From Zero To Infinity

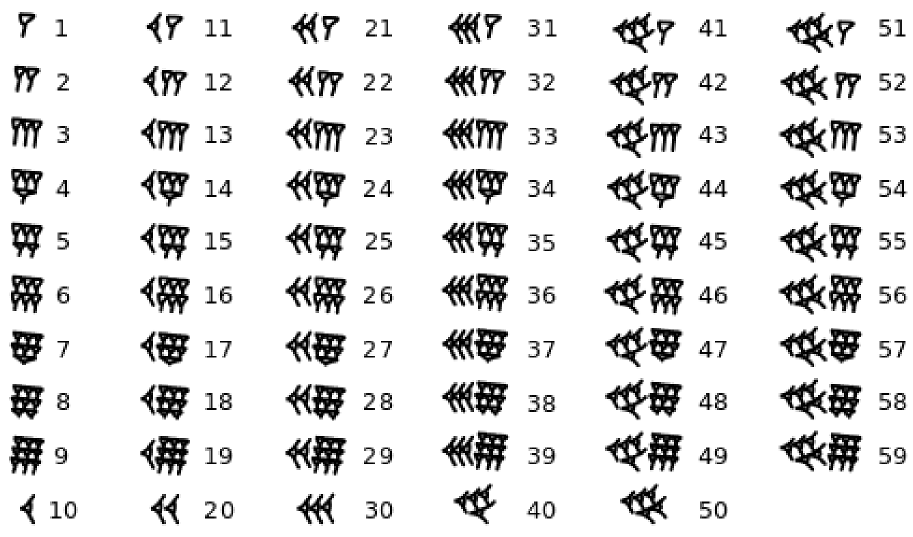
Aditya Karnataki

"To fall in love, is the greatest change in life." - Socrates, 435 BC

While it was my wish to open this article with a quotation from a master such as above, I am forced to admit that the quote is not at all factual. The thought in fact occurred to me while thinking about paradigm shifts. Being in love changes the perspective towards life in various ways. A paradigm shift could be described as such a fundamental change that frequently occurs by challenging the axioms in a field. To think of an example, both Ptolemy and Copernicus tried to understand the planetary motion by describing the planetary orbits. Both of them provided reasonable explanations of the phenomenon. But Ptolemy remained within the realm of the presiding thought, which was the assumption that planetary motion must be described by perfect circles; while Copernicus disposed of this axiom and used other geometrical shapes such as an elliptical orbit to describe the same. This introduced a new paradigm in the world of astronomy. Various streams of thought have experienced such shifts of paradigm, or revolutionary changes, over decades and centuries. I would like to summarize some of the paradigm shifts in Mathematics below.

Place Value and the Discovery of Zero

It is known that primitive humans had discovered methods of counting, one of them being to place marks on a tree bark to count animals in a herd. Evidence exists that humans knew how to count roughly 50000 years ago. While the concept of a number itself represents a big abstraction, here we focus on the subsequent problem of writing them down meaningfully. 'Babylonian numerals' is widely accepted to be the first such method. Established at 3000 BC, this method was based on a place value system, similar to the decimal system we use today. It used 60 as its base, that is, it had independent symbols for numbers from 1 to 59.



Babylonian Symbols

Being the first method to write numbers and also being the first one to use a form of place value system, it can be said that this represented a new paradigm for humanity. However, the Babylonians would leave empty space to represent 'zero'. But this introduced a unique problem. It was hard to understand where a number ended because of this empty space. For instance, the number 2 could be read as 2×60 or $2 \times 60 \times 60$. There were no rules to be found for arithmetic involving zero. Thus, while Babylonians understood the importance of zero as a place-holder, they did not think of it as a symbol or a concept. Similar observations could be made about other systems that followed such as the Greek method or the Roman numerals.

In such an environment, a paradigm shift was brought by Indian mathematicians. They recognized zero as a concept and assigned the symbol 0 to it. Brahmagupta is the first mathematician to state properties of zero as a number. $a + 0 = a, a - 0 = a, a \times 0 = 0, 0 - a = -a$ are some of the rules stated by Brahmagupta in his seminal work *Brahmasphutsiddhanta* in 638 AD [1]. Work of Aryabhatta is also of importance in this regard. His work on place value system proclaims, "*sthaanat sthaanam dashagunam syaat*" meaning "if you move a digit to the next place, its value gets multiplied by ten". Work of these mathematicians resulted in the first steps taken by the Hindu-Arabic numeral system, which was improved upon by Arabic mathematicians and which is the system the world uses today. These discoveries were essential to the development of modern mathematics. Their consequences are multidimensional and far-reaching.

Geometry : Euclid and Beyond

While this was happening in the world of numbers, other fields of mathematics weren't to be left behind. At around 300 AD, Euclid of Alexandria was active in mathematics. Euclid proved the Fundamental Theorem of Arithmetic in number theory. His work in geometry is perhaps the most celebrated mathematical book of all times. His book named 'Elements' is still influential. It was so comprehensibly ahead of other such books, that its predecessors were not preserved and are lost to the flow of Time. Some of his theorems were known to others before, but his monumental success lies in the unifying logical framework he brought to mathematics. The axiomatic Framework introduced by him showed the first instances of rigorous proof in mathematics. Thus Euclid brought a paradigm shift to the world of geometry. The postulates on which his geometry is based are well-known :

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the **parallel postulate** [2].

Other than these axioms, Euclid defines many geometrical objects. Aided with these definitions and axioms, he sets out to prove his theorems. These postulates are extremely minimal in nature. For instance, to draw a line segment having length equal to the length of a given line segment, we usually use the compass by putting both ends on the endpoints of the given segment and then lift it to draw on a different line. But this is not allowed in Euclid's world, as the compass is rendered useless as soon as both its ends leave the paper! Even so, it is entirely possible to perform this action only using the above postulates. The reader is invited to try this. Even if Euclid had included this as a postulate instead of proving it, not many would find issue with it. But Euclid maintained his insistence to take as axioms only statements that he truly found himself unable to prove. I like to think that he was in tune with the core principles of mathematics while doing this.

What are these core principles that I mention? I like to think of mathematics as a quest for truth rooted in beauty. Of course, the truth is most important. One cannot accept an incorrect proof just because it is beautiful to behold, but no matter how artificial, difficult to appreciate a proof might seem, it is acceptable if there is no logical flaw in it. But a mathematician's mind doesn't usually stop at finding proofs. It is her attempt to fill beautiful colors in the big picture of mathematics. Newer definitions and philosophies that aim to simplify a difficult proof or to

[1] Brahmagupta however made a mistake in stating the rules for division by zero. This in fact represents yet another paradigm shift in mathematics and I may write about it in a separate article.

[2] Only in the presence of first four postulates.

illuminate a fact in new light are sought out by mathematicians. These open new avenues of exploration. Euclid was one of the first such mathematicians, and hence I feel indebted to him.

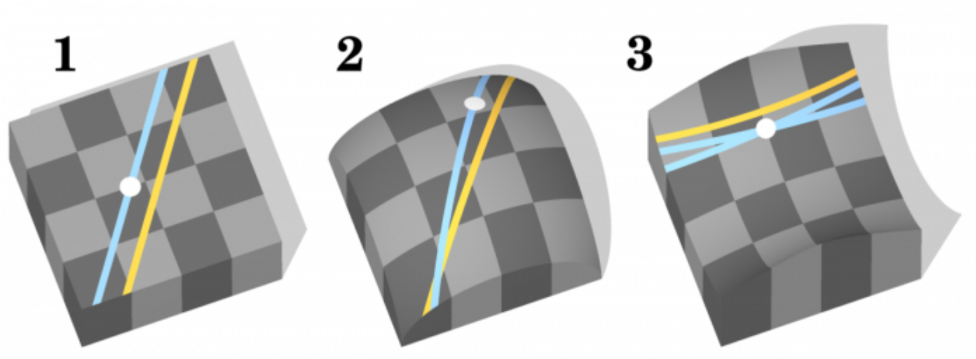
Euclid decided on these five postulates, but he wasn't satisfied. Indeed, the first four assertions are so simple and fundamental that it seems impossible to prove them out of nowhere. They feel self-evident. But the fifth one doesn't seem to be so at a first glance. Euclid tried a lot to prove it using the first four, but finally had to accept it as a fifth postulate. It involves passage to a distant infinity of a line and one is at a loss of explaining why such a statement might hold true.

Many mathematicians tried for centuries in vain to prove this postulate. An interesting factoid in this regard is that Omar Khayyam, who wrote the famous *rubaiyats*, was a mathematician and he also tried his hand at it. He in fact inadvertently proved some theorems from what was later to be known as non-euclidean geometry, but could not understand their significance and dismissed them.

Many equivalent formulations of the fifth postulate emerged out of these considerations. One important formulation which led to deeper understanding is as follows :

- Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended.

As mentioned before, this involves happenings at infinity and thus objections were ultimately raised by saying that perhaps there doesn't exist a parallel line or perhaps there are more lines possible. It might seem unbelievable, but these possibilities certainly occur on surfaces that are not plane. A diagram illustrating this is shown below :



The realization, that the aforementioned objections represent a gap in knowledge and not just a tiny logical snag, finally happened in the eighteenth and nineteenth century. Carl Friedrich Gauss, known as the Prince of Mathematicians thought about this question, but never published his subsequent work. In few years after him, mathematicians such as Janos Bolyai, Nikolai Lobachevsky, Bernhard Riemann, Henri Poincare contributed to discovery of 'Elliptical Geometry' and 'Hyperbolic Geometry' which are geometries that satisfy the first four postulates but not the fifth. These are examples of non-Euclidean geometry. The second figure in the above diagram represents elliptical geometry, while the third represents hyperbolic geometry. These do not prove Euclid's geometry 'wrong' as one might believe hastily, but instead show that Euclid's geometry does not work in every scenario. In this manner, these represent a paradigm shift. It is instructive to see how a paradigm shift (non-Euclidean geometry) was built out of a theory (Euclidean geometry) which by itself was a paradigm shift in its own right.

To top it all, Beltrami showed in 1868 that the fifth postulate is indeed independent of the first four which provided a worthy ending to a search conducted over centuries.

Mathematical Analysis and Rigor

As geometry and the theory of numbers were making great strides, the political and social landscape in Europe was undergoing great changes. The winds of renaissance were beginning to pick up, study of sciences such as physics was making progress. Rene Descartes had discovered his 'Analytical Geometry' which was a valuable tool for mathematicians and scientists alike. (In fact, a mathematician was a philosopher and a scientist most of the time, it was hard not to be and it still is.)

These events prompted mathematicians to search for Mathematics of Change. (How can I not write on Mathematics of Change in an article on Changes in Mathematics?) Since the time of ancient Greek mathematicians, questions about continuous processes were known, some of them illustrated beautifully in “Zeno’s Paradoxes”. Finally stalwarts such as Newton and Leibniz created Calculus and answered these questions. They studied infinitesimal quantities and essentially introduced the concept of a limit in this process. Roughly, the change in a system could essentially be described as a function and then one would study the effect on such a function by varying its input infinitesimally.

The concept of a limit seems very intuitive when stated in such a rough manner, but it is not at all rigorous. The idea of ’convergence’ forms a big part of this system, which is not defined. Though one may feel at ease with it by way of one’s intuition, frequently it is not enough to understand the mathematics of infinitesimal quantities. Many influential mathematicians grappled with this concept. Leonhard Euler has written down statements such as

$$1 - 1 + 1 - 1 + 1 - \cdots = 1/2 \tag{1}$$

and

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots = 1/4 \tag{2}$$

which seem nonsensical, but are only a result of not having defined the idea of convergence.

The beginning of understanding such concepts of ’infinity’ began with Bernhard Bolzano. He could be said to have founded the basis of Mathematical Analysis, a branch widely used in real world today. He rejected Leibniz’s definition of continuity which wasn’t precise enough and gave a new definition involving ’epsilon-delta’ which is used in modern mathematics. He proved many fundamental theorems in analysis. Unfortunately his work only became well-known much later, and even his definition wasn’t known to all.

After Bolzano, Augustin Cauchy followed suit. He discovered the same definition independently and proved many theorems using it. He also repaired proofs of old theorems which weren’t logically rigorous. He also could be said to have discovered ’Complex Analysis’, which relates to Complex Numbers. Cauchy is an influential figure in mathematics with many such as G. H. Hardy acknowledging his debt.

Even Cauchy had made some errors in understanding convergence. (Illustrative of the slippery slope that the concept still was!) Finally Karl Weierstrass filled these gaps. He defined the concept of ’Uniform Convergence’ which led to many deep theorems in analysis. He also discovered Bolzano’s work and gave deserved credit. He generalized some of Bolzano’s work.

These mathematicians thus changed perspectives and paved way for modern mathematics in the twentieth century. This was a big paradigm shift in mathematics.

Epilogue

The word ’revolutionary’ often includes the understanding that it is fast. The definition of ’fast’ is often reliant on context. A change in a person’s life may occur within a day, while a change in some business conglomerate may take a few years to happen ... But the changes we have seen seemingly have been happening for centuries. The audacity and creativity of the mathematicians who were the architects of these is of course to be honored with highest praise, and their importance cannot be overstated, and yet it is also important to remember that every mathematician on the way kept this flame burning. It is seldom possible to understand the boundaries of unexplored land without first having pushed all the way through the known territory to test our knowledge. As Newton famously said - “If I have seen further it is by standing on ye sholders of Giants.”

Those who were so gloriously part of the human quest for truth are immortals in the lore of mathematics. I wish to salute them and the human thirst for knowledge through this article.