COHERENT SHEAVES FOR THE STEINBERG PARAMETER OF PGL_2

Contents

1. Introduction

Let k be an algebraically closed field of characteristic 0. In this note, we study the derived category $\mathcal{D}_{\mathbb{G}_m}(R)$ of $\mathbb{G}_m$-equivariant coherent sheaves on the ring $R = k[x,y]/(xy)$, with trivial $\mathbb{G}_m$-action on x and $t \cdot y = t^2 y$. Our reason for studying these sheaves is that they should correspond by the categorical conjecture to the various sheaves on $\mathrm{Bun}_{\mathrm{PGL}_2}$ with Fargues–Scholze parameter equal to the semi-simplification of the L-parameter of the Steinberg representation.

2. Coherent Sheaves

Definition 2.1. We let R_n, (resp. $\mathcal{L}_n$) $\in \mathcal{D}_{\mathbb{G}_m}(R)$ be the object equal to R (resp. $R/(x)$) as an R-module with $\mathbb{G}_m$-action given by $t \cdot 1 \mapsto t^n$ for $n \in \mathbb{Z}$.

As we shall see, the sheaves $\mathcal{L}_n$ are the fundamental objects of study.

Lemma 2.2. *Let T be a reductive group. A T-equivariant R-module is projective if its underlying R-module is.*

Proof. Let M be a T-equivariant R-module. By definition, M is projective if $\mathrm{Hom}_{R \rtimes T}(M, \_)$ is exact. But we have an equality of functors

$$\mathrm{Hom}_{R \rtimes T}(M, \_) = (-)^T \circ \mathrm{Hom}_R(M, \_).$$

Moreover, $(-)^T$ is exact since algebraic T-representations are completely reducible. If M is projective as an R-module, then $\mathrm{Hom}_R(M, \_)$ is exact, which therefore implies it is projective in the equivariant category. $\square$

As a consequence of Lemma 2.2, we have that R_n is projective. Thus, we have the following projective resolution of $\mathcal{L}_n$.

$$\ldots \xrightarrow{\cdot y} R_{n+2} \xrightarrow{\cdot x} R_{n+2} \xrightarrow{\cdot y} R_n \xrightarrow{\cdot x} R_n \to \mathcal{L}_n. \tag{2.1}$$

From this, it is easy to compute that the maps between the various $\mathcal{L}_n$ sheaves are as follows. We use the notation that $\mathrm{Hom}^i(\mathcal{F}, \mathcal{G})$ is defined to be $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[i])$ and denote $\bigoplus_n \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ by $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G})$. Then, if m and n are of different parity, we have $\mathrm{Hom}^\bullet(\mathcal{L}_m, \mathcal{L}_n) = 0$. If they are of the same parity and $m \geqslant n$, then we have a *lowering operator*

$$L_{m,n} : \mathcal{L}_m \to \mathcal{L}_n,$$

given by multiplication by y^{m-n}. In fact, for $m \geqslant n$, we have

$$\mathrm{Hom}^i(\mathcal{L}_m, \mathcal{L}_n) = \begin{cases} 0 & i \neq 0 \\ k & i = 0 \end{cases}$$

We normalize the identification of $\mathrm{Hom}^0(\mathcal{L}_m, \mathcal{L}_n)$ and k such that $L_{m,n}$ corresponds to 1. If $m < n$ and are of the same parity, then it is easy to verify from (2.1) that

$$\mathrm{Hom}^i(\mathcal{L}_m, \mathcal{L}_n) = \begin{cases} 0 & i \neq n-m \\ k & i = n-m. \end{cases}$$

We define a *raising operator*

$$R_{m,n} : \mathcal{L}_m \to \mathcal{L}_n[n-m],$$

given by the identity map between the modules in the projective resolutions. We normalize the identification of $\mathrm{Hom}^{n-m}(\mathcal{L}_m, \mathcal{L}_n)$ and k such that $R_{m,n}$ corresponds to 1.

Definition 2.3. We define sheaves δ_n and ϵ_n for each $n \in \mathbb{Z}$ up to isomorphism via the following triangles

$$\mathcal{L}_{n+2} \xrightarrow{L_{n+2,n}} \mathcal{L}_n \to \delta_n, \qquad \epsilon_n \to \mathcal{L}_n \xrightarrow{R_{n+2,n}} \mathcal{L}_{n+2}[2]. \tag{2.2}$$

We now describe how these sheaves are supposed to relate to the Categorical Conjecture. Recall that the Kottwitz set $B(\mathrm{PGL}_2)$ has two basic elements that we denote b_0 and $b_{1/2}$ in accordance with the slopes of their Newton polygons. Then the closure of b_0 contains non-basic elements $b_2, b_4, b_6, \ldots$ satisfying that $b_{n+2} \in \overline{b_n}$. Similarly, the closure of $b_{1/2}$ contains non-basic elements $b_1, b_3, b_5, \ldots$.

Conjecture 2.4. *Under the hypothetical correspondence between ℓ-adic sheaves on Bun_G and (Ind-)coherent sheaves on $Z^1(W, \hat{G})/\hat{G}$, the stack of L-parameters, we expect the following for $n > 0$:*

$$\begin{aligned}
\mathcal{L}_0 &\longleftrightarrow i^{\mathrm{ren}}_{b_0!}\mathrm{St} \\
\mathcal{L}_2[1] &\longleftrightarrow i^{\mathrm{ren}}_{b_0!}1 \\
\delta_0 &\longleftrightarrow i^{\mathrm{ren}}_{b_0!}\mathrm{Ind}(\delta^{1/2}) \\
\epsilon_0 &\longleftrightarrow i^{\mathrm{ren}}_{b_0!}\mathrm{Ind}(\delta^{-1/2}) \\
\mathcal{L}_1 &\longleftrightarrow i^{\mathrm{ren}}_{b_{1/2}!}1 \\
\delta_n &\longleftrightarrow i^{\mathrm{ren}}_{b_n\#}\delta^{-1/2} \\
\epsilon_{-n} &\longleftrightarrow i^{\mathrm{ren}}_{b_n\#}\delta^{1/2} \\
\delta_{-n} &\longleftrightarrow i^{\mathrm{ren}}_{b_n!}\delta^{1/2} \\
\epsilon_n &\longleftrightarrow i^{\mathrm{ren}}_{b_n!}\delta^{-1/2}
\end{aligned}$$

From the triangles (2.2), one can compute the following Hom-table (where we assume that m and n have the same parity)

$H^i(A_m, B_n)$	$\mathcal{L}_n$	δ_n	ϵ_n
$\mathcal{L}_m$	$m \geqslant n : H^0 = k$ $m < n : H^{n-m} = k$	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^{n-m} = H^{n-m+1} = k$	$m > n : H^0 = H^{-1} = k$ $m \leqslant n : H^i = 0, \forall i$
δ_m	$m \geqslant n : H^i = 0, \forall i$ $m < n : H^{n-m} = H^{n-m-1} = k$	$m > n : H^i = 0, \forall i$ $m = n : H^0 = H^1 = k$ $m < n : H^{n-m+1} = H^{n-m-1} = k, H^{n-m} = k^2$	$m > n : H^i = 0, \forall i$ $m = n : H^0 = H^1 = k$ $m < n : H^i = 0, \forall i$
ϵ_m	$m \geqslant n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$	$m > n : H^1 = H^{-1} = k, H^0 = k^2$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$

3. Ind-Sheaves

To go further, we must study the Ind-completion of $\mathcal{D}_{\mathbb{G}_m}(R)$, which we denote $\mathrm{Ind}(\mathcal{D}_{\mathbb{G}_m}(R))$. There is a natural injective system

$$\mathcal{L}_n \xrightarrow{L_{n,n-2}} \mathcal{L}_{n-2} \xrightarrow{L_{n-2,n-4}} \mathcal{L}_{n-4} \xrightarrow{L_{n-4,n-6}} . \tag{3.1}$$

The commutative diagram

$$\begin{array}{ccccccc} \mathcal{L}_n & \longrightarrow & \mathcal{L}_{n-2} & \longrightarrow & \mathcal{L}_{n-4} & \longrightarrow & \dots \\ & \searrow & \| & & \| & & \\ & & \mathcal{L}_{n-2} & \longrightarrow & \mathcal{L}_{n-4} & \longrightarrow & \dots \end{array}$$

implies that the limit of this system in $\mathrm{Int}(\mathcal{D}_{\mathbb{G}_m}(R))$ only depends on the parity of n. Hence, the above system defines two objects $\mathcal{L}_e^-, \mathcal{L}_o^- \in \mathrm{Ind}(\mathcal{D}_{\mathbb{G}_m}(R))$ based on whether n is even or odd, respectively.

We also have the injective system

$$\mathcal{L}_n \xrightarrow{R_{n,n+2}} \mathcal{L}_{n+2}[2] \xrightarrow{R_{n+2,n+4}} \mathcal{L}_{n+4}[4] \xrightarrow{L_{n+4,n+6}} . \tag{3.2}$$

The commutative diagram

$$\begin{array}{ccccccc} \mathcal{L}_n & \longrightarrow & \mathcal{L}_{n+2}[2] & \longrightarrow & \mathcal{L}_{n+4}[4] & \longrightarrow & \dots \\ & \searrow & \| & & \| & & \\ & & \mathcal{L}_{n+2}[2] & \longrightarrow & \mathcal{L}_{n+4}[4] & \longrightarrow & \dots \end{array}$$

shows that this system only depends on the parity of n up to a shift. We denote by $\mathcal{L}_e^+$ (resp. $\mathcal{L}_o^+[-1]$) the limit of the system starting with $n = 0$ (resp. $n = 1$). Using our computation of $\mathrm{Hom}^\bullet(\mathcal{L}_m, \mathcal{L}_n)$, we can compute that for even n we have

$H^i(A_m, B_n)$	$\mathcal{L}_n$	$\mathcal{L}_e^+$	$\mathcal{L}_e^-$
$\mathcal{L}_m$	$m \geqslant n : H^0 = k$ $m < n : H^{n-m} = k$	$H^{-m} = k$	$H^0 = k$
$\mathcal{L}_e^+$	$H^i = 0, \forall i$	$H^0 = k$	$H^i = 0, \forall i$
$\mathcal{L}_e^-$	$H^i = 0, \forall i$	$H^i = 0, \forall i$	$H^0 = k$

and in fact, we have the same table when n is odd but we must replace each e with an o.

The presentation of the sheaves given by the next definition was suggested to us by an email exchange with T. Koshikawa.

Definition 3.1. For n even, we define $\mathcal{M}_n \in \mathrm{Ind}(\mathcal{D}_{\mathbb{G}_m}(R))$ up to isomorphism by pushout of $\mathcal{L}_{n+2}$ along the natural maps to $\mathcal{L}_e^+[-n-2]$ and $\mathcal{L}_e^-$. In other words, we have a triangle

$$\mathcal{L}_{n+2} \xrightarrow{(1,-1)} \mathcal{L}_e^- \oplus \mathcal{L}_e^+[-n-2] \to \mathcal{M}_n,$$

where the first map is the sum of the natural map to $\mathcal{L}_e^-$ and the negative of the natural map to $\mathcal{L}_e^+[-n-2]$. When n is odd, we use the same construction, but replace each instance of e by o.

The sheaves $\mathcal{M}_n$ seem to behave quite analogously to the sheaves $\mathcal{L}_n$. For instance, we can compute that we have canonically

$$\mathrm{Hom}^\bullet(\mathcal{L}_m, \mathcal{L}_n) = \mathrm{Hom}^\bullet(\mathcal{M}_m, \mathcal{M}_n).$$

We let $L'_{m,n}$ and $R'_{m,n}$ denote the maps between the $\mathcal{M}$-sheaves under this correspondence, defined such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{M}_n & \xrightarrow{L'_{n,n-2}} & \mathcal{M}_{n-2} \\ \downarrow & \searrow & \downarrow \\ \mathcal{L}_{n+2}[1] & \xrightarrow[L_{n+2,n}[1]]{} & \mathcal{L}_n[1] \end{array} \qquad \begin{array}{ccc} \mathcal{M}_{n-2} & \xrightarrow{R'_{n-2,n}} & \mathcal{M}_n[2] \\ \downarrow & \searrow & \downarrow \\ \mathcal{L}_n[1] & \xrightarrow[R_{n,n+2}[1]]{} & \mathcal{L}_{n+2}[3] \end{array}$$

We record the following lemma which is useful when working with the $\mathcal{M}_n$ sheaves.

Lemma 3.2. *Given objects A, B, C and a map $A \to B \oplus C$, we have a triangle*

$$C \to \mathrm{Cone}(A \to B \oplus C) \to \mathrm{Cone}(A \to B). \tag{3.3}$$

Alternatively, if we have a map $B \oplus C \to A$, we have a triangle

$$B \to \mathrm{Cone}(C \to A) \to \mathrm{Cone}(B \oplus C \to A). \tag{3.4}$$

Proof. These are easy applications of the octahedral axiom using the factorings

$$A \to A \oplus C \to B \oplus C,$$

and

$$B \oplus C \to B \oplus A \to A.$$

□

We can now make the following definition.

Definition 3.3. We define the sheaves $\alpha_n, \beta_n \in \mathrm{Ind}(\mathcal{D}_{\mathbb{G}_m}(R))$ via the following triangles

$$\alpha_n \to \mathcal{M}_n \xrightarrow{L'_{n,n-2}} \mathcal{M}_{n-2}, \qquad \mathcal{M}_{n-2}[-2] \xrightarrow{R'_{n-2,n}[-2]} \mathcal{M}_n \to \beta_n.$$

Using these triangles, we deduce the following Hom-table

$H^i(A_m, B_n)$	$\mathcal{M}_n$	α_n	β_n
$\mathcal{M}_m$	$m \geqslant n : H^0 = k$ $m < n : H^{n-m} = k$	$m \geqslant n : H^i = 0, \forall i$ $m < n : H^{n-m} = H^{n-m-1} = k$	$m \geqslant n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$
α_m	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^{n-m} = H^{n-m+1} = k$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^{n-m-1} = H^{n-m+1} = k, H^{n-m} = k^2$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$
β_m	$m > n : H^0 = H^{-1} = k$ $m \leqslant n : H^i = 0, \forall i$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$	$m > n : H^1 = H^{-1} = k, H^0 = k^2$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$

We are about to state another conjecture, but before doing so, we define another pushforward functor $i_{b\flat}^{\mathrm{ren}} : \mathcal{D}(G_b(E), \Lambda)^{\mathrm{ULA}} \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G)$ given by

$$\pi \mapsto D_{\mathrm{Verd}}(i_{b\#} D_{\mathrm{sm}}(\pi)).$$

The Verdier dual of the canonical map $i_{b\#}^{\mathrm{ren}}\pi \to i_{b!}^{\mathrm{ren}}\pi$ is a map $i_{b*}^{\mathrm{ren}}\pi \to i_{b\flat}^{\mathrm{ren}}\pi$. Altogether therefore, we have canonical maps

$$i_{b\#}^{\mathrm{ren}}\pi \to i_{b!}^{\mathrm{ren}}\pi \to i_{b*}^{\mathrm{ren}}\pi \to i_{b\flat}^{\mathrm{ren}}\pi.$$

Conjecture 3.4. *Under the hypothetical correspondence between ℓ-adic sheaves on Bun_G and (Ind-)coherent sheaves on $Z^1(W, \hat{G})/\hat{G}$, the stack of L-parameters, we expect the following for $n > 0$:*

$$\begin{aligned}
\mathcal{M}_0 &\longleftrightarrow i_{b_0*}^{\mathrm{ren}}\mathrm{St} \\
\mathcal{M}_{-2}[-1] &\longleftrightarrow i_{b_0*}^{\mathrm{ren}}1 \\
\mathcal{M}_{-1} &\longleftrightarrow i_{b_{1/2}*}^{\mathrm{ren}}1 \\
\alpha_0 &\longleftrightarrow i_{b_0*}^{\mathrm{ren}}\mathrm{Ind}(\delta^{-1/2}) \\
\beta_0 &\longleftrightarrow i_{b_0*}^{\mathrm{ren}}\mathrm{Ind}(\delta^{1/2}) \\
\alpha_n &\longleftrightarrow i_{b_n*}^{\mathrm{ren}}\delta^{-1/2} \\
\beta_{-n} &\longleftrightarrow i_{b_n*}^{\mathrm{ren}}\delta^{1/2} \\
\alpha_{-n} &\longleftrightarrow i_{b_n\flat}^{\mathrm{ren}}\delta^{1/2} \\
\beta_n &\longleftrightarrow i_{b_n\flat}^{\mathrm{ren}}\delta^{-1/2}
\end{aligned}$$

Finally, in order to understand the t-structure corresponding to the perverse t-structure on Bun_G, we need to define some other sheaves.

Definition 3.5. Let n be even (the odd definitions are analogous). We define the sheaf ϵ_n^+ such that it fits into a triangle

$$\epsilon_n^+ \to \mathcal{L}_n \to \mathcal{L}_e^+[-n].$$

We define δ_n^- such that it fits into a triangle

$$\mathcal{L}_{n+2} \to \mathcal{L}_e^- \to \delta_n^-.$$

The composition $\mathcal{L}_{n+2} \to \mathcal{L}_n \to \mathcal{L}_e^-$ implies a triangle

$$\delta_n \to \delta_n^- \to \delta_{n-2}^-. \tag{3.5}$$

Similarly, the composition $\mathcal{L}_n \to \mathcal{L}_{n+2}[2] \to \mathcal{L}_e^+[-n]$ implies a triangle

$$\epsilon_n \to \epsilon_n^+ \to \epsilon_{n+2}^+[2]. \tag{3.6}$$

In particular, applying (3.3) to the definition of $\mathcal{M}_n$ yields the triangles

$$\mathcal{L}_e^+[-n-2] \to \mathcal{M}_n \to \delta_n^-,$$

and

$$\epsilon_n^+ \to \mathcal{L}_e^- \to \mathcal{M}_{n-2}.$$

The compositions $\mathcal{L}_e^+[-n] \to \mathcal{M}_{n-2} \to \mathcal{M}_n[2]$ and $\mathcal{L}_e^- \to \mathcal{M}_n \to \mathcal{M}_{n-2}$ give triangles

$$\delta_{n-2}^-[-2] \to \delta_n^- \to \beta_n \tag{3.7}$$

$$\epsilon_{n+2}^+ \to \epsilon_n^+ \to \alpha_n. \tag{3.8}$$

4. Exceptional t-structures

The sheaves we have defined yield at least three interesting t-structures, two of which are described by dualizable exceptional sets. We briefly recall these following [Bez03].

Definition 4.1. An exceptional set is a totally ordered collection $\{\nabla^i\}_{i\in I}$ satisfying the following properties

(1) $\mathrm{Hom}^\bullet(\nabla^i, \nabla^j) = 0$ for $i < j$,
(2) $\mathrm{Hom}^{<0}(\nabla^i, \nabla^i) = 0$ for all i,
(3) $\mathrm{End}(\nabla^i) = k$ for all i.

We furthermore say that an exceptional set is *dualizable* if there exists a collection $\{\Delta_i\}_{i\in I}$ satisfying

(1) $\mathrm{Hom}^\bullet(\Delta_n, \nabla^i) = 0$ for $n > i$,
(2) there exists an isomorphism $\Delta_n \cong \nabla_n \mod \mathcal{D}_{<n}$,

where $\mathcal{D}_{<n}$ denotes the triangulated category generated by all shifts of the ∇^i for $i < n$.

Proposition 4.2. *The collection*

$$\mathcal{L}_0, \mathcal{L}_1, \epsilon_0, \delta_{-1}, \epsilon_1, \delta_{-2}, \epsilon_2, \ldots$$

is a dualizable exceptional set with dual equal to

$$\mathcal{L}_0, \mathcal{L}_1, \delta_0, \epsilon_{-1}, \delta_1, \epsilon_{-2}, \delta_2, \ldots.$$

Similarly, the collection

$$\mathcal{M}_0, \mathcal{M}_{-1}, \alpha_0, \beta_1, \alpha_{-1}, \beta_2, \alpha_{-2}, \ldots$$

is a dualizable exceptional set with dual equal to

$$\mathcal{M}_0, \mathcal{M}_{-1}, \beta_0, \alpha_1, \beta_{-1}, \alpha_2, \beta_{-2}, \ldots.$$

Proof. In fact, one can immediately verify all the conditions except for the last from our Hom-tables. We now explain how to verify the last property. From Definition 2.3, we get maps

$$\delta_n \to \mathcal{L}_{n+2}[1] \to \epsilon_n \qquad \epsilon_n \to \mathcal{L}_n \to \delta_n.$$

Consider $n \geqslant 0$ and apply the octahedral axiom to the first composition, we get a triangle

$$\mathcal{L}_n[1] \to \mathrm{Cone}(\delta_n \to \epsilon_n) \to \mathcal{L}_n \to \mathcal{L}_n[2].$$

In particular, if we can show that $\mathcal{L}_n \in \mathcal{D}_{<2n+2}$, then we would get $\delta_n \cong \epsilon_n \mod \mathcal{D}_{<2n+2}$ as desired. We automatically have $\mathcal{L}_0, \mathcal{L}_1 \in \mathcal{D}_{<2n+2}$ and by inductive assumption, this category contains $\mathcal{L}_k$ and ϵ_k for $0 \geqslant k < n$. But then the triangle

$$\mathcal{L}_{n-2}[-1] \to \mathcal{L}_n[1] \to \epsilon_{n-2},$$

implies $\mathcal{L}_n \in \mathcal{D}_{<2n+2}$. The case when $n < 0$ uses $\epsilon_n \to \mathcal{L}_n \to \delta_n$ but is otherwise analogous. Finally, we use the same argument for the second exceptional set but now with the compositions

$$\alpha_n \to \mathcal{M}_n \to \beta_n \qquad \beta_n \to \mathcal{M}_{n-2}[-1] \to \alpha_n.$$

$\square$

Given a dualizable exceptional set generating a triangulated category $\mathcal{D}$, we get by [Bez03, Proposition 1] a unique t-structure such that $\nabla^i \in \mathcal{D}^{\geqslant 0}$ and $\Delta_i \in \mathcal{D}^{\leqslant 0}$. In our case, we denote the relevant triangulated categories by $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{M}}$.

Conjecture 4.3. *Under the categorical conjecture, we expect that the exceptional t-structure on $\mathcal{D}_{\mathcal{L}}$ corresponds to the* Hadal *t-structure and that the exceptional t-structure on $\mathcal{D}_{\mathcal{M}}$ corresponds to the Verdier dual of the Hadal t-structure.*

By [Bez03, Proposition 2], the irreducible elements in the heart of an exceptional t-structure are precisely the $\mathrm{im} H^0(\Delta_i \to \nabla_i)$. Our notation here is that H^0 is the cohomological functor valued in the heart and given by the truncations $\tau_{\leqslant 0} \circ \tau_{\geqslant 0}$. We now describe the irreducible objects of each t-structure.

We discuss $\mathcal{D}_{\mathcal{L}}$ first. From the Hom-tables, it is easy to check that $\mathcal{L}_{m+2}[1]$, δ_n, $\mathcal{L}_{-m+1}$, ϵ_0, ϵ_1 lie in the heart for $m \geqslant 0$ and each n. To compute $H^0(\epsilon_n)$ we look at the triangle

$$\mathcal{L}_n[-1] \to \mathcal{L}_{n+2}[1] \to \epsilon_n \to \mathcal{L}_n \to \mathcal{L}_{n+2}[2].$$

When $n \geqslant 2$, we can take H^0 everywhere and deduce that $H^0(\epsilon_n) \cong \mathcal{L}_{n+2}[1]$. When $n < 0$, we deduce that $H^0(\epsilon_n) \cong \mathcal{L}_n$. Thus the irreducible objects are as follows. We have $\mathcal{L}_0, \mathcal{L}_1$ are irreducible. We have short exact sequences in the heart

$$0 \to \mathcal{L}_0 \hookrightarrow \delta_0 \twoheadrightarrow \mathcal{L}_2[1] \to 0 \qquad 0 \to \mathcal{L}_2[1] \hookrightarrow \epsilon_0 \twoheadrightarrow \mathcal{L}_0 \to 0,$$

and

$$0 \to \mathcal{L}_1 \hookrightarrow \delta_1 \twoheadrightarrow \mathcal{L}_3[1] \to 0 \qquad 0 \to \mathcal{L}_3[1] \hookrightarrow \epsilon_1 \twoheadrightarrow \mathcal{L}_1 \to 0,$$

and hence $\mathcal{L}_2[1]$ and $\mathcal{L}_3[1]$ are irreducible. When $n < 0$, the map $H^0(\epsilon_n \to \delta_n)$ becomes $\mathcal{L}_n \to \delta_n$ which fits into an exact sequence in the heart:

$$0 \to \mathcal{L}_{n+2} \hookrightarrow \mathcal{L}_n \twoheadrightarrow \delta_n \to 0,$$

and hence δ_n is irreducible. When $n \geqslant 2$, $H^0(\epsilon_n) = \mathcal{L}_{n+2}[1]$ and so $H^0(\delta_n \to \epsilon_n)$ becomes $\delta_n \to \mathcal{L}_{n+2}[1]$ which fits into an exact sequence in the heart

$$0 \to \delta_n \hookrightarrow \mathcal{L}_{n+2}[1] \twoheadrightarrow \mathcal{L}_n[1] \to 0,$$

and hence δ_n is irreducible. In all, the irreducible objects are therefore $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2[1]$, $\mathcal{L}_3[1]$, δ_n for $n \neq 0, 1$.

We now study $\mathcal{D}_{\mathcal{M}}$. It is easy to check that $\mathcal{M}_{m-1}$, $\mathcal{M}_{-m-2}[-1]$, α_n, β_0, β_{-1} lie in the heart for $m \geqslant 0$ and each n. We compute that $H^0(\beta_n) \cong \mathcal{M}_n$ for $n \geqslant 1$ and $H^0(\beta_n) \cong \mathcal{M}_{n-2}[-1]$ for $n \leqslant -2$. Then for $n = 0, -1$, we have short exact sequences in the heart

$$0 \to \mathcal{M}_{-2}[-1] \hookrightarrow \alpha_0 \twoheadrightarrow \mathcal{M}_0 \to 0 \qquad 0 \to \mathcal{M}_0 \hookrightarrow \beta_0 \twoheadrightarrow \mathcal{M}_{-2}[-1]$$

and

$$0 \to \mathcal{M}_{-3}[-1] \hookrightarrow \alpha_{-1} \twoheadrightarrow \mathcal{M}_{-1} \to 0 \qquad 0 \to \mathcal{M}_{-1} \hookrightarrow \beta_{-1} \twoheadrightarrow \mathcal{M}_{-3}[-1].$$

Thus, $\mathcal{M}_{-2}[-1]$ and $\mathcal{M}_{-3}[-1]$ are irreducible. For $n \geqslant 1$ we have a short exact sequence in the heart

$$0 \to \alpha_n \hookrightarrow \mathcal{M}_n \twoheadrightarrow \mathcal{M}_{n-2} \to 0$$

and for $n \leqslant -2$, we have the the short exact sequence

$$0 \to \mathcal{M}_n[-1] \hookrightarrow \mathcal{M}_{n-2}[-1] \twoheadrightarrow \alpha_n \to 0.$$

We conclude the irreducible objects are $\mathcal{M}_0$, $\mathcal{M}_{-1}$, $\mathcal{M}_{-2}[-1]$, $\mathcal{M}_{-3}[-1]$, α_n for $n \neq 0, -1$. We have therefore proven

Proposition 4.4. *The irreducible objects in the heart of the exceptional t-structure on $\mathcal{D}_{\mathcal{L}}$ (resp. $\mathcal{D}_{\mathcal{M}}$) are $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2[1], \mathcal{L}_3[1], \delta_n$ for $n \neq 0, 1$ (resp. $\mathcal{M}_0$, $\mathcal{M}_{-1}$, $\mathcal{M}_{-2}[-1]$, $\mathcal{M}_{-3}[-1]$, α_n for $n \neq 0, -1$).*

5. The perverse t-structure

We now describe the t-structure that should conjecturally correspond to the perverse t-structure on Bun_G. The category $\mathcal{D}^{\leqslant 0}$ is generated by $\mathcal{L}_0[n]$, $\mathcal{L}_1[n]$, $\mathcal{L}_2[n+1]$, $\delta_{-m}[n]$, $\epsilon_m[n]$ for $m > 0$ and $n \geqslant 0$. The category $\mathcal{D}^{\geqslant 0}$ is generated by $\mathcal{M}_0[-n]$, $\mathcal{M}_{-2}[-n-1]$, $\mathcal{M}_{-1}[-n]$, $\alpha_m[-n]$, $\beta_{-m}[-n]$ for $m > 0$ and $n \geqslant 0$.

The following Hom-table is useful for computations. As usual, we always assume m and n have the same parity, and any "e" can be replaced with an "o".

$H^i(A_m, B_n)$	$\mathcal{M}_n$	$\mathcal{L}_e^+$	$\mathcal{L}_e^-$	α_n	β_n
$\mathcal{L}_m$	$m > n : H^{n-m+2} = k$ $m \leqslant n : H^0 = k$	$H^{-m} = k$	$H^0 = k$	$m > n : H^{n-m+1} = H^{n-m+2} = k$ $m \leqslant n : H^i = 0, \forall i$	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^1 = H^0 = k$
$\mathcal{L}_e^+$	$H^{n+2} = k$	$H^0 = k$	$H^i = 0, \forall i$	$H^{n+1} = H^{n+2} = k$	$H^i = 0, \forall i$
$\mathcal{L}_e^-$	$H^0 = k$	$H^i = 0, \forall i$	$H^0 = k$	$H^i = 0, \forall i$	$H^1 = H^0 = k$
δ_m	$m > n : H^{n-m+2} = H^{n-m+1} = k$ $m \leqslant n : H^i = 0, \forall i$	$H^{-m} = H^{-m-1} = k$	$H^i = 0, \forall i$	$m > n : H^{n-m} = H^{n-m+2} = k, H^{n-m+1} = k^2$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$
ϵ_m	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^1 = H^0 = k$	$H^i = 0, \forall i$	$H^1 = H^0 = k$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$	$m > n : H^i = 0, \forall i$ $m = n : H^1 = H^0 = k$ $m < n : H^2 = H^0 = k, H^1 = k^2$

We also record the following

$H^i(A_m, B_n)$	$\mathcal{M}_n$	α_n	β_n
δ_m^-	$m > n : H^0 = H^{n-m+1} = k$ $m \leqslant n : H^i = 0, \forall i$	$m \geqslant n : H^{m-n} = H^{m-n+1} = k$ $m < n : H^i = 0, \forall i$	$m \geqslant n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$
ϵ_m^+	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^0 = H^{n-m+1} = k$	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^{n-m} = H^{n-m+1} = k$	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^1 = H^0 = k$

and

$H^i(A_m, B_n)$	δ_n^-	ϵ_n^+
$\mathcal{L}_m$	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^0 = H^{n-m+1} = k$	$m > n : H^0 = H^{n-m+1} = k$ $m \leqslant n : H^i = 0, \forall i$
δ_m	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^{n-m+1} = H^{n-m} = k$	$m \geqslant n : H^{n-m+1} = H^{n-m} = k$ $m < n : H^i = 0, \forall i$
ϵ_m	$m > n : H^i = 0, \forall i$ $m \leqslant n : H^1 = H^0 = k$	$m \geqslant n : H^1 = H^0 = k$ $m < n : H^i = 0, \forall i$

We can now verify that a number of sheaves lie in the heart. In particular, $\mathcal{L}_e^-$, $\mathcal{L}_o^-$, $\mathcal{L}_e^+[-1]$, $\mathcal{L}_o^+[-1]$, $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{M}_0$, $\mathcal{M}_{-1}$, α_n, δ_{-n}, ϵ_n^+, δ_{-n}^- for $n \geqslant 0$ are all perverse. We can now describe the irreducible objects assuming Conjectures 2.4 and 3.4. We describe $H^0(\mathcal{L}_0 \to \mathcal{M}_0)$. To do so, we define the sheaf Z_e to be the pushout of $\mathcal{L}_2$ along the natural maps to $\mathcal{L}_0$ and $\mathcal{L}_e^+[-2]$, such that there is a triangle

$$\mathcal{L}_2 \xrightarrow{(1,-1)} \mathcal{L}_0 \oplus \mathcal{L}_e^+[-2] \to Z_e,$$

and the first map is given as the sum of $\mathcal{L}_2 \to \mathcal{L}_0$ and the canonical map $\mathcal{L}_2 \to \mathcal{L}_e^+[-2]$. Via the composition

$$\mathcal{L}_2 \to \mathcal{L}_0 \oplus \mathcal{L}_e^+[-2] \to \mathcal{L}_e^- \oplus \mathcal{L}_e^+[-2],$$

and an application of the octahedral axiom, we deduce a triangle

$$Z_e \to \mathcal{M}_0 \to \delta_{-2}^-.$$

from the definition of Z_e, and using (3.3), we deduce a triangle

$$\epsilon_2^+ \to \mathcal{L}_0 \to Z_e.$$

Since both δ_{-2}^- and ϵ_2^+ are perverse, this implies that Z_e is. We therefore deduce that the above triangles are short exact sequences in the heart. Hence we have maps

$$\mathcal{L}_0 \twoheadrightarrow Z_e \hookrightarrow \mathcal{M}_0,$$

and so $Z_e \cong \mathrm{im}H^0(\mathcal{L}_0 \to \mathcal{M}_0)$ and is therefore irreducible. There is another sheaf in the picture which is $\mathcal{L}_e^-$. First, we observe that the definition of $\mathcal{M}_0$ and (3.3), implies a triangle

$$\mathcal{L}_e^- \to \mathcal{M}_0 \to \epsilon_2^+[1].$$

Then, from the facts that δ_{-2}^- and ϵ_2^+ are perverse, we deduce that we have a sequence

$$\mathcal{L}_0 \hookrightarrow \mathcal{L}_e^- \twoheadrightarrow \mathcal{M}_0.$$

Remark 5.1. The sheaf $\mathcal{F}_e^-$ corresponding to $\mathcal{L}_e^-$ under the categorical conjecture will be a tilting extension of St along $i : \mathrm{Bun}_G^{b_0} \hookrightarrow \mathrm{Bun}_G$ in the sense that the restriction to $\mathrm{Bun}_G^{b_0}$ of $\mathcal{F}_e^-$ is St and the following canonical maps are respectively surjective and injective.

$$\mathcal{F}_e^- \twoheadrightarrow i_*i^*\mathcal{F}_e^- \qquad i_!i^*\mathcal{F}_e^- \hookrightarrow \mathcal{F}_e^-$$

See [BBM04, §1].

We now study $\mathrm{im}H^0(\mathcal{L}_1 \to \mathcal{M}_{-1})$. We define Z_o as the pushout of $\mathcal{L}_1$ along the maps to $\mathcal{L}_{-1}$ and $\mathcal{L}_o^+[-1]$. Hence we have a triangle

$$\mathcal{L}_1 \xrightarrow{(1,-1)} \mathcal{L}_{-1} \oplus \mathcal{L}_o^+[-1] \to Z_o.$$

One can check that the map $\mathcal{L}_1 \to \mathcal{M}_{-1}$ factors in both of the following ways:

$$\mathcal{L}_1 \to \mathcal{L}_{-1} \to \mathcal{L}_o^- \to \mathcal{M}_{-1},$$

and

$$\mathcal{L}_1 \to \mathcal{L}_o^+[-1] \to Z_o \to \mathcal{M}_{-1}.$$

We claim we have the following short exact sequences in the heart

$$0 \to \epsilon_1^+ \hookrightarrow \mathcal{L}_1 \twoheadrightarrow \mathcal{L}_o^+[-1] \to 0,$$
$$0 \to \mathcal{L}_1 \hookrightarrow \mathcal{L}_{-1} \twoheadrightarrow \delta_{-1} \to 0,$$
$$0 \to \epsilon_1^+ \hookrightarrow \mathcal{L}_o^- \twoheadrightarrow \mathcal{M}_{-1} \to 0,$$
$$0 \to \mathcal{L}_o^+[-1] \hookrightarrow Z_o \twoheadrightarrow \delta_{-1} \to 0,$$
$$0 \to Z_o \hookrightarrow \mathcal{M}_{-1} \twoheadrightarrow \delta_{-3}^- \to 0.$$

These imply that we have the following maps in the heart:

$$\mathcal{L}_1 \hookrightarrow \mathcal{L}_o^- \twoheadrightarrow \mathcal{M}_{-1},$$

and

$$\mathcal{L}_1 \twoheadrightarrow \mathcal{L}_o^+[-1] \hookrightarrow \mathcal{M}_{-1}.$$

Hence, the $\mathcal{L}_o^+[-1]$ is an IC-sheaf and $\mathcal{L}_o^-$ corresponds to a tilting extension.

We now study the map $\mathcal{L}_2[1] \to \mathcal{M}_{-2}[-1]$. This map factors through $\mathcal{L}_e^+[-1]$ and one can deduce that $H^0(\mathcal{L}_2[1] \to \mathcal{M}_{-2}[-1])$ is the identity map on $\mathcal{L}_e^+[-1]$.

Next we compute $H^0(\delta_{-n} \to \beta_{-n})$. The sheaf δ_{-n} is perverse and the sequence (3.7) implies that $H^0(\beta_{-n}) = \delta_{-n}^-$. Then (3.5) implies that δ_{-n} injects into δ_{-n}^-, so δ_{-n} is the IC-sheaf.

Finally we study $H^0(\epsilon_n \to \alpha_n)$. We deduce from (3.6) that $H^0(\epsilon_n) = \epsilon_n^+$. Then (3.8) implies that ϵ_n^+ surjects onto α_n and hence α_n is the IC-sheaf.

Proposition 5.2. *The irreducible objects in the heart of the perverse t-structure are Z_e, $\mathcal{L}_o^+[-1]$, $\mathcal{L}_e^+[-1]$, δ_{-n}, α_n. Additionally, $\mathcal{L}_o^-$ and $\mathcal{L}_e^-$ correspond to tilting extensions.*

References

[BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirkovic. *Tilting exercises.* 2004. arXiv: `math/0301098 [math.RT]`.

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