A Characterization of the Supercuspidal Local Langlands Correspondence

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Notation

- Fix $F/\mathbb{Q}_p$ finite.
- Fix G/F connected reductive (eg $G = GL_n, Sp_{2n}, U_n(E/F)$).
- Let $\widehat{G}$ denote the dual group of G over $\mathbb{C}$ ($\widehat{GL_n} = GL_n(\mathbb{C}), \widehat{Sp_{2n}} = SO_{2n+1}(\mathbb{C})$).
- Let W_F be the Weil group of F.
- Define the "L-group" of G to be ${}^L G := \widehat{G} \rtimes W_F$.

The Local Langlands Correspondence (LLC)

- Idea: Relates "nice" irreducible representations of $G(F)$ and "nice" finite dimensional representations of W_F valued in $\widehat{G}$.
- Simplest Case: LCFT = LLC for $\mathbb{G}_m$! The local Artin map

$$Art : W_F^{ab} \cong F^\times$$

induces a bijection

$$\left\{ \begin{matrix} \text{Continuous characters} \\ \text{of } \mathbb{G}_m(F) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Continuous homs} \\ W_F \to \mathbb{G}_m(\mathbb{C}) = \widehat{\mathbb{G}_m} \end{matrix} \right\}$$

General Case

- Exists finite to one map

$$R : \mathcal{A}_F(G) \to \mathcal{G}_F(G)$$

- $\mathcal{A}_F(G)$ the set of equivalence classes of irreducible smooth $G(F)$-representations
- $\mathcal{G}_F(G)$ equivalence classes of "L-parameters" : $\phi : W_F \times SL_2(\mathbb{C}) \to {}^L G$.
- Fibers $\Pi(\phi) := R^{-1}(\phi)$ called "L-packets".

Key Question of Talk: How to characterize R?

Our goal: Describe new characterization generalizing work of Scholze (2013).

The GL_n Case

- GL_n case is special: The local Langlands map $R : \mathcal{A}_F(G) \to \mathcal{G}_F(G)$ is a bijection.
- R constructed by Harris–Taylor, Henniart.
- Characterized by Henniart using L, ϵ factors.
- In 2013, Scholze gave a new characterization coming from geometry.

Beyond GL_n Case

- GSp_4 Gan–Takeda.
- Sp_{2n}, SO_{2n+1}, and SO_{2n}(almost) due to Arthur.
- Quasisplit $U_n(E/F)$ Mok.
- Inner forms of $U_n(E/F)$ Kaletha–Minguez–Shin–White.
- Inner forms of SL_n Hiraga–Saito.
- Supercuspidal case for "almost all" groups Kaletha.
- Characterization typically by compatibility with GL_n.

Scholze's Construction

- Given $\tau \in W_F^+$ and $h \in C^\infty(GL_n(\mathcal{O}_F))$ constructs $f_{\tau,h} \in C_c^\infty(GL_n(F))$.
- Constructs (via Shimura varieties) R for supercuspidals satisfying **Key Equation**

$$\mathrm{tr}(\pi \mid f_{\tau,h}) = \mathrm{tr}(R(\pi)|\cdot|^{\frac{1-n}{2}} \mid \tau)\mathrm{tr}(\pi \mid h).$$

- Extends to all of $\mathcal{A}_F(G)$ by proving compatibility with parabolic induction.

Scholze's Characterization in the Supercuspidal Case

- Suppose $R_1, R_2 : \mathcal{A}_F(G) \to \mathcal{G}_F(G)$ satisfy **Key Equation**:

$$\mathrm{tr}(\pi \mid f_{\tau,h}) = \mathrm{tr}(R_i(\pi)|\cdot|^{\frac{1-n}{2}} \mid \tau)\mathrm{tr}(\pi \mid h).$$

- Pick $\pi \in \mathcal{A}_F(GL_n)$ and $h \in C^\infty(GL_n(\mathcal{O}_F))$ such that $\mathrm{tr}(\pi \mid h) \neq 0$.
- We have

$$\mathrm{tr}(R_1(\pi)|\cdot|^{\frac{1-n}{2}} \mid \tau) = \frac{\mathrm{tr}(\pi \mid f_{\tau,h})}{\mathrm{tr}(\pi \mid h)} = \mathrm{tr}(R_2(\pi)|\cdot|^{\frac{1-n}{2}} \mid \tau).$$

- Implies $R_1(\pi) \sim R_2(\pi)$.

Work of Scholze–Shin

- Scholze–Shin (2011) extend construction of $f_{\tau,h}$ to unramified "PEL type" and get a function $f^{\mu}_{\tau,h} \in C_c^{\infty}(G(F))$ for each:
  - $\tau \in W_F^{+}$
  - $h \in C_c^{\infty}(\mathcal{G}(\mathcal{O}_F))$, (where $\mathcal{G}(\mathcal{O}_F)$ is hyperspecial)
  - $\mu \in X^*(\widehat{G})$ minuscule
- Youcis (thesis) defines $f^{\mu}_{\tau,h}$ in "Abelian type" cases.
- $f^{\mu}_{\tau,h}$ described by cohomology of tubular neighborhoods inside of Rapoport–Zink spaces.

Scholze–Shin Conjecture (No Endoscopy Case)

- Let $\phi : W_F \to {}^L G$ be a supercuspidal L-parameter, G unramified.
- Let $S\Theta_\phi :\approx \sum_{\pi \in \Pi(\phi)} \Theta_\pi$ be the "stable distribution of ϕ" $(\Theta_\pi(f) := \mathrm{tr}(\pi \mid f))$

- Conjecture (**Scholze–Shin Equation**)

  We have the following trace identity:

$$S\Theta_\phi(f^\mu_{\tau,h}) = \mathrm{tr}(r_{-\mu} \circ \phi| \cdot |^{-\langle \mu, \rho \rangle} \mid \tau) S\Theta_\phi(h).$$

- Known cases
  - EL, some PEL cases (Scholze, Scholze–Shin)
  - $G = D^\times$ appropriately interpreted (Shen)
  - Unramified $U_n(E/F)$ (BM, Youcis)

Hint of Proof

- Fix global group $\mathbf{G}/\mathbf{F}$ such that $\mathbf{G}_p = G$ and exists nice Shimura datum $(\mathbf{G}, X)$.
- Langlands–Kottwitz–Scholze method: for $\mathbf{K} \subset \mathbf{G}(\mathbf{F})$ compact

$$\mathrm{tr}(\tau \times f^p h \mid H^*(Sh_{\mathbf{K}})) = \sum SO(f^p f_\infty f^\mu_{\tau,h})$$

- Study of cohomology of Shimura varieties (Kottwitz and others) gives:

$$\sum \mathrm{tr}(\pi \mid f^p h)\mathrm{tr}(r_{-\mu} \circ \phi_\pi \mid \tau) \approx \mathrm{tr}(\tau \times f^p h \mid H^*(Sh_{\mathbf{K}}))$$

- Stable trace formula gives:

$$\sum SO(f^p f_\infty f^\mu_{\tau,h}) \approx \sum \mathrm{tr}(\pi \mid f^p f_\infty f^\mu_{\tau,h})$$

- "Localize at p" to get result.

Supercuspidal Parameters

- From now on, assume G quasisplit (for simplicity)
- L-parameter $\phi : W_F \times SL_2(\mathbb{C}) \to {}^LG$ *supercuspidal* if trivial on SL_2 part and doesn't factor through a Levi subgroup of ${}^LG$.
- Reasons for supercuspidal parameters:
  - Easy to work with (behaves well with elliptic endoscopy)
  - Can prove Scholze–Shin equations
  - Considered in literature (Kaletha, Scholze)
- Need "Backwards LLC"

$$\Pi : \left\{\begin{matrix}\text{Supercuspidal}\\ \text{L-Parameters}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Finite Subsets of}\\ \text{supercuspidal } G(F) \text{ reps}\end{matrix}\right\}$$

$$\phi \longmapsto \Pi(\phi)$$

Desired Properties

- **Dis**: $\Pi(\phi) \cap \Pi(\phi') \neq \varnothing$ implies $\phi \sim \phi'$.
- **Bij**: Each Whittaker datum $\mathfrak{w}$ gives a bijection

$$i_{\mathfrak{w}} : \Pi(\phi) \cong \mathrm{Irr}(\overline{C_\phi}),$$

  where $\overline{C_\phi} = Z_{\widehat{G}}(\mathrm{im}\phi)/Z(\widehat{G})^{\Gamma_F}$.
- **Stab**: $S\Theta_\phi := \sum_{\pi \in \Pi(\phi)} \dim(i_{\mathfrak{w}}(\pi))\Theta_\pi$ is stable.
- **SS**: Each ϕ satisfies the Scholze–Shin equations.

$$S\Theta_\phi(f^\mu_{\tau,h}) = \mathrm{tr}(r_{-\mu} \circ \phi| \cdot |^{-\langle \mu,\rho\rangle} \mid \tau) S\Theta_\phi(h).$$

- We will need to assume G is "good": If $\mathrm{tr}(r_\mu \circ \phi \mid \tau) = \mathrm{tr}(r_\mu \circ \phi' \mid \tau)$ for all μ, τ then $\phi \sim \phi'$.

Main Theorem(Imprecise Version)

Theorem (BM-Youcis)

For G a "good" reductive group, a supercuspidal LLC is characterized by **Dis**, **Bij**, **Stab**, **SS**, $+$ *compatibility with endoscopy.*

- **Dis**: Packets are disjoint.
- **Bij**: $i_{\mathfrak{w}} : \Pi(\phi) \cong \mathrm{Irr}(\overline{C_\phi})$
- **Stab**: $S\Theta_\phi$ is stable.
- **SS**: $S\Theta_\phi(f^{\mu}_{\tau,h}) = \mathrm{tr}(r_{-\mu} \circ \phi| \cdot |^{-\langle \mu,\rho \rangle} \mid \tau) S\Theta_\phi(h)$

Proof in the Singleton Packet Case

- Suppose Π_1, Π_2 are supercuspidal LLCs.
- Pick ϕ and suppose $\Pi_1(\phi) = \{\pi\}$ is a singleton.
- If we knew $\Pi_2(\phi') = \{\pi\}$ for some ϕ' then we could compare ϕ, ϕ' using **SS**.
- Need Atomic Stability: If $\Theta = \sum_i a_i \Theta_{\pi_i}$ is stable then Θ is a linear combination of $S\Theta_\phi$s.
- Do NOT need **AtomicStab** axiom (Thanks to Prof. Hiraga!)

Proof Assuming Atomic Stability

- Suppose $\Pi_1(\phi) = \{\pi\}$.
- By **Stab**, we have Θ_π is stable.
- By **AtomicStab** for Π_2 we have $\Pi_2(\phi') = \{\pi\}$ for some ϕ'.
- By **SS**:

$$\mathrm{tr}(r_\mu \circ \phi|\cdot|^{-\langle\mu,\rho\rangle} \mid \tau) = \frac{\mathrm{tr}(\pi \mid f^{\mu}_{\tau,h})}{\mathrm{tr}(\pi \mid h)} = \mathrm{tr}(r_{-\mu} \circ \phi'|\cdot|^{-\langle\mu,\rho\rangle} \mid \tau).$$

- Implies $\phi \sim \phi'$ since G is good.

Introduction to Endoscopy

- Elliptic endoscopic groups of G are auxiliary groups H with a map $\eta : {}^L H \to {}^L G$ and $s \in Z(\widehat{H})^{\Gamma_F}$.
- GL_n only elliptic endoscopic group of GL_n.
- Elliptic endoscopy of $U_n(E/F)$ of the form $U_{n_1}(E/F) \times U_{n_2}(E/F)$ with $n_1 + n_2 = n$.

A Useful Lemma

- Lemma
  If $\phi : W_F \to {}^L G$ is supercuspidal then there is a bijection:

$$\left\{ \begin{matrix} (H, \phi^H) \text{ with} \\ \eta \circ \phi^H = \phi \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{conjugacy classes} \\ \text{in } \overline{C_\phi} := Z_{\widehat{G}}(\mathrm{im}\phi)/Z(\widehat{G})^{\Gamma_F} \end{matrix} \right\}$$

  - In particular, ϕ factors through an elliptic endoscopic $\eta : {}^L H \to {}^L G$ iff $\overline{C_\phi} \neq 1$.
  - By **Bij**, we have $\Pi(\phi)$ a singleton iff ϕ does not factor through non trivial ${}^L H$.
  - Want to induct on dim G using endoscopy.

Elliptic Hyperendoscopy

- An elliptic hyperendoscopic datum is a sequence $({}^LH_1, s_1, \eta_1), ..., ({}^LH_n, s_n, \eta_n)$ so that $({}^LH_1, s_1, \eta_1)$ is an elliptic endoscopic datum for G and $({}^LH_i, s_i, \eta_i)$ an elliptic endoscopic datum for H_{i-1}.
- **ECI**: Let (H, s, η) an elliptic endoscopic datum for G and $f \in C_c^\infty(G(F))$, $f^H \in C_c^\infty(H(F))$ a pair of match of matching functions. Then

$$S\Theta_{\phi^H}(f^H) = \sum_{\pi \in \Pi(\phi)} \mathrm{tr}(i_{\mathfrak{w}}(\pi) \mid s)\Theta_\pi(f)$$

Supercuspidal LLC

- Definition

  A supercuspidal LLC for G is a map for each elliptic hyperendoscopic H:

$$\Pi_H : \left\{ \begin{array}{c} \text{Supercuspidal} \\ \text{L-parameters of } H \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Finite subsets of} \\ \text{supercuspidal } H(F) \text{ reps} \end{array} \right\}$$

- Theorem (BM – Youcis)

  Let G be such that each elliptic hyperendoscopic H is good. Suppose Π_1, Π_2 are supercuspidal LLCs such that $\bigcup_\phi \Pi_{1,H}(\phi) \subset \bigcup_\phi \Pi_{2,H}(\phi)$ for all H and $\Pi_{i,H}$ satisfy **Dis**, **Bij**, **Stab**, **SS**, *and* **ECI**. *Then $\Pi_{1,H} = \Pi_{2,H}$ for all H.*

- Groups with "good" elliptic hyperendoscopy: $PGL_n, GL_n, U_n, GU_n, SO_{2n+1}, G_2$.
- Groups with "bad" elliptic hyperendoscopy: Sp_{2n}, SO_{2n}, E_8.

- Corollary (BM – Youcis)

  LLC for $U_n(E/F)$ as in Mok is characterized by the above.

Sketch of inductive step

- Suppose we have proven that $\Pi_{1,H} = \Pi_{2,H}$ for all elliptic endoscopic H of G.
- Let ϕ be an L-parameter of G. If $\overline{C_\phi} = 1$, done by singleton packet case.
- Otherwise pick $\pi \in \Pi_{1,G}(\phi)$ and $1 \neq s \in \overline{C_\phi}$ such that $\mathrm{tr}(i_{\mathfrak{w}}(\pi) \mid s) \neq 0$ and get (H, ϕ^H) from lemma.
- By **ECI**

$$\sum_{\pi' \in \Pi_{1,G}(\phi)} \mathrm{tr}(i_{\mathfrak{w}}(\pi') \mid s)\Theta_{\pi'}(f) = S\Theta_{\phi^H}(f^H)$$

$$= \sum_{\pi' \in \Pi_{2,G}(\phi)} \mathrm{tr}(i_{\mathfrak{w}}(\pi') \mid s)\Theta_{\pi'}(f)$$

- Hence $\pi \in \Pi_{2,G}(\phi)$ by independence of characters.

Some Questions

- Can one show in a direct way that Kaletha's construction of LLC for supercuspidals satisfies **SS**?
- For GL_n we know this indirectly since Kaletha is compatible with Harris–Taylor (by Oi–Tokimoto) and Harris–Taylor is known to agree with Scholze.
- Can one define a useful version of **SS** that avoids the "good group" assumption? Perhaps this would look like Genestier-Lafforgue's characterization in terms of Bernstein center elements: $\{\mathfrak{z}_{I,f,(\gamma_i)_{i\in I}}\}$.