Teaching Differential Equations With a Dynamical Systems Viewpoint

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Revising the Differential Equations Course

There are a number of issues that should be addressed by any revision of the differential equations course. Most important is an issue that we face in all of our courses: What do the students take away from the course? Just as with the calculus sequence, we find that many of our students become proficient at certain symbolic calculations. However, if we ask them general questions about the fundamental mathematics involved, we find that there is little understanding of the basic concepts. We need to encourage students to see the forest as well as they see the trees.

A related issue is our students' ability to apply what they learn in this course to other courses and in their careers after college. This is a particularly important consideration for our engineering students, the vast majority of the students who take this course. Teaching students how to find explicit solutions to a number of carefully chosen equations is far less important than teaching them how to recognize a problem that involves differential equations. Moreover, our students should be able to interpret their results in the terms of the original problem. Formulation and interpretation are just as important as technique, and this aspect of the subject should not be left solely to the scientists and engineers who teach related courses.

The "nonlinear revolution" that is occurring throughout science, engineering, and mathematics dictates that this course should present nonlinear systems on the same footing as linear systems. The process of linearization is certainly less prevalent in the sciences and in engineering than it once was. Therefore, students should be able to recognize the difference between linear and nonlinear systems, and they must be familiar with techniques for analyzing nonlinear systems.

Finally there is no other mathematics course for which the current technological revolution is more significant. It is no longer necessary to restrict our attention to those equations for which a closed form solution is available. Indeed, it is desirable for students to study many differential equations that do not possess explicit solutions. Since computers and graphing calculators can readily graph approximate solutions, students must be prepared to interpret what they see and evaluate the validity of their computations. In addition, it is important that we teach our students to think graphically as well as analytically since graphical illustration of quantitative information is much more common in today's society. It is also important to recognize the fact that many of the standard computational techniques that are part of the current course are also embedded in computer-based mathematics systems such as *Derive*, *Maple*, and *Mathematica*. This gives us the freedom to concentrate more on the underlying mathematics and less on the symbolic manipulation.

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The Boston University Revision

The Dynamical Systems Group at Boston University has begun to address these issues with a project whose goal is to produce a course that is appropriate for a variety of institutions. Although we are not formally related to any of the major calculus reform efforts, we find a significant overlap in philosophy. Of course, since we have significant experience teaching the study of dynamical systems, we also bring a particular viewpoint to the issues at hand.

Most importantly we feel that there should be a balance between analytic, numeric, and graphical techniques throughout the course and that students should occasionally be asked to explain what they are doing as they do it. We stress qualitative theory throughout our course. We illustrate the significance of linearity by presenting a detailed treatment of both linear and nonlinear equations and systems, and we use technology as a tool for illustration, experimentation, and discovery. Our students are required to set up and solve their own systems/equations, and most importantly, they are required to explain their solutions in everyday language.

We feel particularly strongly about two aspects of our approach — the use of technology and the presence of a significant modeling component. We believe that technology serves as a vehicle for changing the nature of the course from one where students passively receive information to one where students actively participate in their education. Computer experiments and the associated lab reports allow students to pursue open-ended investigations of nontrivial systems. They can experiment with parameters, observe patterns, and modify their models. This modification is especially desirable since this is the type of modeling that they will undoubtedly do in their scientific careers.

It is also important to recognize that technology (in the form of computer graphics) has revolutionized the way in which we "see" solutions of differential equations. Throughout the course, we stress the many different ways in which we view solutions, e.g., as graphs, as phase plots, as bifurcation plots, and as time series.

Of course, the idea of having a significant discussion of modeling in this course is not a new one. There are already many interesting differential equations textbooks that adopt a modeling approach. But our use of models differs in emphasis from many of these approaches. For us, modeling is an important aspect of the course, but we do not feel that a significant modeling component alone addresses all of the issues mentioned above.

As mentioned above, we also bring our backgrounds in dynamical systems theory to the project. For example, we feel that discrete models of dynamical systems are more prevalent, and the inclusion of such models is appropriate at this level. Not only do discrete systems provide a smooth introduction to modeling, but they also provide a nice link between the continuous theory and iterative topics such as numerical methods and logistic population models.

More significantly, the great advances in dynamical systems theory in the past few decades provides a rare opportunity to expose students at this level to current topics of interest in mathematics. With computer graphics as a tool, students can easily investigate such fascinating systems as the Lorenz attractor, the double pendulum, and chaotic behavior in logistic models. When students learn that these systems are not completely understood or that the mathematical ideas they are using were developed in their lifetimes, they change their opinions on the nature of mathematics. Rather than a collection of tricks from centuries past, mathematics becomes an alive and thriving discipline. We should always strive to give our students a glimpse of what is new and exciting in mathematics, and we take that opportunity in our approach.

Outline of the Course at Boston University

We have class tested our ideas in seven courses over four semesters. Three of the courses have had small enrollments (15–40 students) and four of the courses have been large lecture courses (100–200 students). A preliminary outline has emerged.

Introduction. In order to emphasize that the acts of formulation and interpretation will be important in our course, we start with a number of examples that include a heavy dose of modeling. We delay the discussion of topics such as existence, uniqueness, and the order of an equation so that we can focus on the fundamental concept that a differential equation (along with an initial value) determines a function by specifying its derivative. Our goal here is to have students formulate simple equations and analyze the solutions, and we avoid examples that involve detailed symbolic computation. We build simple models of diverse phenomena such as population growth, epidemics, arms races, the national economy, and the dilution of drugs in the blood stream.

Since modeling is a difficult topic for students at this level, we include a discussion of difference equations. In the theory of dynamical systems, it is always easier and often more informative to begin with discrete systems and then to follow with a corresponding analysis of the continuous case. We feel that the same principle applies at this level. If we start the course with the students thinking about how to describe the difference between successive generations of a discrete system, then they develop a better sense that the course involves the use of equations to specify the "evolution" of a function. We also believe that it is much easier for a student initially to formulate difference equations rather than it is to formulate differential equations.

Our introduction also lays the foundation that enables us to break from the first order equation, second order equation, n-th order equation, linear systems, nonlinear systems progression of the traditional course. Many interesting and simple examples of differential equations are nonlinear, autonomous systems in the plane, and our introduction includes discussion and lab work on some of these systems. We want to illustrate the fact that understanding the qualitative behavior of solutions is just as important as finding closed form expressions for them.

Our introduction relies significantly on the use of the computer. Our initial lab work requires that the students investigate the long-term behavior of the solutions both to difference and differential equations using elementary computer techniques such as iteration and numerical solution via Euler's method. However, we avoid a full-scale examination of numerical methods at this point in the course. Rather, we discuss Euler's method because it is the numerical method that is most closely related to the concept of a differential equation. A more detailed examination of numerical techniques is presented at the end of the course.

Here are some of our favorite exercises that the students face immediately at the beginning of the course:

1. For the data set given in Table 1, use a model of the form

$$\frac{dP}{dt} = kP$$

(an exponential growth model) to make predictions about the years 2010, 2050 and 2100.

   a. Solve the initial value problem,

   b. Determine the constant k,

   c. Compute the predicted populations,

   d. Compare the solution to the actual data. Do you believe your prediction?

   The data in the following table is the land area in Australia colonized by the American marine toad (*Bufo Marinis*).

Year	Cumulative Area Occupied (km^2)
1939	32,800
1944	55,800
1949	73,600
1954	138,000
1959	202,000
1964	257,000
1969	301,000
1974	584,000

Table 1

[Note that the area of Queensland is 1,728,000 km^2 and the area of Australia is 7,619,000 km^2. All data taken from *Cumulative Geographical Range of Bufo Marinis in Queensland, Australia from 1935 to 1974*, by Michael D. Sabath, Walter C. Boughton and Simon Easteal, in **Copeia**, no. 3, 1981, pp. 676–680.]

Remark: The American marine toad was introduced to Australia to control sugar cane beetles and in the words of J.W. Hedgpath (see *Science*, July 93 and *The New York Times*, 6 July 1993)

> "Unfortunately the toads are nocturnal feeders and the beetles are abroad by day, while the toads sleep under rocks, boards and burrows. By night the toads flourish, reproduce phenomenally well and eat up everything they can find. The cane growers were warned by Walter W. Froggart, president of the New South Wales Naturalist Society, that the introduction was not a good idea and that the toads would eat the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right."

2. For the following predator-prey systems, identify which dependent variable, x or y, is the prey population and which is the predator population. State whether the prey has other limits to its growth than just the supply of prey and whether the predator has other food sources than just the prey. (All parameters are positive.)

a. $\frac{dx}{dt} = -ax + bxy$

$\frac{dy}{dt} = cy - dxy$

b. $\frac{dx}{dt} = ax - \frac{ax^2}{N} - bxy$

$\frac{dy}{dt} = cy + dxy$

3. For the following predator-prey population models (x = prey, y = predator),

a. $\frac{dx}{dt} = 2x - 3xy$

$\frac{dy}{dt} = -y + \frac{1}{2}xy$

b. $\frac{dx}{dt} = x - 4xy$

$\frac{dy}{dt} = -2y + 3xy$

   a. In which system does the prey reproduce more quickly?
   b. In which system is the predator more successful at catching prey?
   c. In which system does the predator require more prey to increase its growth rate by a fixed amount?

4. Consider the initial value problem

$$\frac{dy}{dt} = (3 - y)(y + 1), \quad y(0) = 0.$$

   a. Use Euler's method with the step size $\Delta t = 0.5$ to approximate the solution over the time interval $0 \leq t \leq 5$. Your answer should include a table of the approximate values of y. It should also include a sketch of the graph of the approximate solution.
   b. Do a qualitative analysis of the solution and compare your analysis with your results from Part a. What's wrong?

5. Suppose a species of fish has a population in a particular lake that is modeled by the logistic population model with growth rate k and carrying capacity N. Adjust the model to take into account each of the following situations:
   a. Fishermen take 1000 fish/year from the lake.
   b. Fishermen take one third of the fish population per year from the lake.
   c. The number of fish taken from the lake per year by fishermen is proportional to the square root of the number of fish in the lake.

First Order Equations. After the introduction, we return to a somewhat more traditional approach to first order equations. We discuss linear equations and separation of variables. But students also sketch direction fields and interpret existence and uniqueness results in graphical terms. The most substantial difference from the traditional techniques is our detailed discussion of autonomous equations. We discuss the phase line in depth. Therefore, our students are comfortable with the idea of an equilibrium point, and we also require that the students learn to move between sketching the phase line and sketching graphs of solutions. At this point, it is possible to introduce some bifurcation theory in order to address the important distinction between a variable and a parameter. In this part of the course, students are required to do lab work on first order equations that are not "solvable" by traditional techniques. Thus, we can emphasize that a differential equation may yield a solution even if the solution is not expressible in terms of elementary functions.

The following exercises highlight the differences between our presentation and the standard discussion of first order equations.

6. Consider the following 8 first order equations:

$$1.\ \frac{dy}{dt} = t - 1 \qquad 2.\ \frac{dy}{dt} = t + 1 \qquad 3.\ \frac{dy}{dt} = y + 1 \qquad 4.\ \frac{dy}{dt} = 1 - y$$
$$5.\ \frac{dy}{dt} = y^2 + y \qquad 6.\ \frac{dy}{dt} = y(y^2 - 1) \qquad 7.\ \frac{dy}{dt} = y - t \qquad 8.\ \frac{dy}{dt} = y + t$$

Four of the associated slope fields are shown in Figure 1. Pair the slope fields with their associated equations. Provide a brief justification for your choice.

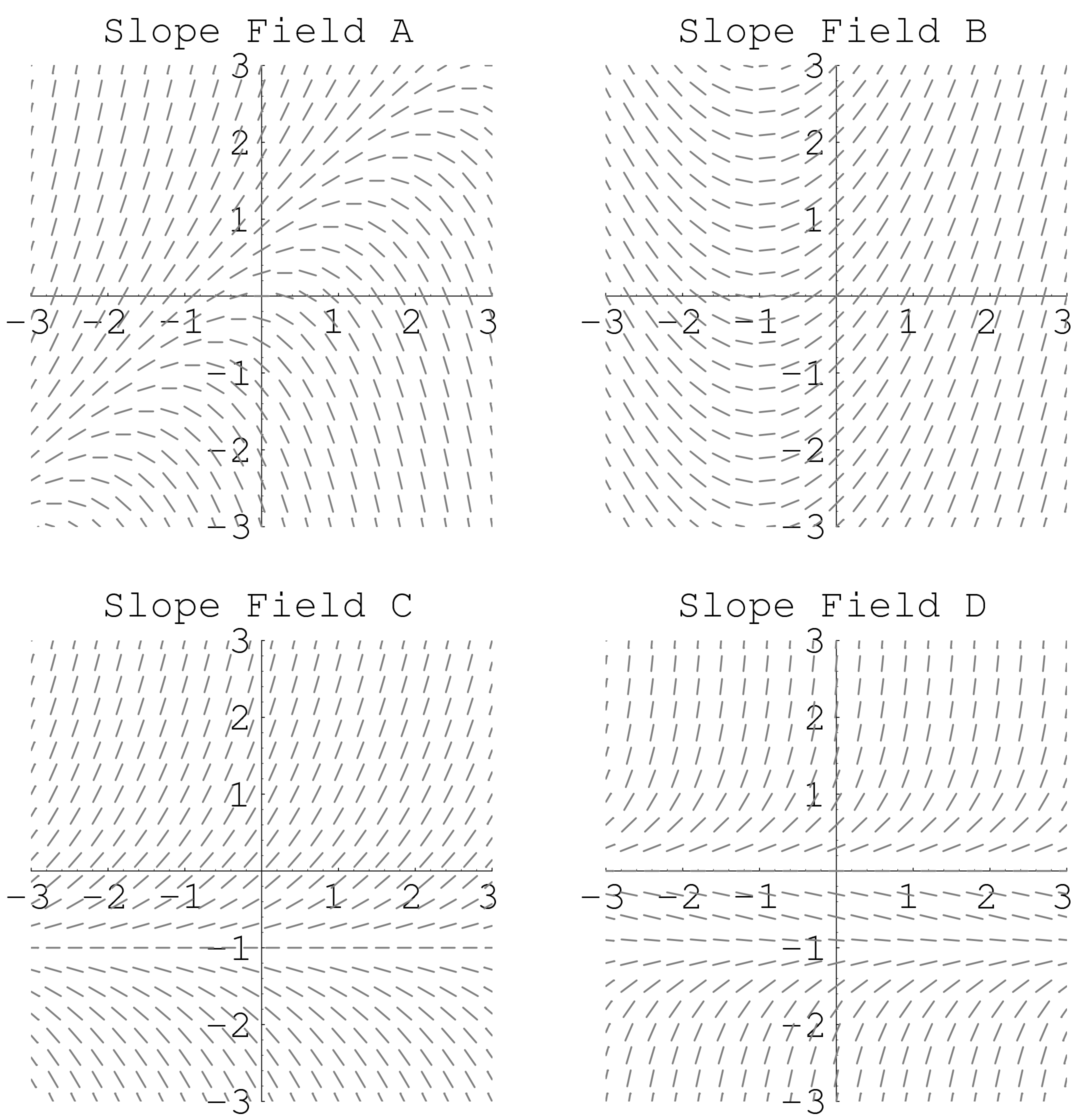

Figure 1

A variation on Exercise 6 is a question in which the student is given the slope field and an initial condition and asked to give a rough sketch of the graph of the corresponding solution.

7. Consider the first order, autonomous equation

$$\frac{dy}{dt} = y(4 - y^2).$$

a. Plot the phase line for this equation. Classify the equilibria (e.g., sink, ...).

b. Give rough sketches of the graphs of the solutions that satisfy the following four initial conditions: $y(0) = -1$; $y(0) = 0$; $y(0) = 1$; $y(0) = 3$.

8. Suppose we know that the graph below is the graph of a solution to $\dfrac{dy}{dt} = f(y)$.

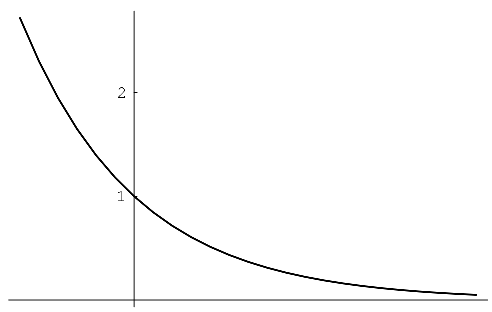

Figure 2

a. How much of the slope field can you sketch from this information? [Hint: Note that the equation is autonomous, i.e,. f is a function of y only.]

b. What can you say about the solution with $y(0) = 2$ (i.e., sketch this solution)?

9. Consider the autonomous differential equation

$$\frac{dy}{dt} = 3y(1 - y).$$

a. Sketch its phase line identify the equilibrium points as sinks, sources or nodes.

b. Sketch the graphs of the solutions satisfying the initial conditions $y(0) = 1$, $y(2) = -1$, $y(0) = 1/2$, $y(0) = 2$. You should put all of your graphs on one plot.

In Exercises 10–11, the graph of a function $f(y)$ is given. Sketch the phase line for the autonomous differential equation $\dfrac{dy}{dt} = f(y)$.

10.

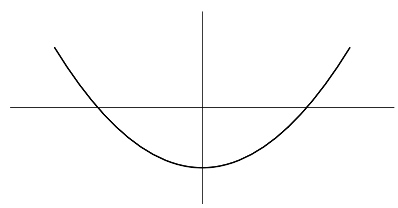

Figure 3

11.

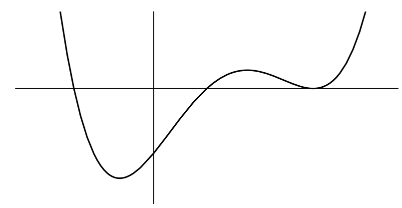

Figure 4

12. Consider the following population model for a species of fish in a lake,

$$\frac{dP}{dt} = -P^2 + 100P.$$

Suppose it is decided that fishing will be allowed in the lake, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 3 fish per year (these are hard fish to catch!).

a. What is the largest number if licenses that can be issued if the fish are to have a chance to survive in the lake?

b. Suppose the number of fishing licenses in Part a are issued. What will happen to the fish population (i.e., how does the behavior of the population depend on the initial population)?

c. Given that fish can occasionally be subject to epidemics of diseases (just like people), what do you think will happen if the fishing level is kept at the level in Part b for a long time (i.e., will the fish population remain fixed)?

Linear Systems. Just as in the traditional course, linear equations/systems are central to our course. Since numerical approximation is also an important part of the course, we consider linear systems at the same time as linear equations. Although we continue to emphasize geometric concepts such as the phase plane, our discussion of linear systems is traditional (eigenvalues, eigenvectors, etc.) However, we expect the students to be able to sketch tx- and ty-plots using what they have learned about the phase portraits for linear systems in the plane.

Again we ask students to match vector fields and direction fields with corresponding systems (similar to Exercise 6 above) and to sketch solution curves satisfying given initial conditions directly on the direction fields. In this way, students learn to interpret the equations geometrically. They also learn to associate phase portraits with second order equations in a natural fashion.

Here are two other exercises that illustrate our approach:

13. A small object of mass 1 kg is attached to a spring with spring constant 2 N/m. This mass-spring system is immersed in a viscous medium (e.g., lentil soup) with damping constant 3 N·s/m. Suppose that the mass is lowered $\frac{1}{2}$ m below its equilibrium position and released. Determine the long-term behavior of the mass. Sketch the phase portrait associated to this system along with the solution curve that corresponds to this initial condition.
14. Here is the slope field corresponding to the system

$$\frac{dx}{dt} = -x - 4y$$
$$\frac{dy}{dt} = -2x + y$$

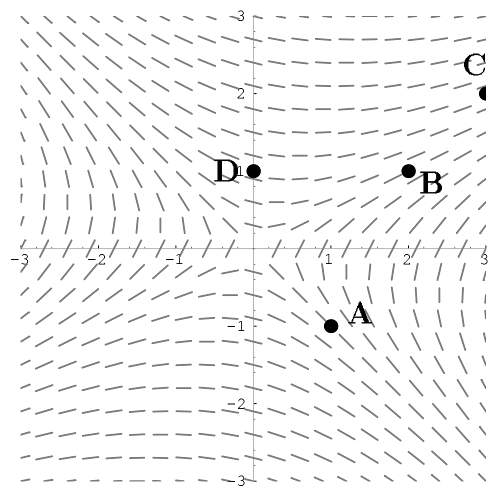

Figure 5

a. Determine the type of the equilibrium point at the origin and find all straight line (in the phase plane) solutions.
b. Plot the tx- and ty-graphs for $t \geq 0$ for the initial conditions $A = (1, -1)$, $B = (2, 1)$, $C = (3, 2)$, and $D = (0, 1)$.

Nonlinear Systems. At this point, the students are ready to tackle autonomous, nonlinear systems. They are able to formulate and interpret such systems and their solutions. We concentrate on the analysis of equilibria via linearization. Then we introduce techniques that yield global qualitative results (nullclines, regions where dx/dt and dy/dt have constant sign, saddle connections, and closed orbits). For lab work, they do more on bifurcations. Finally, this part of the course ends with a discussion of chaotic verses nonchaotic behavior as illustrated by the double pendulum and the Lorenz equations. Our analysis of the Lorenz equations contains a discussion of the relevant one-dimensional map, which provides a nice connection with the discussion of difference equations at the beginning of the course.

As mentioned above, we have had great success with laboratory assignments in this course. Here is a very successful lab that can be assigned at this point of the course:

Predator-Prey Lab

In this laboratory exercise, you will study a nonlinear, first-order system which is a generalization of the predator-prey equations that we have discussed in class. The equations are

$$\frac{dx}{dt} = (9 - \alpha x - 3y)x$$

$$\frac{dy}{dt} = (-2 + x)y$$

where $\alpha \geq 0$ is a parameter. In other words, for different values of α we have different systems. The variable x is the population (in some scaled units) of prey (rabbits), and y is the population of predators (foxes). For a given value of α, we want to understand what happens to these populations as $t \to \infty$.

Using both the mathematical techniques we have developed and the computer, you should investigate the phase portraits of these equations for various values of α in the interval $0 \leq \alpha \leq 5$. To get started, you might want to try $\alpha = 0$, 1, 2, 3, 4, and 5. Think about what the phase portrait means in terms of the evolution of the x and y populations. Where are the equilibrium points? What types are they? What happens to a typical solution curve? Also, consider the behavior of the special solutions where either $x = 0$ or $y = 0$ (i.e., solution curves lying on the x- or y-axes).

Determine the bifurcation values of α. That is, determine the values of α where nearby α's lead to "different" behaviors in the phase portrait. For example, $\alpha = 0$ is a bifurcation value because, for $\alpha = 0$, the long term behavior of the populations is dramatically different than the long term behavior of the populations if α is slightly positive. (*Hint:* If the type of an equilibrium point changes at a certain value of α, then that value of α may be a bifurcation value.)

Laboratory report: After you have determined all of the bifurcation values for α in the interval $0 \leq \alpha \leq 5$, study enough specific values of α to be able to discuss all of the various population evolution scenarios for these systems. In your report, you should describe these scenarios using the phase portraits, tx- and ty-graphs. We are especially interested in your interpretation of the computer output in terms of the populations. Fundamental calculations should be included in your essay.

Your report should include:

1. A brief discussion of the significance of the various terms in the system. For example, what does the 9 represent? What does the $3y$ term represent?
2. A discussion of all bifurcations including the bifurcation at $\alpha = 0$. For example, a bifurcation occurs between $\alpha = 3$ and $\alpha = 5$. What does this bifurcation mean for the predator population?
3. All fundamental calculations. Details can be included in an appendix.

The text of your report should be no longer than four typewritten pages. You may write mathematical formulae by hand. You may provide as many illustrations from the computer as you wish, but the relevance of each illustration must be explained in your report. Illustrations are not included in the four page limit.

Short Topics. Much of the course focuses on autonomous equations, but it concludes with various topics including solutions to a number of nonautonomous equations. For example, we study topics such as forcing, resonance, and Laplace transforms. Finally, the course ends with an examination of numerical methods along with a discussion of what can go wrong with these techniques.

The Future of This Project

In the near term, we will continue to develop our materials and test them here at Boston University. Class testing at other sites will begin in earnest in Fall of 1994. We have established an email address (odes@math.bu.edu) for this project, and if you would like to receive mailings as they become available, you can contact us via email. The author can also be reached by US mail. The address is: Department of Mathematics, 111 Cummington St., Boston University, Boston, MA 02215.