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Levy and set theory

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Abstract

Azriel Levy (1934–) did fundamental work in set theory when it was transmuting into a modern, sophisticated field of mathematics, a formative period of over a decade straddling Cohen's 1963 founding of forcing. The terms "Levy collapse", "Levy hierarchy", and "Levy absoluteness" will live on in set theory, and his technique of relative constructibility and connections established between forcing and definability will continue to be basic to the subject. What follows is a detailed account and analysis of Levy's work and contributions to set theory.

Keywords: Levy collapse; Levy hierarchy; Levy absoluteness; Relative constructibility; Reflection principles; Ackermann's set theory; Urelements; Fraenkel–Mostowski method; Forcing; Measurable cardinals; Indescribable cardinals.

Azriel Levy (1934–) did fundamental work in set theory when it was transmuting into a modern, sophisticated field of mathematics, a formative period of over a decade straddling Cohen's 1963 founding of forcing. The terms "Levy collapse", "Levy hierarchy", and "Levy absoluteness" will live on in set theory, and his technique of relative constructibility and connections established between forcing and definability will continue to be basic to the subject. Levy came into his prime at what was also a formative time for the State of Israel and has been a pivotal figure between generations in the flowering of mathematical logic at the Hebrew University of Jerusalem.[1] There was initially Abraham Fraenkel, and then Yehoshua Bar-Hillel, Abraham Robinson, and Michael Rabin. With Levy subsequently established at the university in the 1960s there would then be Saharon Shelah, Levy's student and current university president Menachem Magidor, and Ehud Hrushovski, all together with a stream of students who would achieve worldwide prominence in set theory.

What follows is a detailed account and analysis of Levy's work and contributions to set theory. Levy's work has featured several broad, interconnected themes coincident with those in the pioneering work of Fraenkel: Axiom of Choice independences using urelements, the investigation of axioms and the comparative strengths of set theories, and the study of formal definability. To these Levy brought in concerted uses of model-theoretic reflection arguments

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[1] See Bentwich [49] for a history of the Hebrew University to 1960.

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and the method of forcing, and he thereby played an important role in raising the level of set-theoretic investigation through metamathematical means to a new height of sophistication.

Already in his M.Sc. work under Fraenkel's supervision, Levy [2,3] used urelement models to establish independences for several definitions of finiteness. Levy's ground-breaking 1958 Hebrew University dissertation, *Contributions to the metamathematics of set theory* (in Hebrew with an English summary), under the joint supervision of Fraenkel and Robinson featured work in three directions: relative constructibility, reflection principles, and Ackermann's set theory. Each of these is discussed in turn in the first three sections. Section 4 discusses Levy's hierarchy of formulas of set theory and his well-known absoluteness result. This work as well as the further development of the thesis topics were undertaken when Levy was a Sloan postdoctoral fellow at the Massachusetts Institute of Technology 1958–1959 and a visiting assistant professor at the University of California at Berkeley 1959–1961. At Berkeley Levy through shared interests associated particularly with Robert Vaught. From 1961 Levy had academic positions at the Hebrew University, eventually becoming professor of mathematics there while continuing to have active visiting positions in the United States.

In the early 1960s Levy focused on independence results, first with Fraenkel's urelement approach as developed by Mostowski, and as soon as it appeared, with Cohen's method of forcing. These are discussed in Sections 5 and 6, and in Sections 7 and 8 Levy's work on definability using forcing is presented. The latter section is focused on the Levy collapse of an inaccessible cardinal, and in Section 9 his further work on large cardinals, measurable and indescribable cardinals, is described. Levy put capstones to his work in different directions in the late 1960s, and these are taken up in Section 10. There remains an envoi.

In such an account as this it is perhaps inevitable that the more basic results are accorded more of an airing and the subsequent developments are summarily sketched, giving the impression that the latter are routine emanations. This is far from the case, but on the other hand a detailed analysis of basic results becomes a natural undertaking when discussing Levy's work, since one sees and wants to bring out how they have become part and parcel of our understanding and investigation of set theory.

1. Relative constructibility

Set theory was launched on an independent course as a distinctive field of mathematics by Gödel's work [63,64] on the inner model L of *constructible* sets. Not only did this work establish the relative consistency of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH), but it promoted a new relativism about the notion of set as mediated by first-order logic, which beyond its sufficiency as a logical framework for mathematics was seen to have considerable operational efficacy. Gödel's work however stood as an isolated monument for quite a number of years, the world war having a negative impact on mathematical progress. In retrospect, another inhibitory factor may have been the formal presentation of L in Gödel's monograph [65]. There he pointedly avoided the use of the satisfaction predicate and, following John von Neumann's lead, used a class theory to cast, in effect, definability. This presentation of L, however, tended to obfuscate the metamathematical ideas, especially the reflection Skolem closure argument for the GCH. Levy's work in general would serve to encourage the use and exhibit the efficacy of metamathematical methods in set theory.

In the next generation, András Hajnal, Joseph Shoenfield and Levy came to generalize Gödel's construction in order to establish conditional independence results. Their presentations would be couched in the formalism of Gödel [65], but the metamathematical ideas would soon become clear and accessible. Hajnal [73,74] in his Hungarian dissertation essentially formulated for a given set A the *constructible closure* $L(A)$, the smallest inner model M of ZF such that $A \in M$. To summarize, for any structure N and subset y of its domain, y is *definable over* N *iff* there is a (first-order) formula $\psi(v_0, v_1, v_2, \ldots, v_n)$ in the free variables as displayed and $a_1, a_2, \ldots, a_n$ in the domain of N such that

$$y = \{z \mid N \models \psi[z, a_1, \ldots, a_n]\},$$

where $N \models \psi[z, a_1, \ldots, a_n]$ asserts that the formula ψ is satisfied in N when v_0 is interpreted as z and each v_i is interpreted as a_i for $i \geq 1$. For any set x now define

$$\mathrm{def}(x) = \{y \subseteq x \mid y \text{ is definable over } \langle x, \in\rangle\}.$$

Finally, with $\mathrm{tc}(x)$ denoting the transitive closure of set x, define:

$$L_0(A) = \mathrm{tc}(\{A\});\ L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(A));\ L_\delta(A) = \bigcup_{\alpha<\delta} L_\alpha(A) \text{ for limit } \delta > 0;$$

and

$$L(A) = \bigcup_\alpha L_\alpha(A).$$

$L(\emptyset)$ is just Gödel's L, and the general construction starts instead with $L_0(A)$ as an "urelement" basis. Although $L(A)$ is indeed an inner model of ZF, unless $\mathrm{tc}(\{A\})$ has a well-ordering in $L(A)$, $L(A)$ does not satisfy the Axiom of Choice. This general situation was not broached by Hajnal, who used sets A of ordinals to establish the first relative consistency results about cardinal arithmetic after Gödel. Hajnal showed that in $L(A)$, if $A \subseteq \kappa^+$, then for every cardinal $\lambda \geq \kappa$, $2^\lambda = \lambda^+$, the case $\lambda = \kappa$ requiring some refinement of Gödel's original argument. In particular, if the Continuum Hypothesis (CH) fails, one can use an $A \subseteq \omega_2$ coding $\aleph_2$ sets of natural numbers and injections: $\alpha \to \omega_1$ for every $\alpha < \omega_2$ (so that $\omega_2^{L(A)} = \omega_2$ and $(2^{\aleph_0} = 2^{\aleph_1} = \omega_2)^{L(A)}$) to establish Con(ZFC $+ \neg$CH) implies Con(ZFC $+ \neg\mathrm{CH} + \forall\alpha \geq 1(2^{\aleph_\alpha} = \aleph_{\alpha+1})$), or more dramatically, if one can prove $2^{\aleph_0} \neq \aleph_2$ or $2^{\aleph_0} \neq 2^{\aleph_1}$, then one can prove the Continuum Hypothesis.

Levy [1,8] on the other hand developed for a given class A the inner model $L[A]$ of sets *constructible relative to* A, i.e. the smallest inner model M such that for every $x \in M$, $A \cap x \in M$. While $L(A)$ realizes the algebraic idea of building up a model starting from a basis of generators, $L[A]$ realizes the idea of building up a model using A construed as a predicate. Let

$$\mathrm{def}^A(x) = \{y \subseteq x \mid y \text{ is definable over } \langle x, \in, A \cap x\rangle\},$$

incorporating $A \cap x$ as a unary relation for definitions. Now define

$$L_0[A] = \emptyset;\ L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A]);\ L_\delta[A] = \bigcup_{\alpha<\delta} L_\alpha[A] \text{ for limit } \delta > 0;$$

and

$$L[A] = \bigcup_\alpha L_\alpha[A].$$

For sets A, unlike for $L(A)$ what remains of A is only $A \cap L[A] \in L[A]$, so that for example $L[\mathbb{R}] = L$ for the reals $\mathbb{R}$. $L[A]$ is more constructive since knowledge of A is incorporated through the hierarchy of definitions, and like L, $L[A]$ satisfies the Axiom of Choice for every A. Levy pointed out an important absoluteness, that with $\overline{A} = A \cap L[A]$, $L[A] = L[\overline{A}]$.

Shoenfield [111,112] had separately sketched a special case of this construction to establish partial results toward the conditional independence Con(ZFC $+ V \neq L$) implies Con(ZFC $+$ GCH $+ V \neq L$). Levy [8] then established this result by refining Shoenfield's argument to show that the full GCH can be proved in an appropriate $L[A]$.

The work of Hajnal and of Levy on constructibility elicited some interest in the correspondence between Paul Bernays and Gödel [71, 151–5].[2] Although differing in their formal presentations, since Hajnal and Levy both worked with sets A of ordinals so that $L[A] = L(A)$, distinctions would surface only later. Because of its intrinsic absoluteness, Levy's construction $L[A]$ would become basic for the inner model theory of large cardinals. Hajnal's construction $L(A)$ has also become basic, particularly with the constructible closure of the reals $L(\mathbb{R})$ becoming the focal inner model for the Axiom of Determinacy.

2. Reflection principles

Reflection has been an abiding motif in set theory, with its first appearance in a proof occurring in Gödel's proof of the GCH in L. Gödel himself saw roots in Russell's Axiom of Reducibility and in Zermelo's Axiom of Separation,

[2] In later correspondence, Bernays (Gödel [71, 199]) referred Gödel to a review of Levy [1] by Shepherdson [110], but amusingly this review is mistitled and is actually a review of another paper drawn from Levy's dissertation, Levy [4].

writing [66, 178] that: "...since an axiom of reducibility holds for constructible sets it is not surprising that the axioms of set theory hold for the constructible sets, because the axiom of reducibility or its equivalents, e.g., Zermelo's Aussonderungsaxiom is really the only essential axiom of set theory". A heuristic appearing early on for reflection was that any particular property attributable to the class of all ordinals, since its extent is not characterizable, should already be attributable to some cardinal. This heuristic was at work in the early postulations of large cardinals, cardinals that posit structure in the higher reaches of the cumulative hierarchy and prescribe their own transcendence over smaller cardinals. The weakest of the now standard large cardinals are the *inaccessible* cardinals, those uncountable regular cardinals κ such if $\alpha < \kappa$, then $2^\alpha < \kappa$, so that in the rank hierarchy V_κ models ZFC.

The formalization of reflection properties was one of the early developments of model-theoretic initiatives in set theory. With the basic concepts and methods of model theory being developed by Tarski and his students at Berkeley, Richard Montague [95] in his 1957 Berkeley dissertation had studied reflection properties in set theory and had shown that Replacement is not finitely axiomatizable over Zermelo set theory in a strong sense. Levy [9,10] then exploited the model-theoretic methods to establish the broader significance of reflection principles and the close involvement of the Mahlo hierarchy of large cardinals.

The ZF *Reflection Principle*, drawn from Montague [95, 99] and Levy [9, 234], asserts that for any (first-order) formula $\varphi(v_1, \ldots, v_n)$ in the free variables as displayed and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \text{ iff } \varphi^{V_\alpha}[x_1, \ldots, x_n],$$

where as usual φ^M denotes the relativization of the formula φ to M. The idea is to carry out a Skolem closure argument with the collection of subformulas of φ. Montague showed that the principle holds in ZF, and Levy showed that it is actually equivalent to the Replacement Schema together with the Axiom of Infinity in the presence of the other axioms of ZF. Through this work the ZF Reflection Principle has become well-known as making explicit how reflection is intrinsic to the ZF system.

Levy [9] took the ZF Reflection Principle as motivation for stronger reflection principles. The first in his hierarchy asserts that for any formula $\varphi(v_1, \ldots, v_n)$, there is an inaccessible cardinal α such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \text{ iff } \varphi^{V_\alpha}[x_1, \ldots, x_n].$$

Levy showed that this principle is equivalent to the assertion that the class of inaccessible cardinals is definably stationary, i.e. every definable closed unbounded class of ordinals contains an inaccessible cardinal. Paul Mahlo [90–92] had studied what are now known as the *weakly Mahlo* cardinals, those regular cardinals κ such that the set of smaller regular cardinals is stationary in κ, i.e. every closed unbounded subset of κ contains a regular cardinal. Levy's work thus established an evident connection between Mahlo's cardinals and structural principles about sets. Levy recast Mahlo's concept by replacing regular cardinals by inaccessible cardinals. On the other hand, whereas Mahlo had entertained arbitrary closed unbounded subsets, Levy's principle is restricted to definable closed unbounded classes. Be that as it may, it would be through Levy's work that Mahlo's cardinals would come into use in modern set theory cast as the *strongly Mahlo* cardinals, those regular cardinals κ such that the set of smaller *inaccessible* cardinals is stationary in κ.[3]

Levy proceeded to develop a hierarchy of reflection principles, the next principle being the one above with "inaccessible" replaced by "strongly Mahlo". Mahlo himself had developed a hierarchy of his cardinals, and Levy's work recast it as reflecting reflection: A reflection scheme is first formulated and is then itself reflected. In this way, Levy showed how the iterative formalization of reflection illuminates Mahlo's original scheme, formulated a half-century before.

Levy also substantiated how various reflection principles have proof-theoretic transcendence over each other. He had formulated the following, drawing on his dissertation work (cf. Levy [4]): For theories $T_0 \subseteq T$ in the same language and subsuming enough of arithmetic to encode formal consistency, T is *essentially reflexive* over T_0 if for any sentence σ, $T \vdash \sigma \to \mathrm{Con}(T_0+\sigma)$. This is an elegant formulation of the transcendence of one theory over another; note that no consistent extension of T is finitely axiomatizable over T_0, since for any σ, if $T_0 + \sigma$ were to extend T, we would have $T_0 + \sigma \vdash \mathrm{Con}(T_0 + \sigma)$. Montague [95] had shown in effect that ZF is essentially reflexive over

[3] See Kanamori [83, Sections 1, 6].

Z, Zermelo set theory. Levy [10] considered "partial" reflection principles weaker than the ZF Reflection Principle and studied [15] their minimal models of form V_α; Levy–Vaught [12] showed that these partial principles are also essentially reflexive over Z. In [11] Levy showed that each of the strong reflection principles in his [9] hierarchy à la Mahlo is essentially reflexive over the previous. Moreover, Levy [11] showed that between any two of these reflection principles there is a whole spectrum of theories each essentially reflexive over the previous.

In further ramifications, to a volume dedicated to Fraenkel on the occasion of his 70th birthday Levy contributed a paper [14] that compared the Axiom of Choice with its global form, i.e. there is a class choice function for all sets, and showed that the set consequences of the global form follows from his reflection principle down to inaccessible cardinals. The elder Bernays [50] also contributed a paper to that volume, one in which, inspired by Levy's work on reflection, he developed reflection principles based on second-order formulas which were seen to subsume all of Levy's principles.

The ZF Reflection Principle was foreshadowed in Gödel's remarks [67]; he there introduced the ordinal-definable sets, and to develop their theory requires reflection in some form (cf. Gödel [69, 146]). In his expository article on Cantor's Continuum Problem, Gödel [68, 521] mentioned the Mahlo cardinals in connection with the proposal to search for new large cardinal axioms that would settle the Continuum Hypothesis. Bernays cited the paper Levy [9] in a letter of 12 October 1961 to Gödel (Gödel [71, 196ff]), and Gödel noted in a letter of 13 August 1965 to Cohen (Gödel [71, 385ff]), in a discussion about evidence for inaccessible cardinals, that "Levy's principle might be considered more convincing than analogy [with the integers]". What presumably impressed Gödel was how reflection, a persistent heuristic in his own work, had been newly marshaled to account for Mahlo's cardinals. Finally, Gödel wrote in a letter of 7 July 1967 to Robinson (Gödel [72, 195]):

> . . . I perhaps stimulated work in set theory by my epistemological attitude toward it, and by giving some indications as to the further developments, in my opinion, to be expected and to be aimed at. I did not, however, give any clues as to how these aims were to be attained. This has become possible only due to the entirely new ideas, primarily of Paul J. Cohen and, in the area of axioms of infinity, of the Tarski school and of Azriel Levy.

3. Ackermann's set theory

Ackermann [45] formulated a distinctive axiomatic theory of sets and classes, and this theory quickly came under the scrutiny of Levy whose extended analysis did a great deal to bring it into the fold of the standard ZF axiomatization. Much of the analysis was already present in Levy's dissertation and was subsequently extended in Levy [5] and Levy–Vaught [12]. Gödel wrote to Bernays in a letter of 30 September 1958 (Gödel [71, 155]): "Of the results announced in the introduction to Levy's dissertation, the most interesting seems to me to be that on Ackermann's system of set theory. That really looks very surprising".

Ackermann's theory A is a first-order theory that can be cast as follows: There is one binary relation $\in$ for membership and one constant V; the objects of the theory are to be referred to as classes, and members of V as sets. The axioms of A are the universal closures of:

(1) Extensionality: $\forall z(z \in x \leftrightarrow z \in y) \longrightarrow x = y$.
(2) Comprehension: For each formula ψ not involving t, $\exists t \forall z(z \in t \longleftrightarrow z \in V \wedge \psi)$.
(3) Heredity: $x \in V \wedge (t \in x \vee t \subseteq x) \longrightarrow t \in V$.
(4) Ackermann's Schema: For each formula ψ in free variables $x_1, \ldots, x_n, z$ and having no occurrence of V, $x_1, \ldots, x_n \in V \wedge \forall z(\psi \rightarrow z \in V) \longrightarrow \exists t \in V \forall z(z \in t \leftrightarrow \psi)$.

This last, a comprehension schema for sets, is characteristic of Ackermann's system. It forestalls Russell's Paradox, and its motivation was to allow set formation through properties independent of the whole extension of the set concept and thus to be considered sufficiently definite and delimited.

Ackermann [45] himself argued that every axiom of ZF, when relativized to V, can be proved in A. However, Levy [5] found a mistake in Ackermann's proof of the Replacement Schema, and whether Replacement can be derived from Ackermann's Schema would remain an issue for some time. Toward a closer correlation with ZF, Levy came to the idea of working with

$$A^*\text{: A together with the Axiom of Foundation relativized to } V.$$

As for ZF, Foundation focuses the sets with a stratification into a cumulative hierarchy. Levy [5] showed that, leaving aside the question of Replacement, A* establishes substantial reflection principles. On the other hand, he also showed through a sustained axiomatic analysis that for a sentence σ of set theory (so without V) : *If σ relativized to V is provable in* A*, *then σ is provable in* ZF. The thrust of this work was to show that Ackermann's Schema can be assimilated into ZF — this is presumably what Gödel found surprising — and that ZF and A* have almost the same theorems for sets.

Levy–Vaught [12] later observed by an inner model argument that, as for ZF and Foundation, if A is consistent, then so is A*. They then went on to confirm that the addition of Foundation to A was substantive; they showed that Ackermann's Schema is equivalent to a reflection principle in the presence of the other A^* axioms, and that A* establishes the existence of $\{V\}$ and the power classes $\mathcal{P}(V)$, $\mathcal{P}(\mathcal{P}(V))$, and so forth.

Years later, returning to the original issue about Replacement, William Reinhardt [103] in his 1967 Berkeley dissertation under the supervision of Vaught built on Levy–Vaught [12] to establish for A* what Ackermann could not establish for A: *Every axiom of* ZF*, when relativized to V, can be proved in* A*. Thus, A* and ZF do have exactly the same theorems for sets. Reinhardt also developed a theory of natural models of A*; these are connected to the indescribable cardinals (see Section 9) and led to further large cardinal postulations.[4]

4. Levy hierarchy and absoluteness

In his first work in a distinctive direction from his dissertation, Levy in [6], and much later in full exposition [28], formulated the now standard hierarchy of first-order formulas of the language of set theory. He showed that the hierarchy provides the scaffolding for an efficacious analysis of logical complexity, getting to a substantial absoluteness result that cast reflection in a new light.

For formulating his hierarchy Levy struck on the key idea of discounting *bounded* quantifiers, those that can be rendered as $\forall v \in w$ or $\exists v \in w$, an idea perhaps novel at the time in set theory but now subsumed into its modern sensibilities. There was an antecedent in the discounting of the bounded numerical quantifiers $\forall k < n$ and $\exists k < n$ in Stephen Kleene's [86] formulation of the arithmetical hierarchy over the recursive predicates, but the motivations were rather different, and Levy had to make a conceptual leap because of the arbitrariness of sets.

In brief, a formula of set theory is Σ_0 and Π_0 in the Levy hierarchy if its only quantifiers are bounded. Recursively, a formula is Σ_{n+1} if it is of the form $\exists v_1 \ldots \exists v_k \varphi$ where φ is Π_n, and Π_{n+1} if it is of the form $\forall v_1 \ldots \forall v_k \varphi$ where φ is Σ_n. The classification of definable concepts in this hierarchy depends on the governing theory. For a set theory T, a formula φ is Σ_n^T *iff* for some Σ_n formula φ', $T \vdash \varphi \leftrightarrow \varphi'$; and similarly for Π_n^T. Σ_n^{ZF} and Π_n^{ZF} formulas are equivalent to formulas with blocks of like quantifiers contracted into one through applications of the Pairing Axiom. Also, bounded quantification does not add to complexity in ZF: If φ is Σ_n^{ZF} (respectively, Π_n^{ZF}), then so is $\exists v \in w\varphi$ and $\forall v \in w\varphi$. This depends on Replacement to "push" the bounded quantifiers to the right and is a crucial point about the Levy hierarchy. Finally, that Σ_0^T formulas are wide-ranging yet absolute for transitive models of weak set theories T has become a basic feature of the semantic analysis of set theory.

Levy [6] pointed out that his hierarchy is proper in ZF, i.e. there are formulas in $\Pi_n^{\mathrm{ZF}} - \Sigma_n^{\mathrm{ZF}}$ and in $\Sigma_n^{\mathrm{ZF}} - \Pi_n^{\mathrm{ZF}}$ for every $n > 0$, and that ZF establishes the consistency, for any particular n, of Zermelo set theory plus Replacement restricted to Σ_n formulas. Levy [25,28] worked out for each $n > 0$ a Σ_n (respectively, Π_n) satisfaction formula for the Σ_n (respectively, Π_n) formulas and thereby got careful hierarchy results. The antecedent was the Σ_n^0 universal predicate for the Σ_n^0 predicates in the Kleene arithmetical hierarchy, built directly on the normal forms of recursive predicates. Levy laid out satisfaction sequences à la Tarski level-by-level, once again drawing metamathematical methods into set theory.

The main advance of Levy [6] was a now well-known and basic absoluteness result. Shoenfield [113] had established an absoluteness result seminal for modern descriptive set theory; he showed that, as we now say, every $\mathbf{\Sigma}_2^1$ set of reals is ω_1-Suslin in a constructible way, and concluded in particular that every (lightface) Σ_2^1 set of natural numbers is in L. As detailed in [28] Levy wove in the Shoenfield idea to establish in ZF together with the Axiom of Dependent Choices (DC) that any sentence (without parameters) Σ_1 in his hierarchy, if holding in V, also holds in L. More formally, we have the *Shoenfield–Levy Absoluteness Lemma*:

4 See Kanamori [83, Section 23].

$$\text{For any } \Sigma_1 \text{ sentence } \sigma, \text{ZF + DC} \vdash \sigma \leftrightarrow \sigma^L.$$

Levy readily concluded that any Σ_1 or Π_1 theorem of ZF + $V = L$ is already a theorem of ZF + DC, so that any uses of e.g. GCH in a proof of such a sentence can be eliminated. Levy [6] actually pointed out that L can be replaced by a countable L_γ fixed for all σ, so that any Σ_1 sentence is absolute for every transitive $M \supseteq L_\gamma$.[5] Levy's proof, starting with a Σ_1 sentence σ, first appealed to DC to reflect down from the universe to a countable transitive model of σ. He then got a countable L_α modeling σ by applying Shoenfield's main idea of relying on the absoluteness of well-foundedness, i.e. the equivalence of no infinite descending chains and the existence of a ranking. One can view the Shoenfield and Levy absoluteness results as two sides of the same coin, one in the context of descriptive set theory and the other in the context of general set theory, with either one readily leading to the other.

The Shoenfield–Levy Absoluteness Lemma can be seen as an effective refinement of the ZF Reflection Principle that reflects a restricted sentence down to some countable L_γ, and as such it would find wide use in effective contexts like admissible set theory. Even just Levy's initial reflection down, in effect into the domain of hereditarily countable sets, would become basic to admissible set theory as the *Levy Absoluteness Principle*. In his book on admissible set theory Barwise [47, 77] wrote: "One of the main features of this book (at least from our point of view) is the systematic use of the Levy Absoluteness Principle to simplify results by reducing them to the countable case".

5. Independence with urelements

From the beginning Levy had a steady interest in the independence of choice principles and in the pre-Cohen era established penetrating results based on the Fraenkel–Mostowski method. To establish the independence of AC, Fraenkel had come to the fecund idea of starting with urelements, objects without members yet distinct from each other; building a model of set theory by closing off under set-theoretic operations; and exploiting automorphisms of the model induced by permutations of the urelements. Fraenkel [59] in one construction started with urelements $A = \{a_n \mid n \in \omega\}$ and considered a generated model in which for any set x there is a finite $s \subseteq A$ with the following property: x is fixed by any automorphism of the model induced by a permutation of A that fixes each member of s and at most interchanges pairs within the cells $\{a_{2n}, a_{2n+1}\}$ for $n \in \omega$. There can then be no choice function for the countable set of pairs $\{\{a_{2n}, a_{2n+1}\} \mid n \in \omega\}$ in the model, since for any purported such function one can take some a_{2n}, a_{2n+1} not in its support and apply a permutation interchanging them.

Andrzej Mostowski [97] developed Fraenkel's constructions by imposing algebraic initiatives. First, the set of urelements can be structured e.g. with an ordering; second, a model is built based on invariance with respect to a specified *group* of permutations, group in the algebraic sense with respect to composition; and third, *supports* of sets are closely analyzed, a support of a set to be a set of urelements such that if a permutation fixes each, the induced automorphism also fixes the set. Mostowski in particular established that there is a model in which AC fails but the *Ordering Principle* (OP) holds, where:

(OP) Every set can be linearly ordered.

He began with a countable set of urelements ordered in the ordertype of the rationals; built a model based on the group of order-preserving permutations; established that every set has a $\subseteq$-least finite support; and showed by these means that the set of urelements cannot be well-ordered yet the model itself has a class linear ordering. Levy's initial work [2,3,7] in this direction was directly based on Mostowski's model for OP.

Coming into his own, Levy [16] constructed a model in which C_n holds for every natural number n yet $\mathrm{C}_{<\aleph_0}$ fails, where

(C_κ) Every set consisting of sets of cardinality $\leq \kappa$ has a choice function

($\mathrm{C}_{<\kappa}$) Every set consisting of sets of cardinality $< \kappa$ has a choice function

and moreover the *Axiom of Multiple Choice* (MC) holds, where

(MC) For any set x there is a function f on x such that for any non-empty $y \in x$, $f(y)$ is a non-empty finite subset of y.

[5] In modern terms, γ can be taken to be the least stable ordinal, where δ is *stable iff* $L_\delta \prec_1 L$, i.e. L_δ and L satisfy the same Σ_1 formulas with parameters from L_δ.

Levy began with a countable set of urelements A partitioned as $\bigcup_{k\in\omega} P_k$, where $P_k = \{a_1^k, \ldots, a_{p_k}^k\}$ with p_k the kth prime. Let π_k be that permutation of A fixing every member of $A - P_k$ and such that $\pi_k(a_i^k) = a_{i+1}^k$ for $1 \le i < p_k$ and $\pi(a_{p_k}^k) = a_1^k$. Levy then took the group of permutations generated by the π_k and generated a model based on finite supports. As in the Fraenkel model described above, the set $\{P_k \mid k \in \omega\}$ does not have a choice function so $C_{<\aleph_0}$ fails. That MC holds Levy affirmed with an argument also applicable to the Fraenkel model. Finally, the specifics of Levy's model came into play when he showed with algebraic arguments about cyclic permutations that C_n holds for every natural number n. That OP implies $C_{<\aleph_0}$ is simple to see, so that OP must fail in Levy's model. Hans Läuchli [88] built another model in which $C_{<\aleph_0}$ holds yet OP fails.

Notably, it was later observed that in ZF, MC actually implies AC.[6] Thus, Levy's work shows that having urelements can separate these principles. Levy [17] subsequently applied his [16] model in a considerable analysis of some graph-theoretic propositions studied by Mycielski. Also, Levy [26] developed transfinite versions of his algebraic methods and argued e.g. that $C_{<\kappa}$ does not imply C_κ for limit alephs κ.[7] However, this seems to be the single instance when Levy was proved wrong, but even the error stimulated results.[8]

Levy [22] established independence results for various choice principles indexed by alephs κ (cf. Jech [81, 119ff]):

(DC_κ) Suppose that x is a set and r a binary relation such that for every $\alpha < \kappa$ and $s : \alpha \to x$ there is a $y \in x$ satisfying $s\ r\ y$. Then there is a function $f : \kappa \to x$ such that $f|\alpha\ r\ f(\alpha)$ for every $\alpha < \kappa$.
(AC_κ) Every set x with $|x| = \kappa$ has a choice function.
(W_κ) Every set x is comparable with κ, i.e. $|x| \le \kappa$ or $\kappa \le |x|$.

With DC_κ Levy generalized the Axiom of Dependent Choices, which is $DC_{\aleph_0}$. W_κ generalizes the proposition that every infinite set has a countable subset, which is $W_{\aleph_0}$.[9] DC_κ implies both AC_κ and W_κ. Levy was expanding on work of Mostowski [98], who showed that there is a model satisfying $\neg AC_{\aleph_1} + DC_{\aleph_0}$ (and, as noticed later, $\neg W_{\aleph_1}$).

After drawing implications among these principles for various κ, Levy established several independences. He constructed a basic model by starting with λ urelements, considering all permutations, and working with supports of cardinality $< \lambda$. For a singular cardinal, taking $\lambda = \aleph_{\omega_1}$ as a typicality one gets $\neg AC_{\aleph_1} + DC_{\aleph_0} + \forall\kappa < \aleph_{\omega_1}(W_\kappa) + \neg W_{\aleph_{\omega_1}}$. For a successor cardinal, taking $\lambda = \aleph_1$ one gets:

$$\forall\kappa(AC_\kappa) + DC_{\aleph_0} + \neg W_{\aleph_1} \text{ (and so } \neg DC_{\aleph_1}).$$

In particular, Well-ordered Choice $\forall\kappa(AC_\kappa)$ does not imply $DC_{\aleph_1}$. Jensen [82] later established the surprising result $\forall\kappa(AC_\kappa)$ implies $DC_{\aleph_0}$, that Well-ordered Choice actually implies Dependent Choices.

Levy constructed an interesting, second model, assuming $2^{\aleph_0} = \aleph_1$ and starting with a set of urelements ordered in the ordertype of the reals. He then used the group of order-preserving permutations and supports generated by the "Dedekind cuts" $(-\infty, r)$ to get:

$$\forall\kappa(AC_\kappa) + \neg DC_{\aleph_1} + W_{\aleph_1} + \neg W_{\aleph_2}.$$

Thus, $DC_{\aleph_1}$ and $W_{\aleph_1}$ were separated.

Already in the work on conditional independence results via relative constructibility there was an air of anticipation about possible independences from ZF. This became palatable in the work on choice principles by Fraenkel–Mostowski methods with, e.g. Levy [22, 145] writing: "Even the independence of the axiom of choice itself is still an open problem for systems of set theory which do not admit urelements or non-founded sets. Thus we can hope, for the time being, to prove the above mentioned independence results only for a set theory which admits either urelements or non-founded sets". Of course, independence with respect to ZF was what was really wanted, and this would come about in a dramatic turn of events.

6 See Jech [81, 133]; the observation was first made by David Pincus in his 1969 Harvard dissertation. Working in the post-Cohen era, Pincus's results there (cf. Pincus [101]) also showed how to "transfer" Levy's result to get the consistency relative to ZF of C_n holds for every natural number n and $C_{<\aleph_0}$ fails.

7 Levy [26] was a summary of Fraenkel–Mostowski methods given at a 1963 symposium; in that summary (p. 225) Levy pointed out the open problem of whether $2 \cdot \mathfrak{m} = \mathfrak{m}$ for all infinite cardinals $\mathfrak{m}$ implies AC, and eventually his student Gershon Sageev [104] established that it does not.

8 Paul Howard [79] later established that in every Fraenkel–Mostowski model, $C_{<\aleph_0}$ already implies "C_∞", i.e. that every set consisting of well-orderable sets has a choice function. Pincus [102] and Sageev [104] independently established the ZF independence of $C_{\aleph_0}$ from $C_{<\aleph_0}$.

9 Actually, Levy [22] worked with a more involved proposition $H(\kappa)$, which has the property that W_κ is equivalent to $H(\kappa)$ together with $\forall\lambda < \kappa(W_\lambda)$.

6. ZF independence

Paul Cohen [52,53] in 1963 established the independence of AC from ZF and the independence of the CH from ZFC. That is, complementing Gödel's relative consistency results with L Cohen established that Con(ZF) implies Con(ZF + ¬AC) and that Con(ZFC) implies Con(ZFC + ¬CH). These were, of course, the inaugural examples of *forcing*, a remarkably general and flexible method, with strong intuitive underpinnings, for extending models of set theory. If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's method of forcing began its transformation into a modern, sophisticated one. Cohen's particular achievement lies in devising a concrete procedure for extending well-founded models of set theory in a minimal fashion to well-founded models of set theory with new properties but without altering the ordinals. Set theory had undergone a sea-change, and beyond simply how the subject was enriched, it is difficult to convey the strangeness of it.[10]

Cohen circulated a preprint [51] in April and soon started to give talks on his results. Already evident in the preprint were two features of forcing that resonated with Levy's work: Cohen relied on relative constructibility to extend a model of $V = L$ to one satisfying GCH + $V \neq L$, and Cohen used in effect a reflection argument to show that his beginning with a countable $\in$-model of ZF was formally unnecessary for getting consistency results relative to ZF. Levy first heard the details of Cohen's results at a July model theory conference at Berkeley, and later that summer fully assimilated forcing working with Solomon Feferman. Feferman had been extensively consulted by Cohen at Stanford when he was coming up with forcing, and Feferman [56,57] was the first after Cohen to establish results by forcing. In quick succession several abstracts appeared in the *Notices of the American Mathematical Society*: Levy [18], Feferman–Levy [19], Levy [20], and Levy [21], all received September 3, 1963 and soon after, Halpern–Levy [23], which was actually noted as received August 29, 1963. With this work Levy became the first after Cohen himself to exploit forcing in a sustained fashion to establish a series of significant results. We attend to the AC independence results through the rest of this section; the others, on definability, were seeds for papers which are discussed in the next section. Throughout, we assume familiarity with forcing and give only scant descriptions of the models.

Cohen had shown how to "collapse" a cardinal, i.e. adjoin a generic bijection to a smaller ordinal. Feferman–Levy [19] started with a model of ZFC + GCH and adjoined separate generic bijections between $\aleph_n$ and $\aleph_0$ for every natural number n. In the resulting generated model the former $\aleph_\omega$ becomes the new $\aleph_1$ so that it is singular, and moreover the reals are a countable union of countable sets. $\mathrm{AC}_{\aleph_0}$ thus fails, and it is seen in a drastic way how that principle is necessary to develop Borel and Lebesgue measure. In Cohen's own exposition [55, 143] the Feferman–Levy model was presented in some detail.

James Halpern [75,76] in his 1962 Berkeley dissertation[11] had shown that the *Boolean Prime Ideal Theorem* (BPI) holds in the Mostowski [97] model for OP, where:

(BPI) Every Boolean algebra has a prime ideal.

The significance here is that BPI had become a focal choice principle, one that implies OP. Levy [17] had dealt with BPI, observing that it fails in his [16] model. Halpern's argument required a new sophistication, with a Ramsey-type partition theorem being brought into play.

When Cohen's work appeared, Halpern–Levy [23] showed that in the original Cohen model for the independence of the AC, OP and in fact the *Kinna–Wagner Selection Principle* (KW) holds, where:

(KW) For any set x there is a function f such that whenever $y \in x$ has at least two elements, $f(y)$ is a non-empty proper subset of y.

Kinna–Wagner [85] had formulated KW, a weak form of MC which they showed equivalent to: For any set x there is an ordinal α and a bijection between x and a subset of $\mathcal{P}(\alpha)$. KW implies OP, and though Halpern–Levy [23] emphasized the new independence of AC from OP, there are two observations about their getting KW which are worth mentioning: As noted earlier, the stronger MC implies AC in ZF. Also, KW was effectively unattainable through Fraenkel–Mostowski methods, as Mostowski [99] had observed that KW fails in every Fraenkel–Mostowski model in which the urelements are not well-orderable.

[10] See Moore [96] on the origins of forcing and for what follows.

[11] Halpern [76] wrote that his dissertation was written under the "supervision of" Levy; Levy visited Berkeley in the academic years 1959–1961.

Cohen's original model for the independence of AC was the result of adjoining countably many Cohen reals and the set x consisting of these, so that x has no well-ordering in the resulting model. Halpern–Levy [23] in effect argued in the Cohen model with the Cohen reals acting like urelements. Cohen [54, 40] moreover acknowledged the similarities between his AC independence result and the previous Fraenkel–Mostowski models. In any case, the revelatory Halpern–Levy work initiated the process of "transferring" consistency results with Fraenkel–Mostowski models to ZF consistency results via forcing by correlating urelements with generic sets.

For the BPI, there was no routine transfer of the Halpern [76] independence of AC from BPI. Levy saw the need for a strengthened, "tree" Ramsey-type partition theorem to effect a ZF independence result. Halpern–Läuchli [77] then duly established this result. Finally, Halpern–Levy [35] by 1966 had established that in Cohen's original model, BPI holds. This Halpern–Läuchli–Levy collaboration established a new level of sophistication in effecting a transfer from the Fraenkel–Mostowski context to the Cohen one. Work at this level would soon be pursued by Jech, Sochor, Pincus, and others, and the Halpern–Läuchli partition theorem would lead to an important extension by Richard Laver [89], one also applied to forcing.

7. ZF definability

The abstracts Levy [18,20,21] had to do with formal definability, and the papers Levy [27,34] provided extended accounts in a context of appropriate generality for the proofs. Levy probed the limits of ZFC definability, establishing consistency results about definable sets of reals and well-orderings and in descriptive set theory.

Heralded by Levy [18], Levy [27] established the relative consistency of ZFC + GCH together with there being a non-constructible real yet every definable set is constructible. Here, "definable" meant the broad notion of hereditarily ordinal-definable. A set x is *ordinal-definable iff* there is a formula $\psi(v_0, \ldots, v_n)$ in the free variables as displayed and ordinals $\alpha_1, \ldots, \alpha_n$ such that $x = \{y \mid \psi[y, \alpha_1, \ldots, \alpha_n]\}$. A set x is *hereditarily ordinal definable iff* the transitive closure of $\{x\}$ is ordinal definable. The ordinal-definable sets were introduced by Gödel [67] as mentioned in Section 2, and their theory was developed by John Myhill and Dana Scott by 1964 (cf. Myhill–Scott [100]) with explicit appeal to the ZF Reflection Principle, with which one can replace the informal satisfaction of $x = \{y \mid \psi[y, \alpha_1, \ldots, \alpha_n]\}$ by: for some V_α and ordinals $\alpha_1, \ldots, \alpha_n \in V_\alpha$, $x = \{y \in V_\alpha \mid \psi^{V_\alpha}[y, \alpha_1, \ldots, \alpha_n]\}$. OD denotes the (thus definable) class of ordinal-definable sets, and HOD, the class of hereditarily ordinal-definable sets.

Feferman [56,57] had shown that in Cohen's model which is the result of starting from a model of $V = L$ and adjoining a Cohen real there is no set-theoretically definable well-ordering of the reals, i.e. no formula in two free variables that defines such a well-ordering. Levy [27] showed that the model actually satisfies $V \neq L = \mathrm{HOD}$, so that in particular there is no definable well-ordering of the reals even if ground model parameters are allowed in the definition. The crux is that the partial order of conditions for adjoining a Cohen real is *homogeneous* in the sense that for any pair p, q of conditions there is an automorphism e of the partial order such that p and $e(q)$ are compatible. Hence, for a formula $\varphi(v_1, \ldots, v_n)$ of the forcing language and sets $x_1, \ldots, x_n$ in the ground model, any condition forces $\varphi(\check{x}_1, \ldots, \check{x}_n)$ exactly when all conditions do. That $\mathrm{HOD} = L$ follows by induction on rank. Levy's appeal to homogeneity and automorphism, related to his earlier work with urelements and realigned by the work of Cohen and Feferman, would become a basic motif that connects forcing with definability.

In an eventual sequel, Levy [34] worked out his main delimitative results. He first considered Cohen's model which is the result of starting with a model of $V = L$ and collapsing ω_1^L, i.e. adjoining a generic bijection between ω and ω_1^L. As before one has $V \neq L = \mathrm{HOD}$, but this now easily implies that *every* OD *well-ordering of reals is at most countable*. This confirmed an announcement in Levy [18].

With his next theorem Levy [34] provided an important delimitation for descriptive set theory, confirming an announcement in Levy [20]. Classical descriptive set theory, in its probing into the first levels of the projective hierarchy, had pushed against the limits of axiomatic set theory.[12] Levy presumably had assimilated this work in large part from John Addison at Berkeley (cf. Addison [46]), but in any case Levy quickly saw how to apply forcing to illuminate the central issue of uniformization. For binary relations A and B,

A is *uniformized* by B *iff*

$$B \subseteq A \wedge \forall x(\exists y(\langle x, y\rangle \in A) \longrightarrow \exists! y(\langle x, y\rangle \in B)).$$

[12] See Kanamori [83, Sections 12–14] for the background and basic concepts of descriptive set theory.

$\exists!$ abbreviates the formalizable "there exists exactly one", and so this asserts that A can be refined to a function B. That every relation can be uniformized is a restatement of the Axiom of Choice. A high point of classical descriptive set theory was the result of Motokiti Kondô [87] that, in terms of the projective hierarchy, every $\mathbf{\Pi^1_1}$ relation on reals can uniformized by a $\mathbf{\Pi^1_1}$ relation. This implied via projection that every $\mathbf{\Sigma^1_2}$ relation on reals can be uniformized by a $\mathbf{\Sigma^1_2}$ relation, and by looking at complements that not every $\mathbf{\Pi^1_2}$ relation on reals can be uniformized by a $\mathbf{\Pi^1_2}$ relation. Whether every $\mathbf{\Pi^1_2}$ relation on reals can be uniformized by a projective relation had remained open. Bringing in axiomatics Addison [46] established that assuming $V = L$, for any $n \geq 2$ every $\mathbf{\Sigma^1_n}$ relation on reals can be uniformized by a $\mathbf{\Sigma^1_n}$ relation, so that in particular every $\mathbf{\Pi^1_2}$ relation on reals, being $\mathbf{\Sigma^1_3}$, can be uniformized by a $\mathbf{\Sigma^1_3}$ relation. In contradistinction, Levy established the relative consistency of there being a $\mathbf{\Pi^1_2}$ on reals that cannot be uniformized by any projective relation.

Levy considered Cohen's model which is the result of starting with a model of $V = L$ and adjoining (a sequence of) ω_1^L Cohen reals. Taking the cue from ordinal-definability, say that a set x is *real-ordinal-definable iff* there is a formula $\psi(v_0, \ldots, v_n, v_{n+1})$ in the free variables displayed, a V_α, a real $r \in V_\alpha$, and ordinals $\alpha_1, \ldots, \alpha_n \in V_\alpha$ such that $x = \{y \in V_\alpha \mid \psi^{V_\alpha}[y, \alpha_1, \ldots, \alpha_n, r]\}$. That is, a real parameter is to be allowed in the definition. Levy considered the relation A on reals defined by: $\langle f, g\rangle \in A$ *iff* $g \notin L[f]$. This relation, formulated in terms of his notion of relative constructibility, is $\mathbf{\Pi^1_2}$ by an elaboration of an argument in Addison [46]. Suppose now that B is a real-ordinal-definable relation on reals. By a definability argument the real parameter r in the definition can be taken to be one coding countably many of the Cohen reals. There is then an s such that $\langle r, s\rangle \in A$. If however there were an s such that $\langle r, s\rangle \in B$, then s, like B, would be real-ordinal-definable with real parameter r. But then, a homogeneity argument shows that $s \in L[r]$. Consequently, no real-ordinal-definable relation, and consequently no projective relation, can uniformize A.

8. Levy collapse

The theory of large cardinals was revitalized by pivotal results in the early 1960s, and with Cohen's forcing large cardinals would enter the mainstream of set theory by providing hypotheses and methods to analyze strong set-theoretic propositions. Levy's earlier work on reflection principles had established a central place for Mahlo cardinals; in the post-Cohen era Levy made basic contributions to the fast growing theory of large cardinals.

In the last of the 1963 abstracts Levy [21], Levy announced a result that depended on what we now call the Levy collapse. In general terms, for infinite regular cardinals $\lambda < \kappa$, $\mathrm{Col}(\lambda, \kappa)$ is the partial order for adjoining a κ-sequence of surjections $\lambda \to \alpha$ for $\alpha < \kappa$. If κ is inaccessible, then $\mathrm{Col}(\lambda, \kappa)$ has the κ-chain condition[13]; hence, κ becomes the successor λ^+ of λ in the generic extension. The forcing with $\mathrm{Col}(\lambda, \kappa)$ is then called a *Levy collapse* of κ to λ^+. Already in the first flush of forcing Levy [21,34] used the Levy collapse of an inaccessible cardinal to ω_1 to establish the relative consistency of:

(∗) Every real-ordinal-definable well-ordering of reals is at most countable.

Forcing ω_1 to be countable had led to the consistency of every OD well-ordering of reals being at most countable; forcing every α below an inaccessible to be countable provides enough closure to achieve (∗). Levy also considered the proposition:

(∗∗) Every real-ordinal definable set of reals is either countable or of cardinality $2^{\aleph_0}$.

The Levy collapse of an inaccessible to ω_1 entails CH so that (∗∗) is vacuous, but Levy showed that in any further extension where many Cohen reals adjoined (∗∗) continues to hold, this already inherent in the early abstract [21].

In deploying an inaccessible cardinal Levy was a pioneer in taking the modern approach to large cardinals: They are not novel hypotheses burdened by ontological commitment but are the repository of means for carrying out mathematical arguments. Cohen [55, 147] acknowledged this use of an inaccessible cardinal. As pointed out by Levy [34, 131–2], either (∗) or (∗∗) implies that the ω_1 (of the universe) is an inaccessible cardinal in the sense of L. So, Levy's work was party to the first instance of an important phenomenon in set theory, the derivation

[13] Actually, that $\mathrm{Col}(\lambda, \kappa)$ has the κ-chain condition only requires, by a so-called delta-system argument, that $\alpha < \kappa$ implies that $\alpha^{<\lambda} < \kappa$. Full inaccessibility is typically required in other parts of an argument using the Levy collapse.

of equiconsistency results based on the complementary methods of forcing and inner models.[14] After this heady introduction the Levy collapse would become standard fare in the theory of large cardinals, the way to render a large cardinal accessible yet still with substantial properties to establish the relative consistency of strong combinatorial propositions low in the cumulative hierarchy.

Levy's model was used by Robert Solovay to establish a now famous relative consistency result. Solovay played a prominent role in the forging of forcing as a general method, and he above all in this period raised the level of sophistication of set theory across its breadth from forcing to large cardinals. Already in the spring of 1964 Solovay [116,117] exposed what standard of argument was possible when showing that if an inaccessible cardinal is Levy collapsed to ω_1, *every real-ordinal-definable set of reals is Lebesgue measurable*, and proceeding to the corresponding inner model HROD of the hereditarily real-ordinal-definable sets, that

$$\text{HROD} \models \text{DC} + \text{“Every set of reals is Lebesgue measurable”}.$$

That this model satisfies Dependent Choices bolsters it as a *bona fide* one for mathematical analysis. Solovay thus illuminated the classical measure problem of Henri Lebesgue with the modern technique of forcing. Solovay also showed that the reals in this HROD model have several other substantial properties, one being the perfect set property: *Every set of reals is countable or else has a perfect subset.* This refined Levy's result with $(**)$ above, any perfect set of reals having cardinality $2^{\aleph_0}$, and established the equiconsistency of the perfect set property with the existence of an inaccessible cardinal.[15] For quite some years, it was speculated that an inaccessible cardinal can be avoided for getting all sets of reals to be Lebesgue measurable. However, in 1979 Shelah [108] established that DC + "Every set of reals is Lebesgue measurable" implies that ω_1 is inaccessible in L, vindicating Solovay's use of the Levy collapse for the Lebesgue measurability result.

Solovay [117, 2] announced a joint result with Levy which eventually appeared in Levy–Solovay [38]. Levy and Solovay built on the structure uncovered by Solovay in the Levy collapse model to establish a further "regularity" property about sets of reals, being the union of an $\aleph_1$-sequence of Borel sets. In the classical investigation of the projective hierarchy, though the second level $\mathbf{\Sigma^1_2}$ seemed complicated, Sierpiński [114] had established that *every* $\mathbf{\Sigma^1_2}$ *set of reals is the union of* $\aleph_1$ *Borel sets.* Of course, if CH holds, then every set of reals is the union of $\aleph_1$ Borel sets, namely the singletons of its members. It came to light with the emergence of Martin's Axiom that ¬CH together with the converse of the Sierpiński result, that *every union of* $\aleph_1$ *Borel sets is* $\mathbf{\Sigma^1_2}$, is relatively consistent (Martin–Solovay [93, Section 3]).

Considering arbitrary sets of reals, Levy and Solovay first noted with a simple construction via transfinite recursion that ZFC + ¬CH implies that there is a set of reals which is not the union of any $\aleph_1$ Borel sets, and then went on to show that some substantial use of AC is necessary. As with Levy's $(**)$, Solovay [117] had shown that, with V_1 the Levy collapse model and V_2 a further extension where many Cohen reals are adjoined, the propositions about Lebesgue measurability and so forth for real-ordinal-definable sets that hold in V_1 continue to hold in V_2, thereby establishing the further relative consistency of having a large continuum. Levy–Solovay [38] showed that in any such V_2, every real-ordinal-definable set of reals is the union of $\aleph_1$ Borel sets. Specifically, such a set of reals is the union of Borel sets coded in V_1, and with CH holding in V_1, there are only $\aleph_1$ such Borel sets. Thus, Levy and Solovay established the relative consistency of ZFC + ¬CH + "Every real-ordinal-definable sets of reals is the union of $\aleph_1$ Borel sets". Passing to the inner HROD model Levy and Solovay then established the relative consistency of DC + "Every well-ordering of reals is at most countable" + "Every set of reals is the union of $\aleph_1$ Borel sets". In this way the property of being the union of $\aleph_1$ Borel sets was adjoined to the regularity properties of sets of reals illuminated by the Levy collapse model.

9. Measurable and indescribable cardinals

In addition to their collaborative work on the Levy collapse model, Levy [24] and Solovay [115] independently established a general result about large cardinals that would become much cited in connection with the Continuum Problem. As is well-known, Gödel [68, 520] had speculated that large cardinal postulations might decide CH. He himself

[14] See Kanamori [83, 135ff].

[15] Solovay [117, 45ff] wrote: "Our proof that every [real-ordinal-definable] subset of [the reals] is countable or contains a perfect subset is, essentially, a slight refinement of the following result of Levy [34]: Every uncountable [real-ordinal-definable] subset of [the reals] has power $2^{\aleph_0}$". Compare Levy [34, 140ff].

took CH to be false and made remarks amounting to the observation that those large cardinals consistent with $V = L$ cannot disprove CH. Addressing this issue, Levy–Solovay [29] showed that measurable cardinals κ remain measurable in "mild" forcing extensions, those via partial orders of cardinality $< \kappa$. That this betokened what would become a widely applied observation, that inaccessible large cardinals retain their characteristic properties in mild forcing extensions, is often regarded as springing from Levy–Solovay [29] though they themselves wrote that this was well-known for many large cardinals. At the time there was a particular point in that Scott [106] had dramatically established that the existence of a measurable cardinal contradicts $V = L$, and Levy–Solovay [29] pointed out that measurable cardinals, though loosened from the moorings of constructibility, cannot decide CH and other issues like Suslin's Hypothesis. In years to come, the growing success of the theory of large cardinals led to more and more allusions to "Gödel's Program". The Levy–Solovay result would be consistently cited as a watershed, if only to point to a delimitation to be superseded by other, more subtle invocations of large cardinals in connection with the Continuum Problem.

Levy's last major contribution to the theory of large cardinals dealt with natural extensions of his earlier reflection principles (cf. Section 2) set in a higher order context of Π^m_n and Σ^m_n formulas.[16] William Hanf and Scott in their abstract [78] considered higher-order reflection properties for structures $\langle V_\kappa, \in, R\rangle$ where κ is a cardinal and $R \subseteq V_\kappa$ and thereby provided a hierarchical scheme for large cardinals. For Q either Π^m_n or Σ^m_n,

κ is Q-indescribable iff for any $R \subseteq V_\kappa$ and Q sentence φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.

Including $R \subseteq V_\kappa$ suffices to bolster the concept to accommodate general relational structures; as V_κ is closed under pairing, the definition is equivalent to one where R is replaced by any finite number of finitary relations.[17] Hanf–Scott [78] observed that, with π^m_n the least Π^m_n-indescribable cardinal and σ^m_n the least Σ^m_n-indescribable cardinal, for $m > 0$: $\pi^m_n < \pi^m_{n+1}$ and if $n > 0$, then π^m_n is not Σ^m_n-indescribable. They also pointed out that the Π^1_1-indescribable cardinals are exactly the weakly compact cardinals and that measurable cardinals are Π^2_1-indescribable. This provided probably the earliest proof that below a measurable cardinal there are many weakly compact cardinals. Vaught [119] subsequently pointed out that below a measurable cardinal there is cardinal Π^m_n-indescribable for every $m, n \in \omega$. The evident connection between Levy's earlier reflection principles and the indescribable cardinals had an interconnecting node in the work of Bernays [50], who had extended Levy's principles by positing, in effect, the Π^1_n-indescribability for every $n \in \omega$ of the class of all ordinals.

Levy [36] carried out a systematic study of the sizes of indescribable cardinals, extending aspects of a combinatorial study of large cardinals in Keisler–Tarski [84]. The starting point of Levy's approach was that various large cardinal properties are not only attributable to cardinals, but to their subsets. For $X \subseteq \kappa$ and Q either Π^m_n or Σ^m_n,

X is *Q-indescribable in κ iff* for any $R \subseteq V_\kappa$ and Q sentence φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, there is an $\alpha \in X$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.

This leads to the consideration of $\{X \subseteq \kappa \mid \kappa - X$ is not Q -indescribable in $\kappa\}$, which when κ is Q-indescribable is a (proper) filter, the *Q-indescribable filter over κ*.[18]

Using universal satisfaction formulas Levy showed that these definable filters have a crucial property: *For $m, n > 0$ and Q either Π^m_n or Σ^m_n, the Q-indescribable filter over κ is normal.* Recall that a filter F over a cardinal κ is *normal iff* it is closed under diagonal intersections, i.e. whenever $\{X_\alpha \mid \alpha < \kappa\} \subseteq F$, $\{\xi < \kappa \mid \xi \in \bigcap_{\alpha<\xi} X_\alpha\} \in F$. The previously known normal filters were the closed unbounded filters over regular uncountable cardinals and normal

16 Higher-order languages have typed variables of every finite type (or order), quantifications of these, and beyond the atomic formulas specified by the language, $X \in Y$ and $X = Y$ for any typed variables X and Y. In the intended semantics, if D is the domain of a structure, type 1 variables play the usual role of first-order variables, type 2 variables range over $\mathcal{P}(D)$, and generally, type $i + 1$ variables range over $\mathcal{P}^i(D)$ where $\mathcal{P}^i$ denotes i iterations of the power set operation. A formula is Π^m_n *iff* it starts with a block of universal quantifiers of type $m + 1$ variables, followed by a block of existential quantifiers of type $m + 1$ variables, and so forth with at most n blocks in all, followed afterward by a formula containing variables of type at most $m + 1$ and quantified variables of type at most m. A formula is Σ^m_n *iff* it starts instead with existential quantifiers. Of course, formulas containing only type 1 variables can be construed as the usual first-order formulas. This classification of formulas is cumulative because of the "at most": any Π^m_n or Σ^m_n formula is also Π^r_s and Σ^r_s for any $r > m$, or $r = m$ and $s > n$.

17 Hanf–Scott [78] formulated their concept for inaccessible κ and with κ in place of V_κ, but the difference is inessential as the R can code V_κ.

18 Levy himself called the members of this filter *weakly Q-enforceable* at κ, but we follow the formulation in Kanamori [83, Section 6].

ultrafilters found over a measurable cardinal. Levy further established that $\{\alpha < \kappa \mid \alpha$ is P-indescribable in $\alpha\}$ is in the Q-indescribable filter over κ, where: (a) P is Π^1_n and Q is Π^1_{n+1}; or (b) $m > 1$ and $n > 0$, P is Σ^m_n, and Q is Π^m_n; or (c) $m > 1$ and $n > 0$, P is Π^m_n, and Q is Σ^m_n. These various results showed in concert that, just as normal ultrafilters over a measurable cardinal provide intrinsic senses to how large measurable cardinals are as had been shown in Keisler–Tarski [84], e.g. normal ultrafilters are closed under Mahlo's Operation, so too do indescribable cardinals have inherent transcendence over smaller cardinals, specifically those in the indescribable hierarchy itself. The technique of ascribing large cardinal properties of cardinals also to their subsets has become an important part of large cardinal theory and has been used in particular by James Baumgartner [48] to establish important hierarchical results about the n-subtle and n-ineffable cardinals.

10. Capstones

In the later 1960s, Levy capped his investigations in various directions with papers reflective of previous themes and techniques but also distinctive in how they resolve basic issues in axiomatics.

Levy's last result applying the Fraenkel–Mostowski method concerned Cantor's very notion of cardinality. The problem of cardinal definability in set theory is how to assign to every set x a set $|x|$ such that for every x, y we have $|x| = |y|$ *iff* $x \approx y$, i.e. there is a bijection between x and y. Of course, with AC the initial (von Neumann) ordinals construed as Cantor's alephs serve as such $|x|$. But even without AC, one can use the "trick" of Scott [105] to formulate $|x|$ as the set of sets of least rank bijective with x. Levy [32], in a 1966 conference proceedings, established that relative to ZF, ZF − Foundation + "There is no set-theoretic term τ such that for every x, y we have $\tau(x) = \tau(y)$ *iff* $x \approx y$" is consistent in several strong senses.[19] In this distinctive setting without Foundation the interplay between urelement constructions and forcing is not pertinent. It had been known for over a decade that the Fraenkel–Mostowski method with urelements can be recast, following Specker [118] and Mendelson [94], in ZF − Foundation with sets $a = \{a\}$ in the role of urelements. Levy ultimately relied on this recasting, but worked directly with urelements and automorphisms. In one model he used $\aleph_\omega$ urelements and generated an inner model in which they become a proper class; in another, he proceeded similarly but started from Mostowski's model for OP.[20] With the appearance of the forcing method, one might have thought that Fraenkel–Mostowski methods would be superseded, but in the years to come, there would be a continuing cottage industry investigating Fraenkel–Mostowski models as intrinsically interesting constructions in their own right.[21]

Levy and Georg Kreisel in their [30] provided a detailed exposition that established a central place for proof-theoretic reflection principles in the comparative investigation of theories. Levy brought together his work on the transcendence of theories through set-theoretic reflection principles, and the inimitable Kreisel, whose hand is evident in the sections with the many italicizations, brought to bear his initiatives in the proof theory of arithmetic and analysis. The main unifying motif was the proof-theoretic *Uniform Reflection Principle*:

$$\text{(URP(S))} \qquad \forall p \forall n (\mathrm{Prov}_S(p, s(\ulcorner\varphi\urcorner, n)) \longrightarrow \varphi(n)).$$

Intended for theories sufficient to carry out Gödel numbering, $\mathrm{Prov}_S(x, y)$ is to assert that x is the Gödel number of a proof in the theory S of the formula with Gödel number y; $s(\ulcorner\varphi\urcorner, n)$ is the Gödel number of the sentence obtained by substituting the numeral of the natural number n for the one free variable of φ; and finally, the uniformity has to do with having the parametrization with the numerical variable n. URP(S) is an assertion of soundness; instantiating to a φ refutable in S, URP(S) implies that something is not provable and hence the formal consistency of S. As Kreisel–Levy [30, Section 1] pointed out, URP(S) actually subsumes both the assertion of ω-consistency for S and a general form of induction.

Kreisel and Levy established a strong, general result about how URP(S) leads to transcendence over S in terms of quantifier complexity: *If* U *is a theory in the same language as* S *and* URP(S) *is a theorem of* U, *then for no set* Σ *consisting of sentences of bounded quantifier complexity are the theorems of* U *provable in* S + Σ. For set theories,

19 This was also done, in a strong sense, by Robert Gauntt [61].

20 Pincus [102] later addressed the issue of cardinal *representatives*, i.e. having a set-theoretic term τ such that for every x, y we have $\tau(x) = \tau(y)$ *iff* $x \approx y$, and moreover $|x| \approx x$. Transferring from Mostowski's model for OP Pincus showed that relative to ZF it is consistent to have ZF + "There are no cardinal representatives". Rather surprisingly, he also showed by an iterated forcing argument that relative to ZF it is consistent to have ZF + ¬AC + "There are cardinal representatives".

21 See Howard–Rubin [80].

Σ_n in the Levy hierarchy typifies a set of formulas of bounded complexity. Levy [6] had already announced a result that implied that for any natural number n, no consistent extension of ZF can be obtained by adjoining to Zermelo set theory any set of Σ_n sentences. As noted, various results from Levy's previous work [4,9,11], which had the above form except that the Σ was a finite set of sentences, could now be strengthened. The general result was moreover applicable to Peano Arithmetic, Second-Order Arithmetic (Analysis), and the like to show that these theories cannot be axiomatized over weaker theories using any set of axioms of bounded quantifier complexity.

In the last sections, Kreisel–Levy [30] established direct connections between URP(S) and schemas of transfinite induction. By the classical work of Gerhard Gentzen [62], Peano Arithmetic (PA) establishes the coded schema of transfinite induction up to any particular ordinal less than ϵ_0, the least ordinal α such that $\alpha^\omega = \alpha$ in ordinal arithmetic, yet PA does not establish the schema of transfinite induction up to ϵ_0 itself. Kreisel and Levy showed that over PA, URP(PA) is equivalent to transfinite induction up to ϵ_0, and that over Second-Order Arithmetic Z_1, URP(Z_1) is equivalent to transfinite induction up to ϵ_0. These results provided elegant characterizations that connect two formulations of the consistency of well-known theories.

Levy [25,37] investigated the logical complexity, in terms of his hierarchy of formulas, of basic statements of set theory like AC, GCH, and $V = L$. His result on AC typifies the articulation and argumentation, made possible by forcing. The Axiom of Choice is evidently Π_2. Levy showed that AC is not Σ_2 in the following strong sense: *For any Σ_2 sentence σ, if* ZF $\vdash \sigma \to$ AC, *then* ZF $\vdash \neg\sigma$. The following is the argument in brief:

Suppose that σ is Σ_2, say $\exists x \forall y \varphi(x, y)$ with φ being Σ_0, and ZF $\vdash \sigma \to$ AC. The following can then be formalized to establish ZF $\vdash \neg\sigma$: Assume to the contrary that there is a set x_0 such that $\forall y \varphi(x_0, y)$. Let M be a transitive structure with $x_0 \in M$ and modeling enough of set theory to construct a transitive forcing extension N in which AC fails and the ZF axioms that went into a proof of $\sigma \to$ AC hold. Since $\forall y \varphi(x_0, y)$ holds (in the universe) and φ is Σ_0, $\forall y \varphi(x_0, y)$ and hence σ holds in N. But then, this contradicts having that ZF proof of $\sigma \to$ AC.

This argument not only featured forcing as a model-theoretic method within a proof but also forcing over uncountable structures, for the x_0 above could be arbitrary. Modern set theory would come to incorporate many such tailored uses of forcing, and Levy's application to definability was a remarkably early instance.

Levy [33] provided an analysis of Π_2 statements of set theory in a different direction, one that addresses effectivity in terms of witnessing terms. Suppose that $\forall x \exists y \chi(x, y)$ is Π_2, with $\chi(x, y)$ being Σ_0, and recall that tc(x) denotes the transitive closure of x. Levy established that if ZF $\vdash \forall x \exists y \chi(x, y)$, then there is a set-theoretic term $\tau(u)$ such that

$$\text{ZF} \vdash \forall x (\exists \text{ finite } u \subseteq \text{tc}(x) \wedge \chi(x, \tau(u))).$$

Note that one cannot do much better for the AC assertion $\forall x \exists y (x = \emptyset \vee y \in x)$. Levy also established that if ZFC $\vdash \forall x \exists y \chi(x, y)$, then there is a set-theoretic term $\tau(u)$ such that

$$\text{ZF} \vdash \forall x \forall r (r \text{ is a well-ordering of tc}(x) \longrightarrow \chi(x, \tau(r))).$$

In his final article [40] Levy came full circle back to the bedrock of the main comprehension schemas of ZF to investigate their forms anew. He began with the ZF axioms as most often given, as bi-conditionals with e.g. $\forall u \exists y \forall x (x \in y \longleftrightarrow x \subseteq u)$ for the Power Set Axiom. He first addressed the issue of the parameters allowed in the Separation and Replacement schemas. Let S_0 denote the Separation schema restricted to formulas $\varphi(x)$ with one free variable x, $\forall u \exists y \forall x (x \in y \longleftrightarrow x \in u \wedge \varphi(x))$. By a clever coding argument Levy established the positive result that, over a weak subtheory (Extensionality, Pairs, Union, and a weak form of Power Set), S_0 implies full Separation. He established an analogous result for Replacement. Hence, those universal quantifications of parameters, distracting when learning or teaching set theory, are not formally necessary after all.

Levy's main, negative results addressed another issue of self-refinement in axiomatics and showed that his aforementioned positive result is reasonably sharp. In the presence of Separation the generative axioms are sometimes given parsimoniously in a weaker, conditional form, e.g. $\forall u \exists y \forall x (x \subseteq u \longrightarrow x \in y)$ for the Power Set Axiom. With his positive result, the set T consisting of the usual ZF axioms, but with the Separation schema replaced by S_0 and the Replacement schema replaced by the conditional version, is an axiomatization of ZF. Levy established that full Separation is not a consequence of T if the Power Set axiom is weakened to the conditional form. He also established the analogous results for the conditional version of Union and the conditional version of Pairing.

Levy established these delimitative results by, in effect, taking Cohen's original AC independence model and building appropriate submodels. In this he appealed to the Halpern–Levy [35] work on the Cohen model, the work that effected the first substantial transfer from the Fraenkel–Mostowski context to the Cohen context. Levy's sophisticated results on the independence of Separation are a fitting coda, one that resonates with the work of Fraenkel [58], who long ago and far away, steeped in the Hilbertian axiomatic tradition, established the independence of Separation from Zermelo's other axioms.

11. Envoi

Looking back over Levy's researches in set theory, we see a steady and in fact increasing exploitation of model-theoretic reflection and the method of forcing to establish substantial results about definability and axiomatizations. Levy often saw and developed potentialities after an initial ground-breaking move made by others, and had a way of establishing a full context with systematic, magisterial results. With his work set theory reached a new plateau in the direction of understanding the scope and limits of formal expressibility and derivability. With this assimilated, set theory would move forward over a broad range from the analysis of fine structure to a wealth of objectifications and principles provided by large cardinal hypotheses, becoming infused with more and more combinatorial arguments as well as sophisticated techniques involving forcing and inner models.

Around 1970 Levy turned to the writing of books, works that would establish a broad standard of understanding about set theory. The classic *Foundations of Set Theory* by Fraenkel and Bar-Hillel [60] had become outdated because of the many advances made in the 1960s, and so a "second revised edition" Fraenkel–Bar-Hillel–Levy [39] was brought out. Fraenkel was by then deceased, and Levy in fact carried out an almost complete rewriting of the second chapter. One section was published separately as Levy [41]. Throughout the discussion of the axiomatic foundations one sees how the subject has become more elucidated by Levy's own work.

Levy's distinctive book *Basic Set Theory* [42], largely written when he was a visiting professor at Yale University 1971–1972 and at the University of California at Los Angeles 1976–1977, provided a systematic presentation of the broad swath of set theory between elementary beginnings and advanced topics. Levy deliberately set out the extent of set theory before the use of model-theoretic methods and forcing, working out the extensive combinatorial development in a classical setting as rigorized by axiomatic foundations. In a way, it is quite remarkable that Levy forestalled the inclusion of most of his own work by insisting on this middle way. On the other hand, the book is a singular achievement of detailed exposition about what there is in set theory up to the use of the satisfaction predicate. The account of trees is typical, on the one hand a bit idiosyncratic in dealing with generalities but on the other hand broaching an interesting concept, that of a *thin* tree: Trees that have been studied on uncountable cardinals $\kappa > \omega_1$ usually have the property that their αth level has cardinality 2^α, but Levy raised the issue about having cardinality at most $|\alpha|$. Throughout the book, the specialist is treated to a reckoning of the historical sources for the concepts. And the student finds full and patient treatment of topics on which other texts might leave one queasy, like the set theorist's view of the reals. Descriptive set theorists from early on converted from the traditional construal of the real numbers as the continuum to the function space ${}^\omega\omega$ of functions: $\omega \to \omega$, and Levy explained in extensive detail the topological and measure-theoretic connections among the various "real" spaces. What is perhaps most notable is the appendix to the book, where at last Levy's own rigorous and axiomatic approach to set theory casts a telling light. There, he allayed another source of queasiness in accounts of set theory by showing, in some of the most thoroughgoing arguments in the book, that introduced class terms can be formally eliminated in regression back to the primitive language of set theory.

In the development of a mathematical field, a modest turn of events sometimes has an unforeseen effect and achieves a folkloric distinction. Shelah had considerably developed his general concept of *proper forcing* by the early 1980s, a concept amenable to iteration schemes and having remarkably wide applications. Shelah lectured at the Hebrew University on proper forcing, and Levy took systematic notes. These notes, refined and edited, eventually became the bulk of the first three chapters of the monograph Shelah [107] and of the subsequent book Shelah [109] as therein acknowledged. For a generation of set theorists the Levy exposition was the entré into proper forcing; once drawn in, the return of Shelah's inimitable hand in subsequent chapters led to new realizations.

In later years, Levy wrote two texts [43,44] in Hebrew on mathematical logic. The writing of books is an important venture for the advancement of mathematical fields, and as one who has written a book knows well, it is a difficult and time-consuming undertaking, especially when one aspires to codify mathematical knowledge over a broad range and to make its many facets available to succeeding generations.

Turning at last to his teaching and administrative work at the Hebrew University, there is a remarkable legacy of renowned students, as listed below. Levy himself has served as the Dean of the Faculty of Science and the Chairman of Institute of Mathematics and Computer Science, among other positions both in the university and with the academic union. And, as mentioned before, his student Menachem Magidor is currently president of the university.

Soon after his 60th birthday, a 1996 issue of the *Archive for Mathematical Logic* (vol. 35, no. 5–6), was dedicated to Levy, with the following words in the dedication (p. 279):

> Azriel, besides being the important mathematician he is, is also a unique human being. His friends know him as the epitome of wisdom. He can always be approached for good advice, which is given without any personal interest, but purely out of a desire to help. His contribution to the public are innumerable. In any capacity he has held—University administrator, in the educational system in Israel, as a member of usual important editorial boards—you could always rely on his common sense, wisdom and devotion. Azriel is a symbol of intellectual honesty and integrity. His former students will always remember him as a devoted and inspiring teacher.

To this we add the words of Ecclesiastes 9:7–9 (King James Version):

> Go thy way, eat thy bread with joy, and drink thy wine with a merry heart; for God now accepteth thy works.
> Let thy garments be always white; and let thy head lack no ointment.
> Live joyfully with the wife whom thou lovest all the days . . . which he hath given thee under the sun . . .

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Gadi Moran, *Size direction games over the real line*, 1972.
Menachem Magidor, *On supercompact cardinals*, 1972.
Gershon Sageev, *An independence result concerning the axiom of choice*, 1973.
Moti Gitik, *All uncountable cardinals can be singular*, 1979. Second supervisor: Menachem Magidor.
Uri Abraham, *Isomorphisms of Aronszajn trees and forcing without the generalized continuum hypothesis*, 1979. Second supervisor: Saharon Shelah.
Aharon Beller, *Applications of Jensen's coding technique*, 1980.
Ami Litman, *Combinatorial characterization of definable properties in the constructible universe*, 1981. Second supervisor: Menachem Magidor.
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