mathbb{C}) & \\
\mathcal{A}(G) \ni \pi \arrow[r, mapsto] & {} & & {} \arrow[r, mapsto] & \Pi \in \mathcal{A}(\mathrm{GL}_m) \\
 & & W'_F \arrow[uul, "\phi"] \arrow[uur, "\Phi = r \circ \phi"'] & &
\end{tikzcd}
$$

Definition

$L(s, \pi, r) := L(s, r \circ \phi)$ via Local Langlands Correspondence.

- $L(s, \pi, r) = L(s, \Pi)$.

One Basic Example

- $G = \mathrm{GL}_n$. Then G acts on $\mathfrak{sl}_n(\mathbb{C})$ by conjugation, inducing

$$r = \mathrm{Ad}: \ \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{n^2-1}(\mathbb{C}).$$

- Can see another expression of the L-function

$$L(s,\pi,r) = L(s,\pi,\mathrm{Ad}) = \frac{L(s,\pi\times\widetilde{\pi})}{\zeta_F(s)}$$

According to Langlands functoriality, one should expect

Conjecture. (Selberg, 1992)

Let π be a cuspidal representation on $GL(n)$. Then $L(s,\pi,\mathrm{Ad})$ is holomorphic.

Why this conjecture is interesting?

Conjecture. (Selberg, 1992)

Let π be a cuspidal representation on $GL(n)$. Then $L(s, \pi \times \widetilde{\pi})/\zeta_F(s)$ is holomorphic.

- Should be basic results of Langlands philosophy.
- It was not known (since 1975) when $\boxed{n > 2}$ for general π.
- Recall Artin conjecture:

$$\zeta_K(s)/\zeta_F(s) = \prod_{\chi \neq \mathbf{1}} L(s, \chi, K/F)^{\chi(1)}.$$

- According to Langlands, expecting

$$L(s, \pi \times \widetilde{\pi})/\zeta_F(s) = \prod_{\pi \text{ cusp}} L(s, \pi)^{m(\pi)}.$$

What Was Known: $n = 2$

- Shimura (1975) & Zagier (1977):

$$\boxed{L(s, f, \mathrm{Ad}) = L(s, f \times \tilde{f})/\zeta(s) \text{ is holomorphic.}}$$

- Gelbart-Jacquet (1978): Generalized Shimura's method to # field.
- Jacquet-Zagier (1987): Generalized Zagier's method to # field, using a **trace formula identity**.
- Flicker (1992): Partial results on $GL(n)$ by simple trace formula.

Brief Summary

	Algebraic	Analytic
n	$[K:\mathbb{Q}] = n$	$\mathrm{GL}(n)$
Ratio	$\zeta_K(s)/\zeta(s)$	$L(s, \pi \times \widetilde{\pi})/\zeta(s)$
Conjectures	Dedekind Conjecture: $\zeta_K(s)/\zeta(s)$ holomorphic	Selberg Conjecture: $L(s, \pi, \mathrm{Ad}) = L(s, \pi \times \widetilde{\pi})/\zeta(s)$ holomorphic
$n = 2$	Class Field Theory	Shimura & Zagier ($\approx$ 1975)
$n = 3, 4$	Uchida & van der Waall (1975)	$\boxed{\textit{not known}}$
$n \geq 5$	NO IDEA	Wide Open
DREAM	Artin Conjecture	Langlands Conjecture

What We Did

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n	$[K:\mathbb{Q}] = n$	$\mathrm{GL}(n)$
Ratio	$\zeta_K(s)/\zeta(s)$	$L(s, \pi \times \widetilde{\pi})/\zeta(s)$
Conjectures	Dedekind Conjecture: $\zeta_K(s)/\zeta(s)$ holomorphic	Selberg Conjecture: $L(s, \pi, \mathrm{Ad}) = L(s, \pi \times \widetilde{\pi})/\zeta(s)$ holomorphic
$n = 2$	Class Field Theory	Shimura & Zagier ($\approx$ 1975)
$n = 3, 4$	Uchida & van der Waall (1975)	$\boxed{\textit{was not known}}$
$n \geq 5$	NO IDEA	Wide Open
DREAM	Artin Conjecture	Langlands Conjecture

What We Did

- F : global field
- χ : idele class character on $\mathbb{A}_F^\times$
- π : a cuspidal representation on $GL(n, \mathbb{A}_F)$
- $L(s, \pi, \mathrm{Ad} \otimes \chi) := L(s, \pi \times \widetilde{\pi} \otimes \chi)/L(s, \chi)$

Theorem (Y.)

Let $n \leq 4$. Then the complete L-function $L(s, \pi, \mathrm{Ad} \otimes \chi)$ is entire, unless $\chi \neq 1$ and $\pi \otimes \chi \simeq \pi$, in which case $L(s, \pi, \mathrm{Ad} \otimes \chi)$ is meromorphic with only simple poles at $s = 0, 1$.

In particular, Selberg's Conjecture holds for any cuspidal representation π when $n \leq 4$.

Idea

$$\begin{tikzcd} \text{Artin Conjecture} \arrow[d, Rightarrow] & \text{Langlands Philosophy} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\ \underbrace{\text{Dedekind Conjecture}}_{\text{holds for } \textbf{degree } n \le 4} & \underbrace{L(s,\pi,\mathrm{Ad}) \text{ entire}}_{\substack{\textit{Expectation}: \\ \text{holds for } \pi \in \mathcal{A}_0(GL(n)),\ \textbf{rank } n \le 4}} \end{tikzcd}$$

Idea

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Artin Conjecture} \arrow[d, Rightarrow] & \text{Langlands Philosophy} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\underbrace{\text{Dedekind Conjecture}}_{\text{holds for } \textbf{degree } n \leq 4} \arrow[r, "\text{J-Z trace formula}"] & \underbrace{L(s, \pi, \mathrm{Ad}) \text{ entire}}_{\pi \in \mathcal{A}_0(GL(n)), \ \textbf{rank } n \leq 4}
\end{tikzcd}$$

Jacquet-Zagier's Work for $GL(2)$

Recall a trace formula of Jacquet-Zagier type for $\mathrm{GL}(2)$:

$$\boxed{\text{Spectral Side}} = \int_{[GL(2)]} K(x,x)E(x,s)dx = \boxed{\text{Geometric Side}}$$

Explicitly,

$$\underbrace{\sum_{\pi} c_\pi L(s,\pi,\mathrm{Ad})}_{\text{Cusp}} + \underbrace{\Sigma_1}_{\text{Cont}} + \underbrace{\Sigma_2}_{\text{Res}} = \underbrace{\sum_{[E:F]=2} c_E L(s,\eta_{E/F})}_{\text{elliptic regular}} + \underbrace{\boxed{\begin{matrix}\text{finite sum involving}\\ \text{Hecke } L\text{-functions}\end{matrix}}}_{\text{other orbital integrals}}.$$

$$\begin{cases}\text{Compute residues} & :) \\ \text{Convergence issue: they thought the sum over Cont is finite} & :(\end{cases}$$

Jacquet-Zagier Trace Formula for $GL(n)$

Theorem (Y.)

$$\underbrace{\sum_{\pi} c_\pi L(s,\pi,\mathrm{Ad})}_{\textit{infinite sum}} + \underbrace{\Sigma_1}_{\substack{\textit{Rankin-Selberg}\\ \textit{for non-discrete}}} + \underbrace{\Sigma_2 + \cdots + \Sigma_{n-1}}_{\substack{\textit{Langlands-Shahidi}\\ \textit{\& Godement-Jacquet}}} + \underbrace{\Sigma_n}_{\substack{\textit{Intertwining operator}\\ \textit{\& Godement-Jacquet}}}$$

$$= \underbrace{\frac{1}{n}\sum_{[E:F]=n} c_E \cdot \frac{\zeta_E(s)}{\zeta_F(s)}}_{\textit{finite sum of certain Artin L-series}}$$

Dedekind Conjecture

- Convergence issue :)
- All possible poles in $\Re(s) \geq \frac{1}{2}$: $\underbrace{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n-1}{n}}_{\text{at most simple}}, \underbrace{1}_{\text{might be multiple}}$

Thank You!