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Kunen and set theory

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ABSTRACT

This commemorative article provides an account of Kenneth Kunen's work in set theory.

Kenneth Kunen (1943–) figured principally in the development of set theory in all the major directions, this during a formative period for the subject when it was transmuting to a sophisticated field of mathematics. In fact, several of Kunen's results and proofs still frame modern set theory or serve as standards to be reckoned with in its further progress. This is all the more notable as much of the work was done in a short run of about four years from his 1968 thesis. The work has an incisiveness and technical virtuosity remarkable for the time as well as a sense of maturity and definiteness. Typically, others may have started a train of conceptual constructions, but Kunen made great leaps forward advancing the subjects, almost heroically, to what would seem to be the limits achievable at the time. There is a sense of movement from topic to topic and then beyond to applications of set theory, with heights scaled and more heights beckoning.

In what follows, we chronicle Kunen's singular progress over the broad swath of set theory. The work of the years 1968–1972, especially, deserve full airing, and we recall the initiatives of the time as well as describe the ramifications of the advances made. As almost all of this work is in the mainstream of modern set theory, we only set the stage in a cursory way and recall the most immediately relevant concepts, relying in part on the readers' familiarity,[2] but then dwell on the particulars of how ideas and proofs became method. In this way we reaffirm Kunen's work in set theory as central to the subject and of lasting significance for its development.

1. Thesis

Kunen's 1968 thesis [43] was itself a notable landmark in the development of modern set theory, and in its breadth and depth the thesis conveyed a sense of broad reach for the emerging field. For discussing its role and significance, we quickly sketch, though in a deliberate way to our purpose, how set theory was beginning its transmutation into a field of mathematics with sophisticated methods.

With the emergence of the now basic ultraproduct construction in model theory, Dana Scott in 1961 took an ultrapower of the entire set-theoretic universe V to establish that having a measurable cardinal contradicts Gödel's Axiom of Constructibility $V = L$. The ultrapower having brought set theory to the point of entertaining elementary *embeddings* into well-founded models, it was soon transfigured by a new means for getting well-founded *extensions* of well-founded models. This was *forcing*, of course, discovered by Cohen in 1963 and used by him to establish the independence of the Axiom of

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[1] This paper was written whilst the author was a senior fellow at the Lichtenberg-Kolleg at the University of Göttingen. He expresses his gratitude for the productive conditions and support of the kolleg.

[2] See Jech [33] for the set-theoretic concepts and results, and Kanamori [38] for those involving large cardinals.

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Choice (AC) and of the Continuum Hypothesis (CH). With clear intimations of a new, concrete, and flexible way of building models, many rushed into set theory, and, with forcing becoming method, were soon establishing a cornucopia of relative consistency results.

Robert Solovay epitomized this initial period of sudden expansion with his mathematical sophistication and many results over the expanse of set theory. In 1964, he established his now-famous result that if an inaccessible cardinal is Levy collapsed to make it ω_1, there is an inner model of $\mathrm{ZF} + \mathrm{DC}$ in which all sets of reals are Lebesgue measurable.[3] Toward this result, he formulated a new generic real, the random real. In 1965, he worked out with Stanley Tennenbaum the iterated forcing proof of the consistency of Suslin's Hypothesis and therewith of Martin's Axiom (MA). In 1966, Solovay established the equi-consistency of the existence of a measurable cardinal and the real-valued measurability of $2^{\aleph_0}$, i.e. having a (countably additive) measure extending Lebesgue measure to all sets of reals. In the process, he worked out structural consequences of saturated ideals and generic ultrapowers.

While the "forcing king" was forging forth, the University of California at Berkeley became a hotbed of activity in set theory, with Jack Silver's 1966 thesis the initial high point, with its large cardinal analysis of L leading to the set of integers $0^{\#}$ (also isolated by Solovay). To the south, at the University of California at Los Angeles, a huge multi-week conference was held in the summer of 1967 that both summarized the progress already made and focused the energy of a new field opening up.

As a graduate student at Stanford University, Kunen [42] initially developed certain initiatives of Georg Kreisel about effectiveness and compactness of infinitary languages, as did Jon Barwise. Interest in this direction would persist through to a joint Barwise–Kunen study [5] of a cardinal characteristic of infinitary languages. At that 1967 UCLA conference, Kunen [47] presented work on a somewhat related topic, indescribability and the continuum. After the conference, however, Kunen had moved squarely into set theory, with his thesis of a year later having 10 items in its bibliography from the packaged proceedings of the conference.

That thesis, *Inaccessibility properties of cardinals* [43], of August 1968 and with Scott as advisor, consisted of two parts. Part I dealt with strong inaccessibility properties, mainly applications of iterated ultrapowers. Part II discussed weak inaccessibility properties, particularly of the continuum. The next section describes the work of the first part and its extensions, and here we continue with a description of the work of the second.

To the extent that Kunen's part II was disseminated, it served as an early exemplar for the subsequent and wide-ranging analyses of the continuum assuming real-valued measurability or assuming Martin's Axiom. In an introductory section Kunen already stated an important result, a carefully formulated Π^2_1-indescribability for real-valued measurable cardinals. This result was fully aired by Solovay in his eventual real-valued measurability paper [90, §6]. In succeeding sections Kunen addressed three problems going in different directions:

(a) Is every subset of the plane in the σ-algebra generated by arbitrary rectangles (products of subsets of the line)?
(b) Is the Boolean algebra of all subsets of the reals modulo sets of cardinality less than the continuum weakly (ω, ω) distributive?
(c) Must every set of reals of cardinality less than the continuum be of Lebesgue measure 0 (or of first category)?

In addressing these questions, Kunen exhibited a fine and quick understanding of the contexts recently established by the consistency work of Solovay on real-valued measurability and on Martin's Axiom and a remarkable working knowledge of classical set-theoretic work on the continuum.

Question (a) was inspired by a presentation of Richard Mansfield (cf. his [74]) at the UCLA conference. Kunen drew on work from Hausdorff [31], nowadays well known as the source of the "Hausdorff gap", to establish how MA leads to a positive conclusion in a structured sense. He then observed how his indescribability result for real-valued measurability leads to a negative conclusion. R.H. Bing, W.W. Bledsoe and R. Daniel Mauldin [8] subsequently established results about the number of steps needed, starting from the rectangles, to generate all the subsets.

Question (b) Kunen reformulated forthwith in terms of functions $f : \omega \to \omega$ under the eventual dominance ordering, as had already been done classically, and showed that both real-valued measurability and MA entail positive conclusions.

For question (c) Kunen first recalled Solovay's results that under MA any union of fewer than continuum many Lebesgue measure zero sets is again measure zero, and likewise for any union of first category sets. Kunen then went on to observe that MA implies that there is a (generalized) Luzin set, i.e. a set of cardinality the continuum whose intersection with any first category set is of smaller cardinality, and that MA implies that every set of cardinality less than the continuum has strong measure zero. A set of reals has *strong measure zero* if for any sequence $\langle a_i \mid i \in \omega \rangle$, there are open intervals I_i of length at most a_i for $i \in \omega$, whose union covers the set. (Kunen mentioned that Borel in 1919 conjectured that all sets of strong measure zero are countable; with this question recalled in the post-Cohen area, Richard Laver [69], with the first clear use of countable-support iterated forcing, famously established the consistency of Borel's conjecture.) Finally, Kunen showed that under real-valued measurability the answers to question (c) are yes for category, and, applying his indescribability result (Σ^1_2-indescribability suffices), no for measure.

[3] DC is the Axiom of Dependent Choice, a weak form of the Axiom of Choice sufficient for developing the classical theory of measure and category for the reals.

Kunen would make important contributions to the theory of MA itself; see Section 8. See Fremlin [21] and Fremlin [20] for compendium accounts of the mature theory for real-valued measurability and Martin's Axiom respectively, and Kunen's pioneering role situated in context.

2. Iterated ultrapowers

The most sustained of Kunen's work in his early period was on applications of iterated ultrapowers, and this work was to become foundational for modern inner model theory. The infusion of forcing into set theory had induced a broad context extending beyond its applications and sustained by model-theoretic methods, a context which included central developments having their source in Scott's 1961 ultrapower result that measurable cardinals contradict $V = L$. Haim Gaifman [25,26] invented *iterated ultrapowers* and established incisive results about and with the technique, and this work most immediately stimulated the results of Kunen [43,45] on inner models of measurability.

For a normal ultrafilter U over a measurable cardinal κ, the inner model $L[U]$ of sets constructible relative to U is easily seen with $\overline{U} = U \cap L[U]$ to satisfy $L[U] \models$ "$\overline{U}$ is a normal ultrafilter". With no presumption that κ is measurable (in V) and taking $U \in L[U]$ from the beginning, call $\langle L[U], \in, U\rangle$ a *κ-model* iff $\langle L[U], \in, U\rangle \models$ "U is a normal ultrafilter over κ". Solovay observed that in a κ-model, GCH holds above κ by a version of Gödel's argument for L and that κ is the only measurable cardinal by a version of Scott's argument. Silver [84,86] then established that the full GCH holds, thereby establishing the relative consistency of GCH and measurability; Silver's proof turned on a local structure $L_\alpha[U]$ being *acceptable* in the later parlance of inner model theory.

Kunen made Gaifman's technique of iterated ultrapowers integral to the subject of inner models of measurability. For a κ-model $\langle L[U], \in, U\rangle$, the ultrapower of $L[U]$ by U with corresponding elementary embedding j provides a $j(\kappa)$-model $\langle L[j(U)], \in, j(U)\rangle$, and this process can be repeated and the elementary embeddings composed. At limit stages, one can take the direct limit of models, which when well-founded can be identified with the transitive collapse. In terms of the stage thus set, Kunen established in his thesis [43] and subsequent paper [45]:

(a) [43,45] For any κ, there is at most one κ-model.
(b) [45] For any κ-model and κ'-model with $\kappa < \kappa'$, the latter is an iterated ultrapower of the former.

These are the definitive structure results for inner models of measurability, results that argued forcefully for the coherence and consistency of the concept of measurability.

The argument for (b) is similar to that for (a), which we can recapitulate in terms of its essential components as follows: Suppose that $L[U]$ and $L[V]$ are both κ-models. By Gaifman's work, all the iterated ultrapowers of each are well founded, i.e. $L[U]$ and $L[V]$ are *iterable*. Through a helpful representation of the iterates in terms of the successive critical points, for any regular $\nu > \kappa^+$, the νth iterated ultrapower of each works out, quite remarkably, to be $L[\mathcal{C}_\nu]$, where $\mathcal{C}_\nu$ is the closed unbounded filter over ν. In particular, *comparison* of $L[U]$ and $L[V]$ can be carried out by iterating them sufficiently many times. Next, with $j^U_{0\nu}$ the corresponding elementary embedding of $L[U]$ into $L[\mathcal{C}_\nu]$, if Λ is any proper subclass of the image $j^U_{0\nu}$"ON of the ordinals that includes all of κ, then the Skolem hull of Λ in $L[\mathcal{C}_\nu]$ is isomorphic to $L[U]$. The analogous result holds for $j^V_{0\nu}$ and $L[V]$. Finally, the intersection of $j^U_{0\nu}$"ON and $j^V_{0\nu}$"ON is in fact such a Λ—what we here call *stability*—and so $L[U] = L[V]$. Inner model theory would develop from Kunen's work into a mainstream of modern set theory, and the essential components of the above proof—iterability, comparison, stability—would remain at the heart of the subject (cf. the end of this section).

Beyond his structure results, Kunen in his [45] established a range of results about what happens in κ-models to ultrafilters, large cardinals, and so forth. As for generalizations, Kunen [43,45] in 1971 showed that his structure results can be extended to inner models $L[\mathcal{U}]$ where $\mathcal{U}$ is a sequence of normal ultrafilters over various measurable cardinals whose length is less than the least of them. With this, he [43,45] established that if there is a strongly compact cardinal, then there are inner models with arbitrarily many measurable cardinals. Subsequently in 1969, Kunen [48] was able to draw the same conclusion from the existence of a measurable cardinal κ satisfying $\kappa^+ < 2^\kappa$; this conclusion, which applied a combinatorial result of his student Jussi Ketonen about independent functions, provided the first inkling of the strength of this proposition about measurable κ, and popularized it as a focal one.[4]

From the beginning, Kunen had emphasized that iterated ultrapowers can be taken of an inner model M with respect to an ultrafilter U even if $U \notin M$, as long U is an *M-ultrafilter*, i.e. U in addition to having M-related ultrafilter properties also satisfies an "amenability" condition for M. A crucial dividend was a characterization of the existence of $0^\#$ that secured its central importance in inner model theory. Motivated by work of Gaifman [25] on iterated ultrapowers and constructible sets, Silver in his thesis [85,88] had investigated the generation of L with indiscernibles provided by large cardinals, and he and Solovay [89] had independently isolated the set of integers $0^\#$ as encoding the corresponding theory. With $0^\#$, any increasing shift of the Silver indiscernibles provides an elementary embedding $j : L \to L$. Kunen [43, Theorem 4.7] established conversely that any such embedding generates indiscernibles, so that $0^\#$ *exists iff there is a* (*non-identity*) *elementary embedding*

[4] One of the triumphs of modern set theory would be to gauge the consistency strength of having a measurable cardinal κ satisfying $\kappa^+ < 2^\kappa$. Moti Gitik [27,28], building on work of Woodin and Mitchell, showed that this proposition is equi-consistent with Mitchell's $\exists\kappa(o(\kappa) = \kappa^{++})$.

$j : L \to L$. Starting with such an embedding, Kunen defined a corresponding ultrafilter U over the least ordinal moved and showed that U is an L-ultrafilter with which the iterated ultrapowers of L are well-founded. The successive images of the critical point were seen to be indiscernibles for L, giving $0^{\#}$. As inner model theory was to develop, this sort of sharp analysis would become a schematic cornerstone: the "sharp" of an inner model M would encapsulate transcendence over M, and the *non-rigidity* of M, that there is a (non-identity) elementary embedding $j : M \to M$, would provide equivalent structural sense.

Kunen's final application of iterated ultrapowers was to establish a notable result about an infinitary language generalization of constructibility. At that UCLA conference, Chen-Chung Chang [11] had presented for cardinals κ a class C^κ of sets constructible using the infinitary language $L_{\kappa\kappa}$ and observed various generalizations of the properties of L $(= C^\omega)$. Using iterated ultrapowers, Kunen [50] showed that unlike L, these models do not generally have intrinsic well-orderings: *If κ is regular and there are κ^+ measurable cardinals, then the Axiom of Choice fails in C^{κ^+}*. Toward this result Kunen established a striking lemma that revealed an unexpected global constraint on measurable cardinals: *For any ordinal ξ, the set of measurable cardinals κ for which there is a κ-complete ultrafilter over κ such that the corresponding ultrapower embedding moves ξ is finite.* Later, William Fleissner [14] provided a proof that does not use iterated ultrapowers.

It is a notable historical happenstance that the number of normal ultrafilters over a measurable cardinal would stimulate much of the early work in modern inner model theory. Scott had made normality central to the study of measurable cardinals through his ultrapower analysis. Kunen was evidently motivated in large part to consider κ-models in order to establish, which he did, that in any κ-model the defining normal ultrafilter is unique. Already in his thesis Kunen established with forcing that it is relatively consistent for a measurable cardinal κ to have the maximal number 2^{2^κ} of normal ultrafilters. Jeffrey Paris in his thesis [79] independently established this result as well as several of Kunen's results about κ-models. In a joint paper [62] written by Paris, the consistency result was presented as well as some results about saturated ideals, and this early method of forcing that preserved large cardinals came to be called *Kunen–Paris forcing*.

At the end of their paper, Kunen and Paris pointedly asked whether it is consistent to have some intermediate number of normal ultrafilters between 1 and the maximal number 2^{2^κ}, e.g. 2. William Mitchell in 1972, just after completing his own pioneering, Berkeley thesis, considered this question and soon provided the first substantive extension of Kunen's inner model results. Little was known outright about normal ultrafilters, but Kunen [43] did observe that for any measurable cardinal κ, there is always a normal ultrafilter U over κ such that $\{\alpha < \kappa \mid \alpha \text{ is not measurable}\} \in U$. Thus, if there were, rather extravagantly, a normal ultrafilter W over κ such that $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in W$, then U and W would be distinct. Taking this as a beginning point for an inner model construction, Mitchell [78] formulated what is now known as the *Mitchell order* $\lhd$.

For normal ultrafilters U and U' over κ, $U' \lhd U$ *iff* U' is in the ultrapower $\mathrm{Ult}(V, U)$ of the universe by U, i.e. there is an $f : \kappa \to V$ representing U' in the ultrapower, so that $\{\alpha < \kappa \mid f(\alpha) \text{ is a normal ultrafilter over } \alpha\} \in U$ and κ is already a limit of measurable cardinals. $U \lhd U$ always fails, and generally, $\lhd$ is a well-founded relation by a version of Scott's argument that measurable cardinals contradict $V = L$. Consequently, to each U can be recursively assigned a rank $o(U) = \sup\{o(U') + 1 \mid U' \lhd U\}$, and to a cardinal κ, the supremum $o(\kappa) = \sup\{o(U) + 1 \mid U \text{ is a normal ultrafilter over } \kappa\}$. By a cardinality argument, if $2^\kappa = \kappa^+$ then $o(\kappa) \leqslant \kappa^{++}$.

Mitchell [78] devised the concept of a *coherent* sequence of ultrafilters ("measures"), a doubly-indexed sequence that has just enough ultrafilters to witness the $\lhd$ relationships, and was able to establish uniqueness results for inner models $L[\mathcal{U}] \models$ "$\mathcal{U}$ is a coherent sequence of ultrafilters". In this first inner model extension of Kunen's work one sees the closest connection to the essential components of his argument for the uniqueness of κ-models. Mitchell affirmed that these $L[\mathcal{U}]$'s are *iterable* in that arbitrary iterated ultrapowers via ultrafilters in $\mathcal{U}$ and its successive images are always well-founded. He then effected a *comparison*, as any $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ have respective iterated ultrapowers $L[\mathcal{W}_1]$ and $L[\mathcal{W}_2]$ such that $\mathcal{W}_1$ is an initial segment of $\mathcal{W}_2$ or vice versa. This he achieved through a process of *coiteration* of least differences: At each stage, one finds the lexicographically least coordinate at which the current iterated ultrapowers of $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ differ and takes the respective ultrapowers by the differing ultrafilters; the difference is eliminated as ultrafilters never occur in their ultrapowers. Finally, Mitchell applied a generalization of Kunen's *stability* argument to establish e.g. that in $L[\mathcal{U}]$ the only normal ultrafilters over α for any α are those that occur in $\mathcal{U}$. For example, if one starts with a coherent sequence that corresponds to $o(\kappa) = 2$, there are exactly two normal ultrafilters over κ, differentiated as U and W were in the discussion above of Kunen's observation. As inner model theory would develop, coiteration would become embedded as basic for comparison and $\exists\kappa(o(\kappa) = \kappa^{++})$ would become a pivotal large cardinal proposition for gauging consistency strength.

The fine-structural "core" models for larger and larger cardinals would be generated by "mice", local versions of structures that incorporate some large cardinal structure, and always crucial would be their iterability. Iterability allows for the possibility of comparison, now more substantive in that iterates of mice may not coincide but one could be an initial segment of the same relative constructible hierarchy as the other. Iterability and comparison would thus guide the further development of inner model theory for getting canonical structures, and the need for new forms of iteration to get iterable mice and, after that, to effect comparison, would lead to new local and global structures and new procedures. Furthermore, Kunen's stability argument would also surface in new emanations. Loosely speaking, for either the Mitchell core model for coherent sequences of ultrafilters or the more general Steel core model for coherent sequences of extenders, an initial inner model K^c is defined which is "universal" in that it compares at least as long as any other similarly defined model. Then, certain elementary substructures of K^c are considered which are "thick" in that their transitizations are also universal. These thick substructures are Skolem hulls of certain classes of ordinals, which as in Kunen's original argument are actually fixed by all the relevant embeddings. Finally, the "true" core model K is recursively defined as a certain thick elementary submodel of K^c.[5]

Ironically, after four decades of inner model theory and what seems like a lifetime of experience, the Kunen–Paris question of the number of normal ultrafilters was revisited and seen, finally, not to be a matter of stronger hypotheses at all but of stronger methods. In 2009 Sy Friedman and Menachem Magidor [23] showed how to get all the possible values for the number of normal ultrafilters, starting with just one measurable cardinal κ and carrying out forcing that featured adjoining subsets of κ with perfect sets in the style of Sacks forcing.

3. Combinatorial principles

In work quite different from his thesis and yet of fundamental importance for modern set theory, Kunen in 1969, together with Ronald Jensen, established the framing results about now basic combinatorial principles and related large cardinals in the constructible universe L. The consummate master of constructibility was to be Jensen, whose systematic analysis transformed the subject with the introduction of the fine structure theory for L. In 1968 Jensen [34] made his first breakthrough by showing that *$V = L$ implies the failure of Suslin's Hypothesis*, i.e. (there is a Suslin tree)^L, applying L for the first time after Gödel to establish a relative consistency about a classical proposition. Inspired by Jensen's construction, the ubiquitous Solovay established that *$V = L$ implies Kurepa's Hypothesis*, i.e. (there is a Kurepa tree)^L. The combinatorial features of L that enabled these constructions were soon isolated in two combinatorial principles for a regular cardinal κ, $\diamondsuit_\kappa$ ("diamond") and $\diamondsuit^+_\kappa$ ("diamond plus") respectively. To recall just the first:

($\diamondsuit_\kappa$) There is a sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$ such that for any $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.

$\diamondsuit_{\omega_1}$ implies that there is a Suslin tree, and $\diamondsuit^+_{\omega_1}$ implies that there is a Kurepa tree. In notes [35] written at Rockefeller University in 1969, Jensen presented his collaborative work with Kunen on these combinatorial principles and related new large cardinals. As the notes were scrupulous about assigning credit, one sees confirmed Kunen's substantial role in the development of a now basic part of set theory.

First, Kunen established that *$\diamondsuit^+_\kappa$ implies $\diamondsuit_\kappa$*, and this he did by establishing the equivalence of $\diamondsuit_\kappa$ with: ($\diamondsuit'_\kappa$) There is a sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq P(\alpha)$ and $|S_\alpha| \leqslant |\alpha|$ such that for any $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha \in S_\alpha\}$ is stationary in κ. This equivalence is still regarded as a notable combinatorial result, and it spawned the extended investigation of variations of $\diamondsuit_\kappa$.

Second, Kunen and Jensen independently formulated the following now well-known large cardinal concept, a $\forall\exists$ version of $\diamondsuit_\kappa$: A regular cardinal κ is *ineffable* iff for any sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$, there is an $X \subseteq \kappa$ stationary in κ such that for $\alpha < \beta$ both in X, $S_\beta \cap \alpha = S_\alpha$. Kunen forthwith situated ineffability by characterizing it in terms of a strong version of the partition property characterization of weak compactness and showing that ineffability implies Π^1_2-indescribability. He and Jensen showed that the least ineffable cardinal is smaller than the least κ satisfying $\kappa \to (\omega)^{<\omega}$ and is larger than the least totally indescribable cardinal, i.e. one which is Π^n_m-indescribable for every $n, m \in \omega$. They also showed that if κ is ineffable, then (κ is ineffable)^L.

With respect to the motivating connection with combinatorial principles, Jensen and Kunen established that if κ is ineffable, then $\diamondsuit^+_\kappa$ fails. Jensen established the converse under the assumption $V = L$, and hence that *in L, κ is ineffable iff $\diamondsuit^+_\kappa$ fails*.

What about ineffability and $\diamondsuit_\kappa$? The following weak form of ineffability was isolated: A regular cardinal κ is *subtle* iff for every closed unbounded $C \subseteq \kappa$ and sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$, there are $\alpha < \beta$ both in C such that $S_\beta \cap \alpha = S_\alpha$. Kunen, in the last theorem of the notes [35], showed that *if κ is subtle, then $\diamondsuit_\kappa$ holds*.

Ineffable and subtle cardinals would become staple for the theory of large cardinals. In a far-reaching systematic investigation of generalizations of ineffable cardinals, the n-ineffable cardinals, Baumgartner [6] would uncover an elegant closely-knit hierarchy, one in fact that could be taken to have as a basis generalizations of the subtle cardinals, the n-subtle cardinals. Eventually, the existence of n-subtle cardinals would be characterized [37] in terms of having a finite homogeneous set for "regressive" partitions and by even more refined means [22] to establish that certain propositions of finitary mathematics have strong consistency strength.

4. The inconsistency

In 1970 Kunen [46] established an ultimate delimitation for the entire hierarchy of large cardinals by showing that a *prime facie* extension of ideas leads to an outright inconsistency. With large cardinal hypotheses having become central in modern set theory for gauging consistency strength, "Kunen's inconsistency" will live on as an ultimate upper bound on the strength of propositions of set theory and so of mathematics.

In the late 1960s, Solovay and William Reinhardt, as a graduate student at Berkeley, were charting out hypotheses stronger than measurability based on the concept of elementary embedding. Solovay and Reinhardt independently formu-

[5] For details on inner model theory, see the Steel and Schimmerling chapters in the *Handbook of Set Theory* [18].

lated the concept of *supercompactness*, as based on postulating elementary embeddings $j : V \to M$ of the universe V into an inner model M. Assuming always that this notation commits to having j not the identity, it has a critical point $\mathrm{crit}(j)$, i.e. a least ordinal moved by j. Measurability had become focal through the embedding characterization, *κ is measurable iff κ is the critical point of an elementary embedding $j : V \to M$ for some inner model M*, in which case one can through the ultrapower construction assume additionally that ${}^{\kappa}M \subseteq M$, i.e. arbitrary κ-sequences drawn from M are again members of M. Solovay and Reinhardt formulated (cf. [92]) the global concept: κ is *supercompact* iff κ is γ-supercompact for every γ, where κ is *γ-supercompact* iff there is an elementary embedding $j : V \to M$ for some inner model M with $\mathrm{crit}(j) = \kappa$, $\gamma < j(\kappa)$, and ${}^{\gamma}M \subseteq M$. The imposition of strong closure properties on M allowed for strong reflection arguments to take place, and supercompactness has achieved a central place in the large cardinal hierarchy both as the provenance of forcing relative consistency results and as a goal for the development of inner model theory.

At the end of his 1967 Berkeley thesis Reinhardt briefly considered the following postulation as an ultimate extension: There is an elementary embedding $j : V \to V$. Reinhardt's proposal led to a dramatic turn of events. After initial results aroused some suspicion, Kunen [46] established in ZFC: *There is no elementary embedding $j : V \to V$.*

This result delimited the whole large cardinal enterprise. It could have been that $j : V \to V$ would serve as the culmination of the guiding idea of closure conditions on target models of elementary embeddings; a new guiding idea in some orthogonal direction would have been exploited to formulate still stronger hypotheses; and so on. Rather, in quickly resolving the situation with an appropriately simple statement, Kunen's result sharply defined the context and showed that a completion of ZFC in a specific sense exists. The particular forms of the result were intriguing and unexpected, and although the original proof had an *ad hoc* flavor, what it established has not since been superseded by any stronger inconsistency result.

Kunen's original proof applied a result of Paul Erdős and András Hajnal [13] from the partition calculus and about algebras with infinitary operations. As pointed out by Kunen, for the particular case of their result applied, there was a simple recursive construction available, and Kunen's main argument was itself short and transparent. The argument led to two particular forms of the result:

(a) For any δ, there is no elementary embedding $j : V_{\delta+2} \to V_{\delta+2}$.
(b) If $j : V \to M$ is an elementary embedding into some inner model M and δ is the least ordinal above $\mathrm{crit}(j)$ such that $j(\delta) = \delta$, then $\{j(\alpha) \mid \alpha < \delta\} \notin M$.

The first provides a striking, local version, and the second points to a particular set that cannot be in the target model. For both, the δ can be given a concrete sense as emerging from the proof as follows: Setting $\kappa_0 = \mathrm{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$, $\delta = \sup\{\kappa_n \mid n \in \omega\}$.

Several proofs of Kunen's result, as articulated by (a) and (b) above, have emerged, attesting both to the importance of the result and to its resilience as the point of transition to inconsistency. In the late 1980s, Hugh Woodin gave a proof based on splitting stationary sets, and Mikio Harada, on elementarity and closure.[6] In the mid-1990s, Jindřich Zapletal [97] provided a proof using a fundamental consequence of Saharon Shelah's pcf theory, the existence of "scales" on δ^+ for singular cardinals δ.

Arguments for Kunen's result have established limitative consequences in the theory of ideals and generic elementary embeddings, a subject we also broach in Section 7. Kunen's original argument shows e.g. there is no normal, fine, precipitous ideal over $[\aleph_{\omega+1}]^{\aleph_{\omega+1}}$, and Zapletal's, that for singular cardinals λ, there is no λ^+-saturated normal, fine ideal over $[\lambda]^{\lambda}$.[7]

The resilience of Kunen's transition to inconsistency has been affirmed in another, more substantive way, by the use in relative consistency results of, and the development of a coherent theory for, hypotheses just short of the inconsistency. The relative consistency results were the first that were established for strong determinacy hypotheses, a subject to which we turn in Section 6. In 1978 Martin [75] established that if there is an "iterable" elementary embedding $j : V_\delta \to V_\delta$, then $\boldsymbol{\Pi}^1_2$-determinacy holds, i.e. every $\boldsymbol{\Pi}^1_2$ set of reals is determined. Woodin then considered the following possibility, just short of Kunen's inconsistency:

(I0) There is an elementary embedding $j : L(V_{\delta+1}) \to L(V_{\delta+1})$ with $\mathrm{crit}(j) < \delta$.

Here, $L(V_{\delta+1})$ is the constructible closure of $V_{\delta+1}$, the least inner model of set theory containing all the sets of rank at most δ. In 1984 Woodin established that I0 implies $\mathrm{AD}^{L(\mathbb{R})}$, i.e. every set of reals in $L(\mathbb{R})$, the constructible closure of the reals, is determined. The Martin and Woodin results established an initial mooring for strong determinacy hypotheses in the hierarchy of large cardinals; as is well known, by mid-1985 splendid, definitive results were established that secured equi-consistencies.

As to coherent theories for hypotheses just short of Kunen's inconsistency, Laver starting in the late 1980s investigated the collection of elementary embeddings $j : V_\delta \to V_\delta$ for a fixed δ as an algebra satisfying the "left distributive law", taking the cue from identities first applied by Martin in his $\boldsymbol{\Pi}^1_2$ result. Laver's results on a normal form and the solvability of a word

[6] See Kanamori [38, pp. 319–322].
[7] For both results, see Section 6.2 of Foreman's chapter in the *Handbook of Set Theory* [18].

problem spawned a cottage industry, one which through Patrick Dehornoy's infusion of the infinite braid group generated remarkable connections and initiatives.[8] After his initial success with I0, Woodin developed a detailed and coherent theory of $L(V_{\delta+1})$ under I0. (Laver [71,72] summarized this theory and developed it further.) What became evident was a striking analogy between the theory of $L(V_{\delta+1})$ under I0 and the theory of $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$. Woodin has developed this analogy in both directions and has lately speculated about connections with an ultimate inner model for large cardinals (cf. his [96]).

5. Ultrafilters

Kunen would be a mainstay of the Department of Mathematics at the University of Wisconsin, and one of his principal mathematical connections would be with the pioneering initiatives of Mary Ellen Rudin in set-theoretic topology. The first connection involved the Stone–Čech compactification $\beta\mathbb{N}$ of the discrete countable space $\mathbb{N}$, identifiable with the set of ultrafilters over ω topologized by taking as basic open sets $O_X = \{U \mid X \in U\}$ for $X \subseteq \omega$. In a groundbreaking paper, husband Walter Rudin [81] had early on established that $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$ (discarding the distinguishing principal ultrafilters) is not homogeneous under CH, since there are then distinguished points, the *p-points*—those points for which the intersection of any countably many neighborhoods is again a neighborhood. Zdeněk Frolik [24] then established that $\mathbb{N}^*$ is not homogeneous in ZFC, using "integrating sums" of points. Mary Ellen Rudin [80] formatively focused the investigation of $\beta\mathbb{N}$ on partial orders, mainly two, growing out of this earlier work. In the 1969 Wisconsin thesis of David Booth (cf. his [9]), Kunen's first student, the two orders were named the *Rudin–Frolik ordering* and the *Rudin–Keisler ordering*, with the latter defined for ultrafilters in general as follows: For ultrafilters U over I and V over J, $U \leqslant V$ iff there is a function $f : J \to I$ such that for all $X \subseteq I$, $X \in U$ iff $f^{-1}(X) \in V$. (This natural projection ordering had also been considered by the model-theorist H. Jerome Keisler, another mainstay of the Wisconsin department.) Several results of Kunen appeared in [9], and in particular it was he who established (cf. Theorem 4.9) the now familiar characterization of the Rudin–Keisler minimal ultrafilters as the *Ramsey ultrafilters*, p-points of a special sort.

Separately in 1969, Kunen [44,51] showed: *Assuming* CH, (a) *there are points in* $\mathbb{N}^*$ *which are not p-points nor the limits of any countable set*, and establishing a conjecture of Mary Ellen Rudin, (b) *there is a countable* $X \subseteq \mathbb{N}^*$ *such that each point in* X *is a limit point of* X*, yet not a limit of any discrete countable subset of* $\mathbb{N}^*$. In these ways Kunen amply showed that there are Rudin–Frolik minimal points which are not p-points. Investigations under CH and more generally MA would show that they lead to a host of such distinguishing features and a variegated structure for $\mathbb{N}^*$.

An important thrust of Kunen's work on ultrafilters, on the other hand, would be in the direction of the possibilities in just ZFC. Also in 1969, Kunen (cf. [44,49]) established in ZFC, applying a classical, 1930s construction of an independent family of sets to enable a $2^{\aleph_0}$-length recursion, that *there are Rudin–Keisler incomparable ultrafilters over* ω. The proof showed in general that *for any cardinal* κ *there are* 2^κ *pairwise Rudin–Keisler incomparable, uniform ultrafilters over* κ.[9]

The general setting led to a further ZFC result. Keisler had formulated the concept of "good" ultrafilter, and had shown that for a cardinal $\lambda > \omega$, the λ-good, countably incomplete ultrafilters are exactly those that produce λ-saturated ultraproducts of structures for a language of cardinality less than λ. He was only able to show assuming $2^\kappa = \kappa^+$ that there are κ^+-good, countably incomplete ultrafilters over κ. Every countably incomplete ultrafilter is ω_1-good, and so the issue is only substantive for $\kappa > \omega$. Applying an elaboration of independent family of sets leading to an independent family of functions, Kunen [49] established in ZFC that *for any cardinal* κ*, there are* (2^κ) κ^+*-good, countably incomplete ultrafilters over* κ. In the 1974 book *Theory of Ultrafilters* [12], independent families of functions are given conspicuous treatment and Kunen's result on good ultrafilters is showcased as their "fundamental existence theorem".

In a similar vein, Kunen and Karel Prikry [64] established a result in ZFC about descendingly incomplete ultrafilters over uncountable cardinals, one that had previously required the Generalized Continuum Hypothesis (GCH), by appealing to some classical combinatorial constructions. This work too is aired in the aforementioned book.

In the direction of limitations of ZFC for $\beta\mathbb{N}$, already in 1971 Kunen [49, p. 301], [51] in effect had established: *If one adjoins at least* $(2^{\aleph_0})^+$ *random reals, then there are no Ramsey ultrafilters in the extension.*[10] This framed the stage for the issue of whether the more topologically relevant p-points always exist, when in 1977 Shelah (cf. [82,95]) visiting Wisconsin duly established that *it is consistent that there are no p-points in* $\mathbb{N}^*$. Soon afterwards, Kunen focused attention on what arguably could have been the pivotal concept from the beginning by defining a *weak p-point* to be a point which is not the limit of any countable sequence, and establishing in ZFC (cf. [55]) that *there are weak p-points in* $\mathbb{N}^*$. This "effectively" settled the issue of homogeneity for $\mathbb{N}^*$ in ZFC by producing specific, distinguishing points. Kunen's proof was elegant in context; he relaxed the condition of being an ω_2-good ultrafilter (there are no such ultrafilters over ω) to an "ω_1-OK" ultrafilter, which are weak p-points, and then enhanced the independent function argument to establish that there are ω_1-OK ultrafilters. The investigation of $\beta\mathbb{N}$ has been carried forth in such vein, especially by Kunen, Jan van Mill and Alan Dow, but we leave off further discussion of this as situated in general topology.

While on the subject of ultrafilters, we tuck in some 1971 results of Kunen on a partition property for ultrafilters. In that year, Solovay and Telis Menas investigated the natural generalization of the well-known Rowbottom partition property

[8] See Dehornoy's chapter in the *Handbook of Set Theory* [18].

[9] An ultrafilter over κ is *uniform* if every member has cardinality κ.

[10] Much later in Hart and Kunen [29], this result was technically improved with $\aleph_2$ replacing $(2^{\aleph_0})^+$.

for normal ultrafilters over κ to normal ultrafilters over $\mathcal{P}_\kappa\gamma$ for $\gamma \geqslant \kappa$ as given by the supercompactness of κ. Solovay initially showed under GCH that if κ is supercompact, then every normal ultrafilter over $\mathcal{P}_\kappa\gamma$ for small γs like κ^+, κ^{++}, etc. has the partition property. Menas (cf. [76]) showed that for any $\gamma \geqslant \kappa$ there are always the maximal possible number of normal ultrafilters over $\mathcal{P}_\kappa\gamma$ with the partition property, and moreover eliminated the GCH assumption from Solovay's results. On the other hand, Solovay had shown that the partition property does not always hold; e.g. if κ is supercompact and $\lambda > \kappa$ is measurable, then there is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$ without the partition property. With his knowledge of indescribability and ineffability, Kunen (cf. Kunen and Pelletier [63]) improved Solovay's result in these directions: *If κ is supercompact and $\lambda > \kappa$ is ineffable, then there are stationarily many $\gamma < \lambda$ such that there is a normal ultrafilter over $\mathcal{P}_\kappa\gamma$ without the partition property, and moreover, the least such γ is Π^2_1-indescribable.* These results are still the delimitative results for the partition property today.

Kunen's familiarity with ultrafilters contributed substantially to his playing a crucial role in a novel setting without AC, as we describe in the next section.

6. Determinacy

Kunen had a spectacular run through the projective sets under the Axiom of Determinacy (AD) in the spring and summer of 1971. It was at the beginning of the formative period when the structure theory for the projective sets under determinacy hypotheses was being worked out, a theory that would come to be considered their "correct" theory. As with the combinatorial principles (cf. Section 3) Kunen entered the fray fully equal to the task and quickly worked at the forefront as a full-fledged pioneer in the subject. AD seemed to exude remarkable deductive power, and Kunen seemed to get at what was achievable in the initial foray, erecting conceptual constructions that would not be bettered for over a decade.

For $Y \subseteq \mathbb{R}^{k+1}$, the *projection of* Y is $pY = \{\langle x_1, \ldots, x_k\rangle \mid \exists y(\langle x_1, \ldots, x_k, y\rangle \in Y)\}$. With Suslin having essentially noted in 1917 that *a set of reals is analytic iff it is the projection of a Borel subset of* $\mathbb{R}^2$, the early descriptive set theorists had taken the geometric operation of projection to be basic and defined the projective sets to be those in a corresponding hierarchy, which in modern notation is as follows: For $A \subseteq \mathbb{R}^k$, A is $\boldsymbol{\Sigma}^1_1$ *iff* $A = pY$ for some Borel set $Y \subseteq \mathbb{R}^{k+1}$; A is $\boldsymbol{\Pi}^1_n$ *iff* $\mathbb{R}^k - A$ is $\boldsymbol{\Sigma}^1_n$; A is $\boldsymbol{\Sigma}^1_{n+1}$ *iff* $A = pY$ for some $\boldsymbol{\Pi}^1_n$ set $Y \subseteq \mathbb{R}^{k+1}$; and A is $\boldsymbol{\Delta}^1_n$ *iff* A is both $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$. However, the early descriptive set theorists could not make much headway in their structural investigation beyond the first level of the projective hierarchy, and this had to await the coming of a new paradigm, a new way of thinking.

For $A \subseteq {}^\omega\omega$, let $G(A)$ be the following *game*: There are two players I and II. I initially chooses $x(0) \in \omega$, then II chooses $x(1) \in \omega$, then I chooses $x(2) \in \omega$, then II chooses $x(3) \in \omega$, and so forth. With the resulting $x \in {}^\omega\omega$, I wins $G(A)$ *iff* $x \in A$, and otherwise II wins. A *strategy* is a function that tells a player what move to make given the sequence of previous moves, and a *winning strategy* is a strategy such that if a player plays according to it he always wins no matter what his opponent plays. A is *determined* if either I or II has a winning strategy in $G(A)$. With increasing interest in game-theoretic approaches, in 1962 Jan Mycielski and Hugo Steinhaus proposed the Axiom of Determinacy (AD), that *every $A \subseteq {}^\omega\omega$ is determined*. AD postulated in pure form a new kind of dichotomy, one that was seen through coding information to be applicable to a broad range of issues about sets of reals. AD contradicted AC, but from the beginning it was thought that AD could animate $L(\mathbb{R})$, the constructible closure of the reals $\mathbb{R}$ $(= {}^\omega\omega)$, with unfettered uses of AC relegated to the universe at large.

In 1967 two results brought determinacy to the foreground of set theory, one about the transfinite and the other about definable sets of reals. Solovay established that AD *implies that ω_1 is measurable*, injecting emerging large cardinal techniques and results into a novel setting without AC. David Blackwell provided a new, determinacy proof of a classical result of Kuratowski, that *the $\boldsymbol{\Pi}^1_1$ sets have the reduction property*. Martin in particular saw the potentialities in both directions and soon made incisive contributions to investigations with and of determinacy. Martin initially made a simple but crucial observation in the first direction, that *assuming* AD, *the filter over the Turing degrees generated by the Turing cones is an ultrafilter*. This provided another proof of Solovay's measurability result and allowed Solovay to establish in 1968 that AD *implies that ω_2 is measurable*.

The advances in the investigation of definable sets of reals generally under AD would be in terms of their analysis as projections of trees. For purposes of descriptive set theory, T is a *tree on $\omega \times \kappa$* iff (a) T consists of pairs $\langle s, t\rangle$ where s is a finite sequence drawn from ω and t is a finite sequence of the same length drawn from κ, and (b) if $\langle s, t\rangle \in T$, s' is an initial segment of s and t' is an initial segment of t of the same length, then also $\langle s', t'\rangle \in T$. For such T, $[T]$ consists of pairs $\langle f, g\rangle$ corresponding to infinite branches, i.e. f and g are ω-sequences such that for any finite initial segment s of f and finite initial segment t of g of the same length, $\langle s, t\rangle \in T$. In modern terms with ${}^\omega\omega$ taken for the reals $\mathbb{R}$, $A \subseteq {}^\omega\omega$ is *κ-Suslin* iff there is a tree on $\omega \times \kappa$ such that $A = p[T] = \{f \mid \exists g(\langle f, g\rangle \in [T])\}$. $[T]$ is a closed set in the space of $\langle f, g\rangle$'s where $f : \omega \to \omega$ and $g : \omega \to \kappa$, and so otherwise complicated sets of reals, if shown to be κ-Suslin, are newly comprehended as projections of closed sets.

The analytic (i.e. $\boldsymbol{\Sigma}^1_1$) sets of reals are exactly the ω-Suslin sets. Membership in a $\boldsymbol{\Pi}^1_1$ set is thus characterizable as having *no* infinite branch through a tree on $\omega \times \omega$, and this well-foundedness can be converted set-theoretically into *having* an order-preserving map into ω_1, which amounts to having an infinite branch through a tree on $\omega \times \omega_1$. This witnessing possibility can be extended by an existential quantifier, and thus Joseph Shoenfield in 1961 established in ZF that *every $\boldsymbol{\Sigma}^1_2$ set is ω_1-Suslin*. Martin and Solovay analogously "dualized" $\omega \times \omega_1$ trees to get trees of order-preserving maps, maps on which a well-ordering could be imposed by using a measurable cardinal. Martin then refined this Martin–Solovay tree analysis to get a contextually optimal one using Silver-type indiscernibles. He thus established in ZF that *assuming the existence of (Silver-type) indiscernibles for $L[a]$ for every real a, every $\boldsymbol{\Sigma}^1_3$ set is ω_ω-Suslin* with trees having strong "homogeneity" properties, and as a remarkable contingency, that *assuming* AD, *for $2 < n < \omega$, ω_n is singular of cofinality ω_2*!

The structure theory of the projective sets under determinacy hypotheses, what brought the theory to prominence and what the classical descriptive set theorists had aspired for, was the inductive propagation of properties up the projective hierarchy. This propagation was not actually based directly on tree representations, but rather on related properties corresponding to determined games. Moschovakis isolated the *prewellordering property* and the central *scale property*, and established his *periodicity theorems*, from which the structure results flowed. Having defined $\Theta = \sup\{\xi \mid \text{there is a surjection: } {}^\omega\omega \to \xi\}$ to delineate the effect of AD on the transfinite, Moschovakis defined as definability analogues the *projective ordinals* $\boldsymbol{\delta}^1_n = \sup\{\xi \mid \text{there is a } \boldsymbol{\Delta}^1_n \text{ prewellordering of length } \xi\}$.[11] His work established the importance of these ordinals in Suslin representations; e.g. with a corresponding amount of determinacy, *for any $n \in \omega$, the $\boldsymbol{\Sigma}^1_{2n+2}$ sets are exactly the $\boldsymbol{\delta}^1_{2n+1}$-Suslin sets*.

Kunen's entrée into this subject was to establish in early 1971, independently with Martin, a basic theorem using scales, the Kunen–Martin Theorem of ZF + DC: *Every κ-Suslin well-founded relation on the reals has length less than κ^+*. This had as consequences some basic facts under AD about the projective ordinals: For each $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and, with Alexander Kechris having shown that the $\boldsymbol{\Sigma}^1_{2n+1}$ sets are exactly the λ_{2n+1}-Suslin sets for λ_{2n+1} a cardinal of cofinality ω, that $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$.

Martin had extended Solovay's measurability result for ω_1 $(= \boldsymbol{\delta}^1_1)$ to show that *assuming* AD, *for odd $n \in \omega$, $\boldsymbol{\delta}^1_n$ is measurable.* Having arrived at the scene, Kunen provided a uniform proof that *assuming* AD, *for all $n \in \omega$, $\boldsymbol{\delta}^1_n$ is measurable.* At this point, Kunen had quickly provided the facts about the projective ordinals that completed their basic theory.

By the summer of 1971, Kunen had brought to bear his familiarity of ultrafilters to their study under AD. Generalizing the early uses of the Martin cone filter to establish measurability, Kunen observed that under AD, ω_1 is "strongly compact up to Θ": *Assuming* AD, *for any $\lambda < \Theta$, any $\aleph_1$-complete filter over λ can be extended to an $\aleph_1$-complete ultrafilter over λ*. In fact, any such filter is included in a Rudin–Keisler projection of the Martin cone filter. This in turn had the corollary: *Assuming* AD + DC, *for any $\lambda < \Theta$, $\beta\lambda = \{U \mid U \text{ is an ultrafilter over } \lambda\}$ is well-orderable.* Kunen thus pointed out the possibility that under AD there is a great deal of global structure to ultrafilters, and this stimulated the subsequent wide investigation of the structure theory and its applications.

Kunen's main contribution in determinacy was to be to his work toward the determination of the projective ordinals through the use of their ultrafilter theory to code the projective sets. With $\boldsymbol{\delta}^1_1 = \omega_1$ being a classical result, Martin established under AD that $\boldsymbol{\delta}^1_2 = \omega_2$. His aforementioned Suslin analysis of $\boldsymbol{\Sigma}^1_3$ sets established under AD that $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, and thus the Kunen–Martin result entailed that $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. But what is $\boldsymbol{\delta}^1_5$? In a few weeks Kunen provided the contextually optimal ultrafilter analysis up to $\boldsymbol{\delta}^1_3$ and laid out a program for the calculation of $\boldsymbol{\delta}^1_5$, a program that was to be entirely successful, but only more than a decade later.

Much of this theory, already extravagant with the infusion of measurability, was driven by infinite-exponent partition relations. $\kappa \to (\kappa)^\lambda$ asserts that if the increasing functions from λ into κ are partitioned into two cells, then there is an $H \subseteq \kappa$ of cardinality κ such that all the increasing functions from λ into H are in one cell. The *strong partition property for κ* is the assertion $\kappa \to (\kappa)^\kappa$ and the *weak partition property for κ* is the assertion $\forall\lambda < \kappa(\kappa \to (\kappa)^\lambda)$. The connection with ultrafilters was mainly through an observation of Eugene Kleinberg. For $\lambda < \kappa$ both regular, let $\mathcal{C}^\lambda_\kappa$ denote the filter over κ generated by the λ-closed unbounded subsets, i.e. the filter generated by the closed unbounded filter $\mathcal{C}_\kappa$ together with the set $\{\xi < \kappa \mid \mathrm{cf}(\xi) = \lambda\}$. Kleinberg pointed out that in ZF: *If $\lambda < \kappa$ are both regular and $\kappa \to (\kappa)^{\lambda+\lambda}$, then $\mathcal{C}^\lambda_\kappa$ is a normal ultrafilter over κ*.

Martin established, quite dramatically at the time, that *assuming* AD, *ω_1 has the strong partition property*. Solovay had actually shown that the closed unbounded filter $\mathcal{C}_{\omega_1}$ witnesses the measurability of ω_1, and moreover that the ultrapower ${}^{\omega_1}\omega_1/\mathcal{C}_{\omega_1}$ is (order-isomorphic) to ω_2 and has a "canonical" representation property. Martin and Paris in early 1971 applied this to lift ω_1 having the strong partition property to show: *Assuming* AD + DC, *ω_2 has the weak partition property and so there are exactly two normal ultrafilters over ω_2, namely $\mathcal{C}^\omega_{\omega_2}$ and $\mathcal{C}^{\omega_1}_{\omega_2}$*.

Kunen drew his immediate inspiration from the Martin–Paris work, and proceeded to characterize the ultrafilters over ω_2 under AD + DC as those (Rudin–Keisler) equivalent to a product of normal ultrafilters over ω_1 and ω_2, this work also showing that ω_2 does *not* have the strong partition property.

Pushing upward from this in a considerable refinement of the Martin–Paris work, Kunen analyzed ultrapowers by the product ultrafilters $(\mathcal{C}_{\omega_1})^n$ and exploited the strong partition property for ω_1 to get a revelatory representation of subsets of ω_ω as countable unions of "simple" sets, sets comprehendible through an analysis of ultrafilters over ω_ω. This provided a $\boldsymbol{\Delta}^1_3$ coding of the subsets of ω_ω, which in turn established that *assuming* AD + DC, $\boldsymbol{\delta}^1_3$ $(= \omega_{\omega+1})$ *has the weak partition property and so there are exactly three normal ultrafilters over $\boldsymbol{\delta}^1_3$, namely $\mathcal{C}^\omega_{\omega_{\omega+1}}$, $\mathcal{C}^{\omega_1}_{\omega_{\omega+1}}$, and $\mathcal{C}^{\omega_2}_{\omega_{\omega+1}}$*.[12] Proceeding as before, Kunen characterized the ultrafilters over $\omega_{\omega+1}$ under AC + DC as equivalent to products of normal ultrafilters.

For tree representations, Kunen "dualized" the Martin–Solovay tree (mentioned above) to get representations for the $\boldsymbol{\Pi}^1_3$ sets and hence the $\boldsymbol{\Sigma}^1_4$ sets, applying ultrafilters over $\lambda < \omega_{\omega+1}$, and so *assuming* AD + DC, *$\boldsymbol{\Sigma}^1_4$ sets are $\omega_{\omega+1}$-Suslin* with trees

[11] A prewellordering $\preccurlyeq$ is a well-ordering except that there could be distinct x, y such that $x \preccurlyeq y$ and $y \preccurlyeq x$.
[12] See Solovay [91], which provides the details as he worked them out in lectures in 1976–1977.

having strong homogeneity properties as given by the weak partition property for $\omega_{\omega+1}$. Continuing, Kunen could again dualize to get representations for the $\boldsymbol{\Pi}^1_4$ sets and hence the $\boldsymbol{\Sigma}^1_5$ sets, but absent the strong partition property for $\omega_{\omega+1}$, could not get the requisite homogeneity properties to proceed further.

By the later 1970s, a complete structure theory for the projective sets was in place, a resilient edifice founded on determinacy with both strong buttresses and fine details—save for one lacuna, the determination of the projective ordinals. Kunen's 1971 work, communicated in short, pithy notes, remained the high point of the contextually optimal analysis, and the work's several aspects were given extended exposition in the latter 1970s in Kechris [39], Solovay [91], and Kechris [40]. Be that as it may, the ground lay fallow well into the 1980s.

In a veritable *tour de force*, Steve Jackson, a student of Martin, completed the determination of the projective ordinals. With the starting point a new analysis of normal ultrafilters over $\boldsymbol{\delta}^1_3$ by Martin that led to a putative lower bound for $\boldsymbol{\delta}^1_5$, Jackson determined $\boldsymbol{\delta}^1_5$ in his 1983 UCLA thesis. By 1985 Jackson had carried out the determination of all the projective ordinals, with the large part of the upper bound calculations presented in his formidable [32]. Define ordinals $E(n)$ for $n \in \omega$ by $E(0) = 1$, and $E(n+1) = \omega^{E(n)}$ in ordinal exponentiation. Jackson established: *Assuming* $\mathsf{AD} + \mathsf{DC}$*, for* $n \in \omega$*,* $\boldsymbol{\delta}_{2n+3} = \omega_{E(2n+1)+1}$ *and* $\boldsymbol{\delta}^1_{2n+3}$ *has the strong partition property.*

Jackson's proof proceeded by induction, starting at the basis with the weak partition property for $\boldsymbol{\delta}^1_3$ and establishing in turn the upper bound $\boldsymbol{\delta}^1_5 \leqslant \omega_{E(3)+1}$; the strong partition property for $\boldsymbol{\delta}^1_3$ and thus the lower bound $\omega_{E(3)+1} \leqslant \boldsymbol{\delta}^1_5$; and the weak partition property for $\boldsymbol{\delta}^1_5$; and iteratively repeating this cycle. A crucial ingredient of the proof was the Kunen idea of representing sets as countable unions of simple sets, these comprehendible through an analysis of ultrafilters—these being the main complications toward developing tree representations with good homogeneity properties.[13]

7. Saturated ideals

Kunen established results formative for the theory of saturated ideals, with one of the arguments, devised in 1972, becoming a bulwark of method for the modern theory of ideals and generic elementary embeddings. Saturated ideals, particularly $\aleph_1$-saturated ones related to measure, had already occurred in his thesis. His classic paper [53] on the subject, appearing relatively late, set out the various possibilities for saturated ideals and featured two elegant arguments which settled the remaining cases. Here we frame the context and, in turn, describe the workings of the arguments and, of one, the 1972 one, its subsequent reach.

Let I be an ideal over κ.[14] Then I is λ-*saturated* iff for any $\{X_\alpha \mid \alpha < \lambda\} \subseteq P(\kappa) - I$ there are $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in P(\kappa) - I$ (i.e. the corresponding Boolean algebra has no antichains of cardinality λ).

Solovay's 1966 work on real-valued measurable cardinals had brought to the foreground the concepts of saturated ideal, generic ultrapower, and generic elementary embedding. For an ideal I over κ, forcing with the members of $P(\kappa) - I$ as conditions, and p stronger than q when $p - q \in I$, engenders an ultrafilter on the ground model power set $P(\kappa)$. With this, one can construct an ultrapower of the ground model in the generic extension and a corresponding elementary embedding. It turns out that the κ^+-saturation of the ideal ensures that this generic ultrapower is well founded. Thus, a synthesis of forcing and ultrapowers is effected, and this raised enticing possibilities for having such large cardinal-type structure low in the cumulative hierarchy.

The first result of Kunen [53] addressed the relative consistency of having a κ-saturated ideal over an inaccessible cardinal κ in a non-trivial sense. Kunen and Paris [62] had established that starting with a measurable cardinal κ, there is a forcing extension in which $\kappa = 2^{\aleph_0}$ is a regular limit cardinal and there is a κ-saturated ideal over κ, yet no λ-saturated ideal over κ for any $\lambda < \kappa$. Answering a remaining question (cf. his [45, p. 225]) Kunen [53] established: *If* κ *is a measurable cardinal, then there is a cardinal-preserving forcing extension in which* κ *is inaccessible but not measurable and there is a* κ*-saturated ideal over* κ. Kunen cleverly devised a forcing T that adjoins a κ-Suslin tree through a large cardinal κ in such a way that the forcing combined with the further forcing for shooting a cofinal branch through the tree is equivalent to having just added one Cohen subset of κ in the first place. For the actual model, Kunen deftly applied Silver's recently arrived at "reversed Easton" iterated forcing and master condition technique to get to a preliminary forcing extension in which κ is measurable and, moreover, adjoining a further Cohen subset of κ retains measurability. He then followed this up with his T to get his final model. κ is not measurable in this model as there is a κ-Suslin tree. However, by design there is a κ-c.c. forcing over this model which resurrects measurability and so in particular secures a κ-saturated ideal over κ. Finally, a simple lemma about κ-c.c. forcing shows that there was already a κ-saturated ideal in Kunen's model!

The second result of Kunen [53] addressed the relative consistency of having a κ^+-saturated ideal over a successor cardinal κ. By his work [45] with iterated ultrapowers, the consistency strength of having such an ideal was stronger than having a measurable cardinal. To implement his argument Kunen unabashedly appealed to the strongest embedding hypothesis to date for carrying out a relative consistency construction. A cardinal κ is *huge* iff there is an elementary embedding $j : V \to M$ for some inner model M with $\mathrm{crit}(j) = \kappa$ and ${}^{j(\kappa)}M \subseteq M$. Kunen established: *If* κ *is a huge cardinal, then there is a forcing extension in which* $\kappa = \omega_1$ *and there is an* $\aleph_2$*-saturated ideal over* ω_1.

[13] See Sections 4 and 5 of Jackson's Chapter in the *Handbook of Set Theory* [18] for a schematically presented proof.

[14] That is, I is an ideal on the power set $P(\kappa)$, i.e. I is closed under the taking of subsets and of unions. In fact, we always assume that I contains all singletons and is κ-complete, i.e. I is closed under the taking of unions of fewer than κ members.

Huge cardinals are consistency-wise much stronger than supercompact cardinals and were latterly situated at the bottom of a hierarchy, the hierarchy of n-huge cardinals, that reach up to the strong hypotheses just short of Kunen's inconsistency (cf. Section 4). Supercompact cardinals, with their strong reflection properties, would become much applied in relative consistency results, but huge cardinals would remain a landmark only through Kunen's application for quite some time.

With a $j : V \to M$ with critical point κ, $\lambda = j(\kappa)$, and ${}^{\lambda}M \subseteq M$ as given by the hugeness of κ, Kunen collapsed κ to ω_1 and then collapsed λ to ω_2 in such a way so as to be able to define a saturated ideal. The first collapse was a specifically devised, iteratively constructed "universal" collapse P, and the second collapse was a "Silver collapse" S drawn from Silver's relative consistency result (cf. his [87]) for Chang's Conjecture. To show that the resulting forcing extension $V^{P*\dot{S}}$ is as desired, Kunen first used the devised universality of P to show that $P * \dot{S}$ is a subforcing of $j(P)$, and moreover, the rest of the forcing to get $j(P)$ has the λ-c.c., this requiring ${}^{\lambda}M \subseteq M$. He then showed that the Silver collapse allowed for a Silver master condition which enabled the lifting of the embedding j to $V^{P*\dot{S}}$ in the further, λ-c.c. extension. Finally, Kunen was able, as in his previous forcing argument but with some complications, to pull back a corresponding ultrafilter to show because of the λ-c.c. that there is already a λ-saturated ideal in $V^{P*\dot{S}}$.

From the late 1970s on, Kunen's argument, as variously elaborated and amended, would become a prominent tool for producing saturated ideals and other strong phenomena at accessible cardinals. Magidor [73] provided a variation on the theme that only required an "almost" huge cardinal, replacing the master condition with a "master sequence"; on the other hand, Kunen's model satisfies Chang's Conjecture whereas Magidor's does not. Foreman in his thesis (cf. [15]) established, starting from a 2-huge cardinal, the relative consistency of a three-cardinal version of Chang's Conjecture; this involved a considerable complication of the Kunen argument which successively collapsed three cardinals. Foreman [16] subsequently established that if there is a huge cardinal, then in a forcing extension there is a set model of ZFC satisfying "for all regular cardinals κ there is a κ^+-saturated ideal over κ"; this involved a delicate iteration of Kunen's forcing together with Radin forcing. Laver [70] used Kunen's argument with an "Eastonized" version of the Silver collapse to get a stronger saturation property.[15]

As for the proposition that there is an $\aleph_2$-saturated ideal over ω_1 itself, the importance of such ideals grew in the 1980s and with Kunen's result seen as setting an initial high bar for the stalking of its consistency strength, reflective and then definitive results were established. In 1984 Foreman, Magidor, and Shelah [19] established penetrating results that led to a new understanding of strong propositions and the possibilities with forcing. The focus was on a new, maximal forcing axiom, Martin's Maximum (MM), and they showed that *if there is a supercompact cardinal κ, there is a forcing extension in which $\kappa = \omega_2$ and* MM *holds.* They then established that MM *implies that the nonstationary ideal over ω_1 is $\aleph_2$-saturated.* Not only was the upper bound for the consistency strength of having an $\aleph_2$-saturated ideal over ω_1 considerably reduced from Kunen's huge cardinal, but for the first time the consistency of the nonstationary ideal over ω_1 being $\aleph_2$-saturated was established relative to large cardinals. Kunen had naturally enough collapsed a large cardinal to ω_1 in order to transmute strong properties of the cardinal into an $\aleph_2$-saturated ideal over ω_1, and this sort of direct connection had become the rule. The new discovery was that a collapse of a large cardinal to ω_2 instead can provide enough structure to secure such an ideal. In fact, Foreman, Magidor, and Shelah showed that even the usual Levy collapse of a supercompact cardinal to ω_2 engenders an $\aleph_2$-saturated ideal over ω_1. In terms of method, the central point is that the existence of sufficiently large cardinals implies the existence of substantial generic elementary embeddings with small critical points like ω_1.

Woodin in 1984 drew out what turned out to be a critical concept in this direction, that of a *Woodin cardinal*, and as is well known, this concept turned out, remarkably, to play a central role in both establishing the consistency of determinacy hypotheses and in developing inner model theory. Reducing the consistency strength for saturated ideals, in 1985 Shelah [83] established: *If κ is a Woodin cardinal, then there is a forcing extension in which $\kappa = \omega_2$ and the nonstationary ideal over ω_1 is $\aleph_2$-saturated.* Finally, with the inner model theory brought up to this level, John Steel [93] established: *If there is an $\aleph_2$-saturated ideal over ω_1 and there is a measurable cardinal, then there is an inner model with a Woodin cardinal.* Thus, having an $\aleph_2$-saturated ideal over ω_1, first shown relatively consistent by Kunen, has essentially been gauged on the scale of large cardinals, and this at a Woodin cardinal.

As a final note, Kunen had observed by considering where the ordinals go in generic ultrapowers that *there is no uniform,* (*even only*) *$\aleph_1$-complete, $\aleph_2$-saturated ideal over any cardinal between $\aleph_\omega$ and $\aleph_{\omega_1}$*. Foreman [17] has recently generalized this and carried out a systematic study of such "forbidden intervals".

8. Martin's Axiom

From 1973 on, Kunen vigorously pursued research in set-theoretic topology (later, "general topology") and areas reaching into artificial intelligence, but of course, there would continue to be significant results in set theory. Martin's Axiom had been a focal presence in his thesis; in later years, Kunen framed the limits of this axiom, in terms of possibilities both for "gaps" and for when parametrized MA could first fail.

In 1975, Kunen [60] (cf. Baumgartner [7, Theorem 4.2]) provided an incisive analysis of possible "gaps" in $\mathcal{P}(\omega)$ under eventual inclusion $\subseteq^*$, i.e. $X \subseteq^* Y$ iff $|X - Y|$ is finite, which has important MA consequences. Considering pairs $\langle \mathcal{A}, \mathcal{B} \rangle$

[15] See Sections 7.6 through 7.13 of Foreman's chapter in the *Handbook of Set Theory* [18] for a systematic account of Kunen's work as thus variously elaborated and amended.

with $\mathcal{A} = \langle X_\alpha \mid \alpha < \kappa \rangle$ and $\mathcal{B} = \langle Y_\gamma \mid \gamma < \lambda \rangle$ such that $\alpha < \beta < \kappa$ implies $X_\alpha \subseteq^* X_\beta$ and $\gamma < \eta < \lambda$ implies $Y_\eta \subseteq^* Y_\gamma$, $\langle \mathcal{A}, \mathcal{B} \rangle$ *is a* (κ, λ^*)*-gap* iff there is no $Z \subseteq \omega$ such that $\alpha < \kappa$ and $\gamma < \lambda$ implies $X_\alpha \subseteq^* Z \subseteq^* Y_\gamma$. Hausdorff [31] had famously constructed an (ω_1, ω_1^*)-gap which in the post-Cohen area was seen to be c.c.c.-indestructible, i.e. no interpolant Z can be adjoined by any c.c.c. forcing. There is a natural partial order $P(\mathcal{A}, \mathcal{B})$ with finite conditions for adjoining an interpolant Z, and Kunen showed: (a) *if* $\langle \mathcal{A}, \mathcal{B} \rangle$ *is not a* (κ, λ^*)*-gap, then* $P(\mathcal{A}, \mathcal{B})$ *has the c.c.c., and* (b) *if* $\kappa = \lambda = \omega_1$*, then there is a c.c.c. partial order that creates an uncountable antichain in* $P(\mathcal{A}, \mathcal{B})$. It follows that under MA, any $\langle \mathcal{A}, \mathcal{B} \rangle$ with $\kappa = \lambda = \omega_1$ is an (ω_1, ω_1^*)-gap: By (b) the very application of MA shows that the natural way of creating an interpolant does not have the c.c.c., and by (a) there can consequently be no interpolant at all.

Kunen used this pretty stratagem to show in effect: *It is consistent to have* MA $+ 2^{\aleph_0} = \aleph_2 +$ "*there are no* (ω_1, ω_2^*)-, (ω_2, ω_1^*)- *or* (ω_2, ω_2^*)*-gaps*". Starting with CH and $\diamondsuit_{\omega_2}$ for cofinality ω_1 ordinals, he carried out a finite-support c.c.c. iteration using the diamond sequence to anticipate possible (ω_1, ω_2^*)-, (ω_2, ω_1^*)- or (ω_2, ω_2^*)-gaps. These anticipations will have embedded in them (ω_1, ω_1^*)-gaps, which by the above stratagem will henceforth remain unfilled through the iteration, and so could not have anticipated (ω_1, ω_2^*)-, (ω_2, ω_1^*)- or (ω_2, ω_2^*)-gaps after all! There is a nice historical resonance of Kunen's "self-denying" stratagem with the very way that diamond principles were established in L. Kunen's analysis would be pursued in terms of partition relations in the important paper of Abraham, Rubin and Shelah [1].

In 1980, Kunen (cf. [57]) refined the foregoing argument to make his final reckoning of MA, that parametrized MA can first fail at a singular cardinal: *If* θ *is a singular cardinal of cofinality* ω_1*, then there is a c.c.c. partial order forcing* $2^{\aleph_0} > \theta$ *and* "θ *is the least cardinal* κ *such that there is a c.c.c. partial order with a family of* κ *dense subsets for which there is no filter meeting them all*". Kunen now considered pairs $\langle \mathcal{A}, \mathcal{B} \rangle$ with $\mathcal{A}$ and $\mathcal{B}$ no longer necessarily having internal $\subseteq^*$ relationships, and was able to preserve a (θ, θ^*)-gap through the finite-support c.c.c. iteration as a counterexample to MA for meeting θ dense sets.

The Kunen [60] stratagem would engagingly surface after three decades in Hart and Kunen [30, Theorem 5.9] for whether "Cantor trees" have the c.c.c. and just as for gaps, a consistency result with MA $+ 2^{\aleph_0} = \aleph_2$. Furthermore, in the related Kunen–Raghavan [65], which in method has a historical resonance with Kunen [57], another limit on possibilities is set by showing that there are models of MA with the continuum arbitrarily large in which there are "Gregory trees" and in which there are no such trees.

9. Envoi

When one thinks of modern set theory—that autonomous field of mathematics engaged in the continuing investigation of the transfinite numbers and definable sets of reals, employing remarkably elegant and sophisticated methods, and elucidating the consistency strength of strong propositions, indeed the strongest of mathematics—one thinks of Kunen's early work, the work of the years 1968–1972, as crucial for its development and fundamental for its articulation. However, Kunen's subsequent work in set theory, both the expository work and the extended collaborative work, has been considerable and far-ranging, and in its way substantive in its complementarity.

With his early accomplishments in set theory in place, Kunen within a decade provided several magisterial expositions that illuminated different aspects of the subject. For the 1977 *Handbook of Mathematical Logic* [4], edited by his colleague Barwise and the mother of all handbooks in logic, Kunen provided a chapter [52] on combinatorics. Going through the stuff of stationary sets, enumeration principles, trees, almost disjoint sets, partition calculus, and large cardinals, the chapter presented a remarkably integrated view of classical initiatives and modern developments. In 1980, Kunen's book [54] appeared and quickly became the standard for both the basics of the subject through definability as well as independence proofs through forcing. The presentation was precise and to the point, and the articulation of methods, unburdened and accessible. Even to the notation, a generation would imbibe set theory through this careful account. For the 1984 *Handbook of Set-Theoretic Topology* [68] that he himself edited together with Jerry Vaughn, Kunen provided an account [56] of random and Cohen reals. Random reals are, and will always remain, a bit of a mystery for some, but with his command of measure algebras Kunen presented a coordinated, forcing account of both measure and category.

Very recently, in the fullness of time, Kunen has provided a text *The Foundations of Mathematics* [59]. With the sure hand of experience in mathematical logic and computer science, Kunen filled a niche by providing a readable beginning graduate level account of set theory, model theory, and recursion theory.

Kunen's research work after the early 1970s in what would be considered set theory proper was almost all with collaborators. In 1975 Kunen provided a chart about various consistency possibilities for cardinal sizes related to measure and category, and Arnold Miller [77] presented this chart as well as established some results for it. The "Kunen–Miller" chart would elaborate a large part of the later Cichoń diagram, a focal diagram for the burgeoning investigation of the possible orderings of cardinal invariants in the 1980s and 1990s.

Kunen and Tall [66] surveyed the landscape between Martin's Axiom and Suslin's Hypothesis; their division of consequences of MA into "combinatorial" ones that readily imply $2^{\aleph_0} > \aleph_1$ and other "Suslin" type consequences would, with the first, anticipate extensive work on weak forms of MA and, with the second, anticipate Todorčević's [94] Open Coloring Axiom.

Carlson, Kunen and Miller [10] constructed a minimal degree that collapses ω_1. Kunen and Miller [61] and Keisler, Kunen, Leth and Miller [41] investigated descriptive set theory from the point of view of compact sets and over hyperfinite sets respectively. Hart and Kunen [29] studied weak measure extension axioms, axioms that posit that measures on σ-algebras can be extended to encompass a few more sets, and provided (§5) an illuminating random-graph proof of why adding $\aleph_2$ random reals leaves no Ramsey ultrafilters. Kunen and Tall [67] considered reals in elementary submodels of set theory.

In the new millennium, Baker and Kunen [2] generalized Kunen's construction of weak p-points to uniform ultrafilters over regular κ, and Baker and Kunen [3] promulgated a general approach in terms of Stone spaces of Boolean algebras and the unifying concept of a "hatpoint". Juhász and Kunen [36] explored a principle about elementary submodels which holds in models resulting from adding any number of Cohen reals to a ground model of GCH. Kunen [58] investigated a reflection property for compactness of spaces and in its terms characterized the least supercompact cardinal. Hart and Kunen [30] and Kunen and Raghavan [65] were mentioned at the end of the previous section.

The perusal of these later publications raises an overarching issue of subject and technique. Why, after all, are all of these set-theoretic and many others topological? A large body of Kunen's publications have to do with the investigation of general topological spaces using the sophisticated methods and instrumental postulations of modern set theory. But, we cannot say at what point technique begins or where it ends. We see worked in Kunen's hands a large subject, one with classical roots in the investigations of Sierpiński, Hausdorff, and others publishing in *Fundamenta Mathematicae* and conveyed in spirit in the second part of Kunen's thesis—now general topology perhaps, but imbued with modern set theory.

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