Second-Order Equations versus First-Order Systems

As we know, the motion of a harmonic oscillator can be modeled by the second-order equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0,$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, and $b \geq 0$ is the damping coefficient. Since $m \neq 0$, we can also write this equation in the form

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0,$$

where $p = b/m$ and $q = k/m$ are nonnegative constants, and the corresponding linear system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

As we will see in this section, any method to compute the general solution of the second-order equation also gives the general solution of the associated system, and vice versa. In particular we can use the Linearity Principle to produce new solutions from known ones by adding solutions and by multiplying solutions by constants. Therefore second-order equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,$$

where a, b, and c are constants, are said to be **linear**. More precisely, these equations are **homogeneous**, **constant-coefficient**, linear, second-order equations. The constants a, b, and c are the coefficients, and the equation is homogeneous due to the fact that the right-hand side is 0. In Chapter 4 we will study the difference between homogeneous and nonhomogeneous differential equations in detail.

We can find the general solution of the linear system that models the harmonic oscillator by finding the eigenvalues and eigenvectors of the coefficient matrix. The arithmetic is not always pleasant, but the steps are clear. In this section we give a shortcut for finding the general solution of the corresponding second-order equation, and we relate this shortcut to the geometry and qualitative behavior of the solutions of both the second-order equation and the system.

A Free Gift from the Math Department

The shortcut method for finding the general solution of a second-order equation such as

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0,$$

for example, is to guess it. Given what we now know about solutions of the corresponding system, this is not as silly as it sounds. We know that the solutions of the system are made up of terms of the form $e^{\lambda t}\mathbf{V}$, where λ is an eigenvalue and $\mathbf{V}$ is an eigenvector. Hence if we are trying to guess the solution of the second-order equation, the most natural guess is

$$y(t) = e^{st},$$

where s is a constant to be determined. (From our point of view it makes more sense to use λ as the unknown constant. However, s is commonly used in applications, and for this discussion we will follow that practice.) Substituting the guess into the left-hand side of the second-order equation gives

$$\begin{aligned}\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y &= \frac{d^2(e^{st})}{dt^2} + 7\frac{d(e^{st})}{dt} + 10e^{st} \\ &= s^2e^{st} + 7se^{st} + 10e^{st} \\ &= (s^2 + 7s + 10)e^{st}.\end{aligned}$$

Since e^{st} is never zero, we must have

$$s^2 + 7s + 10 = 0$$

in order for $y(t) = e^{st}$ to be a solution. This quadratic equation has roots $s = -5$ and $s = -2$, so we know that $y_1(t) = e^{-5t}$ and $y_2(t) = e^{-2t}$ are solutions of the differential equation. (If this guess-and-test technique seems familiar, it should. We have already used this procedure when we studied analytic techniques for finding solutions to certain systems in Section 2.3.)

Applying the Linearity Principle, we see that any function of the form

$$y(t) = k_1e^{-5t} + k_2e^{-2t}$$

is also a solution for any choice of constants k_1 and k_2 (see Exercise 18 for a direct verification of this assertion). To see that this expression is in fact the general solution of the equation, we note that there is a one-to-one correspondence between solutions of

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

and solutions of the associated system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -10y - 7v.\end{aligned}$$

If we have a solution $\mathbf{Y}(t) = (y(t), v(t))$ to the system, then $y(t)$ is a solution to the second-order equation. If $y(t)$ is a solution to the equation, then we have

$$v(t) = -5k_1e^{-5t} - 2k_2e^{-2t},$$

since $v = dy/dt$. If we form the vector function

$$\mathbf{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} k_1e^{-5t} + k_2e^{-2t} \\ -5k_1e^{-5t} - 2k_2e^{-2t} \end{pmatrix},$$

we have a solution to the system that can be rewritten in the form

$$\mathbf{Y}(t) = k_1e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix} + k_2e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Solving the associated system

This form of the solution looks suspiciously familiar. If we write this second-order equation as a system in matrix notation, we obtain

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix} \mathbf{Y},$$

which has $\lambda^2 + 7\lambda + 10$ as its characteristic polynomial. Note that this is exactly the same quadratic that we obtained earlier when we applied our guess-and-test technique to the the second-order equation (with s replaced by λ). The eigenvalues for this system are $\lambda_1 = -5$ and $\lambda_2 = -2$. Computing the associated eigenvectors, we find that one eigenvector corresponding to λ_1 is $(1, -5)$ and one eigenvector corresponding to λ_2 is $(1, -2)$. Thus using the eigenvalue/eigenvector methods of this chapter, we obtain exactly the same general solution,

$$\mathbf{Y}(t) = k_1 e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

If we had chosen different eigenvectors for the system, we would have obtained a slightly different form of the general solution. For example, $(-1, 5)$ is also an eigenvector corresponding to $\lambda_1 = -5$, and $(2, -4)$ is an eigenvector corresponding to $\lambda_1 = -2$. So our general solution may also be written as

$$\mathbf{Y}(t) = k_1 e^{-5t} \begin{pmatrix} -1 \\ 5 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

But these solutions are precisely the same as those already obtained. (Replace k_1 with $-k_1$ and k_2 with $k_2/2$.) There really is no difference between this guessing method and the eigenvalue/eigenvector method.

Complex Eigenvalues

The method described above works in general for any second-order, linear equation, even those for which the characteristic polynomial has complex roots. For example, consider a harmonic oscillator with mass $m = 1$, damping coefficient $b = 2$, and spring constant $k = 2$. The second-order equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0.$$

As usual, we guess that $y(t) = e^{st}$ is a solution and obtain the characteristic equation

$$s^2 + 2s + 2 = 0.$$

Using the quadratic formula, we obtain the roots

$$s = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$