AKIHIRO KANAMORI

THE EMERGENCE OF DESCRIPTIVE SET THEORY

Descriptive set theory is the definability theory of the continuum, the study of the structural properties of definable sets of reals. Motivated initially by constructivist concerns, a major incentive for the subject was to investigate the extent of the regularity properties, those properties indicative of well-behaved sets of reals. With origins in the work of the French analysts Borel, Baire, and Lebesgue at the turn of the century, the subject developed progressively from Suslin's work on the analytic sets in 1916, until Gödel around 1937 established a delimitative result by showing that if $V = L$, there are simply defined sets of reals that do not possess the regularity properties. In the ensuing years Kleene developed what turned out to be an effective version of the theory as a generalization of his foundational work in recursion theory, and considerably refined the earlier results.

The general impression of the development of set theory during this period is one of preoccupation with foundational issues: analysis of the Axiom of Choice, emerging axiomatics, hypotheses about the transfinite, and eventual formalization in first-order logic. Descriptive set theory on the other hand was a natural outgrowth of Cantor's own work and provided the first systematic study of sets of reals building on his methods, and as such, how it developed deserves to be better known. This article provides a somewhat selective historical account, one that pursues three larger theses: The first is that the transfinite ordinals became incorporated into mathematics, Cantor's metaphysical bent and the ongoing debate about the actual infinite notwithstanding, because they became necessary to provide the requisite length for the analysis of mathematical concepts, particularly those having to do with sets of reals. The second is that later work in recursion theory and set theory emanating from Gödel's results had definite precursors in pre-formal but clearly delineated settings such as descriptive set theory. The third, related to the second, is that as metamathematical methods became incorporated into mathematics, they not only led to extra-theoretic closure results about earlier problems but to intra-theoretic advances to higher levels. The text Moschovalis [1980] serves as the reference for the mathematical

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development of descriptive set theory; the historical bearings established there are elaborated in certain directions here.[1]

As Cantor was summing up his work in what were to be his last publications, the *Beiträge*, it was the French analysts Emile Borel, René Baire, and Henri Lebesgue who were to carry the study of sets of reals to the next level of complexity. As is well-known, they, perhaps influenced by Poincaré, had considerable reservations about the extent of permissible objects and methods in mathematics. And as with later constructivists, their work led to careful analyses of mathematical concepts and a body of distinctive mathematical results. But significantly, the denumerable ordinals, Cantor's second number class, became necessary in their work, as well as the Countable Axiom of Choice, that every countable set of nonempty sets has a choice function.

Soon after completing his thesis Borel in his book (1898, pp. 46–47) considered for his theory of measure those sets of reals obtained by starting with the intervals and closing off under complementation and countable union. The formulation was *axiomatic* and in effect *impredicative*, and seen in this light, bold and imaginative; the sets are now known as the *Borel* sets and quite well understood.

Baire in his thesis (1899) took on a dictum of Dirichlet's that a real function is any arbitrary assignment of reals, and diverging from the 19th Century preoccupation with pathological examples, sought a constructive approach via pointwise limits. He formulated the following classification of real functions: *Baire class 0* consists of the continuous real functions, and for countable ordinals $\alpha > 0$, *Baire class* α consists of those functions f not in any previous class, yet for some sequence $f_0, f_1, f_2, \ldots$ of functions in previous classes f is their pointwise limit, i.e. $f(x) = \lim_{n \to \infty} f_n(x)$ for every real x. The functions in these classes are now known as the *Baire* functions, and this was the first analysis in terms of a *transfinite hierarchy* after Cantor. Baire mainly studied the finite levels, particularly classes 1 and 2; he pointed out in a note (1898) toward his thesis that Dirichlet's function that assigns 1 to rationals and 0 to irrationals is in class 2. He did observe (1899, pp. 70–71) that the Baire functions are closed under pointwise limits (with an implicit use of the Countable Axiom of Choice), and that an appeal to Cantor's cardinality arguments would imply that there are real functions that are not Baire.

Lebesgue's thesis (1902) is, of course, fundamental for modern integration theory as the source of his concept of measurability. Inspired

in part by Borel's ideas, Lebesgue's concept of measurable set subsumed the Borel sets, and his analytic definition of measurable function had the simple consequence of closure under pointwise limits, thereby subsuming the Baire functions (and so Dirichlet's old example). See Hawkins (1975) for more on the development of Lebesgue measurability; Lebesgue's first major work in a distinctive direction was to be the seminal paper in descriptive set theory.

In the memoir (1905) Lebesgue investigated the Baire functions, stressing that they are exactly the functions definable via analytic expressions (in a sense made precise). He first established a correlation with the Borel sets by showing that they are exactly the pre-images $\{x \mid f(x) \in O\}$ of open intervals O by Baire functions f. With this he introduced the first hierarchy for the Borel sets (differing in minor details from the now standard one from Hausdorff (1914)) with his *open sets of class* α being those pre-images of some open interval via some function in Baire class α that are not the pre-images of any open interval via any function in a previous class. After verifying various closure properties and providing characterizations for these classes, Lebesgue established two main results. The first demonstrated the necessity of exhausting the countable ordinals:

(1) The Baire hierarchy is proper, i.e. for every countable ordinal α there is a Baire function of class α, and consequently the corresponding hierarchy for the Borel sets is analogously proper.

The second established transcendence beyond countable closure for his concept of measurability:

(2) There is a Lebesgue measurable function which is not in any Baire class, and consequently a Lebesgue measurable set which is not a Borel set.

The hierarchy result (1) was the first of all such results, and a definite precursor of fundamental work in mathematical logic in that it applied Cantor's *universal enumeration* and *diagonal argument* to achieve a transcendence to a next level. What was missing of course was the formalization in first-order logic of Gödel's Incompleteness Theorem, but what was there was the prior extent of the ordinals, as in Gödel's later construction of L. For the first time, Cantor's second number class provided the necessary length for an individuated analysis of a class of simply defined sets of reals.

Baire (1899) had provided a characterization of Baire class 1, one elaborated by Lebesgue (1904), and had found examples of "effective" functions in class 2 (1898) and class 3 (1906), with a systematic presentation in (1909). In a formal sense, it is necessary to use higher methods to establish the existence of functions in every class.[2] Lebesgue regarded the countable ordinals as an indexing system, "symbols" for classes, but nonetheless he exposed their basic properties, giving probably the first formulation (1905, p. 149) of the concept of proof by transfinite induction. To Borel's credit, it was he (cf. his [1905, note III]) who had broached the idea of applying Cantor's diagonal method; Lebesgue incorporated definability considerations to establish (1).

The transcendence result (2) was also remarkable in that Lebesgue actually provided an explicitly defined set, one that was later seen to be the basic example of an analytic, non-Borel set. For this purpose, the reals were for the first time construed as codes for something else, namely countable well-orderings, and this not only further incorporated the transfinite into the investigation of sets of reals, but foreshadowed the later coding results of mathematical logic.

Lebesgue's results, along with the later work in descriptive set theory, can be viewed as pushing the mathematical frontier of the actual infinite past $\aleph_0$, which arguably had achieved a mathematical domesticity through increasing use in the late 19th century, to $\aleph_1$. The results stand in elegant mathematical contrast to the metaphysical to and fro in the wake of the antimonies and Zermelo's 1904 proof of the Well-Ordering Theorem. Baire in his thesis (1899, p. 36) had viewed the denumerable ordinals and hence his function hierarchy as merely *une façon de parler*, and continued to view infinite concepts only in potentiality. Borel (1898) took a pragmatic approach and seemed to accept the denumerable ordinals. Lebesgue was more equivocal but still accepting, perhaps out of mathematical necessity, although he was to raise objections against arbitrary denumerable choices. (For his (2) above, the example is explicitly defined, but to establish the transcendence the Countable Axiom of Choice was later seen to be necessary.) Poincaré (1906), Shoenflies (1905) and Brouwer (1907) (his dissertation) all objected to the existence of $\aleph_1$, although at least the latter two did accept the denumerable ordinals individually. In any case, mathematics advanced in Hausdorff's work (1908) on transfinite order types: Objecting to all the fuss being made over foundations and pursuing the higher transfinite with vigor, he formulated for the first time the Generalized Continuum Hypothesis, introduced the η_α sets – prototypes for saturated model theory – and broached the possibility of an uncountable regular limit cardinal – the beginning of large cardinal theory. The mathematical advances of the period in set theory were soon codified in the classic text Hausdorff (1914).

During these years, Lebesgue measure became widely accepted as a *regularity property*, a property indicative of well-behaved sets of reals. Two others were discussed: the Baire property and the perfect set property. All three properties were to become of central concern in descriptive set theory for, unlike the Borel sets, there did not seem to be any hierarchical analysis, and indeed the extent of the sets of reals possessing these properties was quite unclear.

The Baire property evolved from the other important concept in Baire's thesis (1899), that of category: A set of reals is *nowhere dense iff* its closure under limits contains no open set; a set of reals is *meager* (or *of first category*) *iff* it is a countable union of nowhere dense sets; and a set of reals has the ***Baire property iff*** it has a meager symmetric difference with some open set. Straightforward arguments show that every Borel set has the Baire property.

The second regularity property has its roots in the very beginnings of set theory: A set of reals is *perfect iff* it is nonempty, closed and contains no isolated points; and a set of reals has the *perfect set property iff* it is either countable or else has a perfect subset. Using his notion of derived set emerging out of his work on the convergence of trigonometric series, Cantor (1883, 1884) and Bendixson (1883) established that every closed set has the perfect set property. Since Cantor (1884) established that every perfect set has the cardinality of the continuum, this provided a more concrete approach to his Continuum Problem: at least no closed set of reals can have an intermediate cardinality between $\aleph_0$ and the cardinality of the continuum. William Young (1903) extended the Cantor-Bendixson result by showing that every G_δ set[3] of reals has the perfect set property. However, unlike for the other regularity properties it was by no means clear that every Borel set has the perfect set property and the verification of this was to only to come a decade later with the shifting of the scene from Paris to Moscow.

The subject of descriptive set theory emerged as a distinct discipline through the initiatives of the Russian mathematician Nikolai Luzin. Through a focal seminar that he began in 1914 at the University of Moscow, he was to establish a prominent school in the theory of functions of a real variable.[4] Luzin had become acquainted with the work and views of the French analysts while he was in Paris as a student, and from the beginning a major topic of his seminar was the "descriptive theory of functions". Significantly, the young Polish mathematician Wacław Sierpiński was an early participant; he had been interned in Moscow in 1915, and Luzin and his teacher Egorov interceded on his behalf to let him live freely until his repatriation to Poland a year later. Not only did this lead to a decade long collaboration between Luzin and Sierpiński, but undoubtedly it encouraged the latter in his efforts toward the founding of the Polish school of mathematics[5] and laid the basis for its interest in descriptive set theory.

In the spring of 1915 Luzin described the cardinality problem for Borel sets (operatively whether they have the perfect set property) to Pavel Aleksandrov, an early member of Luzin's seminar and later a pioneer of modern topology. By that summer Aleksandrov (1916) had established his first important result:

(3) Every Borel set has the perfect set property.

Hausdorff (1916) also established this, after getting a partial result (1914, p. 465). The proof of (3) required a new way of comprehending the Borel sets, as underscored by the passage of a decade after Lebesgue's work. It turns out that the collection of sets having the perfect set property is not closed under complementation, so that an inductive proof of (3) through a hierarchy is not possible. The new, more direct analysis of Borel sets broke the ground for a dramatic development.

Soon afterwards another student of Luzin's, Mikhail Suslin (often rendered Souslin in the French transliteration), began reading Lebesgue (1905). Memoirs of Sierpiński (1950, p. 28ff) recalled how Suslin then made a crucial discovery in the summer of 1916. For $Y \subseteq R^{k+1}$, the *projection of Y* is

$$pY = \{\langle x_1, \ldots, x_k\rangle \mid \exists y(\langle x_1, \ldots, x_k, y\rangle \in Y)\}.$$

Suslin noticed that at one point Lebesgue asserted (1905, pp. 191–192) that the projection of a Borel subset of the plane[6] is also a Borel set. This was based on the mistaken claim that given a countable collection of subsets of the plane the projection of their intersection equals the intersection of their projections. Suslin found a counterexample to Lebesgue's assertion, and this led to his inspired investigation of what are now known as the *analytic* sets. (Lebesgue later ruefully remarked that his assertion was "simple, short, but false" (Luzin [1930, p. vii]); however, it did not affect the main results of his memoir.)

Suslin (1917) formulated the analytic sets as the A-sets (les ensembles (A)), sets resulting from an explicit operation, the Operation (A): A *defining system* is a family $\{X_s\}_s$ of sets indexed by finite sequences s of integers. $A(\{X_s\}_s)$, the result of the Operation (A) on such a system, is that set defined by:

$$x \in A(\{X_s\}_s) \quad \textit{iff} \quad \exists f\colon \omega \to \omega \forall n \in \omega (x \in X_{f|n}).$$

For X a set of reals,

X is *analytic* *iff* $X = A(\{X_s\}_s)$ for some defining system $\{X_s\}_s$ consisting of closed sets of reals.

As Suslin essentially noted, this implies that *a set of reals is analytic iff it is the projection of a G_δ subset of the plane.*[3,6] He announced three main results:

(4) Every Borel set is analytic.

In fact:

(5) A set of reals is Borel *iff* both it and its complement are analytic;

and:

(6) There is an analytic set that is not Borel.

These results are analogous to later, better known results with "recursive" replacing "Borel" and "recursively enumerable" replacing "analytic". (1917) was to be Suslin's sole publication, for he succumbed to typhus in a Moscow epidemic in 1919 at the age of 25. (The whole episode recalls a well-known equivocation by Cauchy and the clarification due to the young Abel that led to the concept of uniform convergence, even to Abel's untimely death.)

In an accompanying note, Luzin (1917) announced the regularity properties for the analytic sets:

(7) Every analytic set is Lebesgue measurable, has the Baire property, and has the perfect set property.

He attributed the last to Suslin.[7]

Whether via the geometric operation of projection of G_δ sets or via the explicit Operation (A) on systems of closed sets, the Russians had hit upon a simple procedure for transcending the Borel sets, one that preserves the regularity properties. Paradigmatic for later hierarchy results, Suslin's (5) provided a dramatically simple characterization from above of a class previously analyzed from below in a hierarchy of length $\aleph_1$, and held the promise of a new method for generating simply defined sets of reals possessing the regularity properties.[8]

The notes of Suslin (1917) and Luzin (1917) were to undergo considerable elaboration in the ensuing years. Proofs of the announced results (4)–(6) appeared in Luzin-Sierpiński (1918, 1923); as for (7), the Lebesgue measurability result was established in the former, the Baire property result in the latter, and the perfect set property had to await Luzin (1926). Luzin-Sierpiński (1923) was a pivotal paper, in that it shifted the emphasis toward *co-analytic* sets, complements of analytic sets, and provided a basic representation for them from which the main results of the period flowed. With it, they established:

(8) Every analytic set is both a union of $\aleph_1$ Borel sets and an intersection of $\aleph_1$ Borel sets.

The representation of co-analytic sets had an evident precedent in Lebesgue's proof of (2); the idea can be conveyed in terms of the Operation (A): Suppose that $Y \subseteq R$ is co-analytic, i.e. $Y = R - X$ for some $X = A(\{X_s\}_s)$, so that

$$x \in Y \quad \textit{iff} \quad x \notin X \quad \textit{iff} \quad \forall f\colon \omega \to \omega \exists n (x \notin X_{f \restriction n}).$$

For finite sequences s_1 and s_2, define: $s_1 \prec s_2$ *iff* s_2 is a proper initial segment of s_1. For a real x define: $T_x = \{s \mid x \in X_t$ for every initial segment t of $s\}$. Then

(9) $x \in Y$ *iff* $\prec$ on T_x is a well-founded relation,

i.e. there is no infinite descending sequence $\ldots \prec s_2 \prec s_1 \prec s_0$.

Thus did well-founded relations enter mathematical praxis. The well-known analysis by von Neumann (1925) and Zermelo (1930) prefigured by Mirimanoff (1917) was particular to the membership relation: this of course led to the Axiom of Foundation and the cumulative hierarchy view of the universe of sets, and crucial as this development was, the main thrust was in the direction of axiomatization of an underlying structural principle.

Luzin and Sierpiński (1918, 1923) linearized their well-founded relations, submerging well-foundedness under the better known framework of well-ordering and getting an ordinal analysis of co-analytic sets. This was natural to do for their results, as various technical aspects became simplified by appeal to the linear comparability of well-ordered sets. The linearization was through none other than what is now known as the "Kleene-Brouwer" ordering. But already in Luzin (1927, p. 50) well-founded relations on the integers were defined explicitly because of their necessary use in his proof of the Borel separability of analytic sets,[9] one route to Suslin's (5). It was only later through the fundamental collapsing isomorphism result of Andrzej Mostowski (1949) that well-founded relations were seen to have a canonical representation via the membership relation that well-orderings have in the correlation with (von Neumann) ordinals. Dana Scott's celebrated result (1961) that if a measurable cardinal exists, then $V \neq L$ can be viewed as a well-founded version of a previous result about well-orderings due to Keisler (1960). Through such reorientations, well-foundedness has achieved a place of prominence in current set theory shoulder-to-shoulder next to well-ordering.

The next conceptual move, a significant advance for the theory, was, to extend the domain of study by taking the operation of projection as basic. Luzin (1925a) and Sierpiński (1925) defined the *projective* sets as those sets obtainable from the Borel sets by the iterated applications of projection and complementation. We have the corresponding *projective hierarchy* in modern notation: For $A \subseteq R^k$,

$$A \text{ is } \Sigma^1_1 \quad \textit{iff} \quad A \text{ is analytic},$$

(defined as for $k = 1$ in terms of a defining system consisting of closed subsets of R^k) and inductively for integers n,

$$A \text{ is } \Pi^1_n \quad \textit{iff} \quad R^k - A \text{ is } \Sigma^1_n, \quad \text{and}$$

$$A \text{ is } \Sigma^1_{n+1} \quad \textit{iff} \quad A = pY \text{ for some } \Pi^1_n \text{ set } Y \subseteq R^{k+1}.$$

Also

$$A \text{ is } \Delta^1_n \quad \textit{iff} \quad A \text{ is both } \Sigma^1_n \text{ and } \Pi^1_n.$$

Luzin (1925c) and Sierpiński (1925) recast Lebesgue's use of the Cantor diagonal argument to show that the projective hierarchy is proper, and soon its basic properties were established by various people, e.g. each of the classes Σ^1_n and Π^1_n is closed under countable union and intersection.

On the other hand, the investigation of projective sets encountered basic obstacles from the beginning. For one thing, unlike for the analytic sets the perfect set property for the Π^1_1, or co-analytic, sets could not be established. Luzin (1917, p. 94) had already noted this difficulty, and it was emphasized as a major problem in Luzin (1925a). In a confident and remarkably prophetic passage, he declared that his efforts towards its resolution led him to a conclusion "totally unexpected", that "one does not know and *one will never know*" of the family of projective sets, although it has the cardinality of the continuum and consists of "effective" sets, whether every member has the cardinality of the continuum if uncountable, has the Baire property, or is even Lebesgue measurable. This speculation from mathematical analysis stands in contrast to the better known anticipation by Skolem (1923, p. 229) of the independence of the Continuum Hypothesis based on metamathematical considerations. Luzin (1925b) pointed out another problem, that of establishing the Lebesgue measurability of Σ^1_2 sets. Both these difficulties at Π^1_1 and Σ^1_2 were also observed by Sierpiński (1925, p. 242), although he was able to show:

(10) Every Σ^1_2 set is the union of $\aleph_1$ Borel sets.

The first wave of progress from Suslin's results having worked itself out, Luzin provided systematic accounts in two expository papers (1926, 1927) and a text (1930). In (1927) and more generally in (1930) he introduced the concepts of *sieves* and *constituents*, implicit in earlier papers. Loosely speaking, a sieve is a version of a defining system for Operation (A), and a constituent is, in terms of (9), a set of form

$$C_\alpha = \{x \in Y \mid \prec \text{ on } T_x \text{ has rank } \alpha\},$$

for some ordinal α. (Every well-founded relation has a rank, its "height", defined by transfinite recursion. These constituents turn out to be Borel sets if the defining system consists of Borel sets, and so the first half of (8) is already evident in $Y = \bigcup_{\alpha < \aleph_1} C_\alpha$.) Sieves and constituents not only became the standard tools for the classical investigation of the

first level of the projective hierarchy, but also became the subjects of considerable study in themselves.

Most extensively in his classic text (1930), Luzin aired the constructivist views of his French predecessors. Not only did he contrive self-effacingly to establish definite precedents for his own work in theirs,[10] but he also espoused their distrust of the unbridled Axiom of Choice and advocated their views on definability, especially analyzing Lebesgue's informal concept of nameability (qu'on peut nommer). He regarded his investigations as motivated by these considerations, as well as by specific new intuitions. For instance, he considered the complementation operation used in the formulation of the projective hierarchy to be "negative" in a sense that he elaborated, its use equivalent to that of all the denumerable ordinals, and that this led to the difficulties. He wrote (1930, p. 196): "Thus, the transfinite can be profoundly hidden in the form of a definition of a negative notion." Related to this, Lebesgue (1905) had ended with question, "Can one name a non-measurable set?", and taking this as a starting point for his own work, Luzin (1930, p. 323) wrote sagaciously: ". . . the author considers the question of whether all projective sets are measurable or not to be unsolvable [insoluble], since in his view the methods of defining the projective sets and Lebesgue measure are not comparable, and consequently, not logically related."

If the projective sets proved intractable with respect to the regularity properties, significant progress was nonetheless made in other directions. In Luzin (1930a) the general problem of *uniformization* was proposed. For $A, B \subseteq R^2$,

$$A \text{ is } \textit{uniformized by } B \textit{ iff } B \subseteq A \text{ and } \forall x(\exists y(\langle x, y\rangle \in A) \leftrightarrow \exists! y(\langle x, y\rangle \in B)).$$

(As usual, $\exists!$ abbreviates the formalizable "there exists exactly one".) Since B is in effect a choice function for an indexed family of sets, asserting the uniformizability of arbitrary $A \subseteq R^2$ is a version of the Axiom of choice. Taking this approach to the problem of definable choices, Luzin announced several results about the uniformizability of analytic sets by like sets. One was affirmed in a sharp form by Novikov (1931), who showed that there is a closed set that cannot be uniformized by any analytic set. It was eventually shown by Yankov (1941) that every analytic set can be uniformized by a set that is a countable intersection of countable unions of differences of analytic sets. Interestingly enough, von Neumann (1949, p. 448ff) also established a less structured uniformization result for analytic sets as part of an extensive study of rings of operators. Presumably because of his difficulties with Π^1_1 sets, Luzin (1930a, p. 351) claimed that there were Π^1_1 sets that could not be uniformized by any "distinguishable" set, and gave a purported example. Notwithstanding, Sierpiński (1930) asked whether every Π^1_1 set can be uniformized by a projective set, and a result of Petr Novikov in Luzin-Novikov (1935) implied that they can, by sets that are at least Σ^1_2. Building on this, the Japanese mathematician Motokiti Kondô (1937, 1939) established the Π^1_1 *Uniformization Theorem*:

(11) Every Π^1_1 subset of R^2 can be uniformized by a Π^1_1 set.

This was the culminating result of the ordinal analysis of Π^1_1 sets. As Kondô noted, his result implies through projections that every Σ^1_2 set can be uniformized by a Σ^1_2 set, but the question of whether every Π^1_2 set can be uniformized by a projective set was left open.

There was also systematic elaboration. Of Luzin's school,[11] Lavrent'ev and Keldysh carried out a deeper investigation of the Borel hierarchy in terms of topological invariance, canonical sets, and constituents of sieves. And Selivanovskii, Novikov, Kolmogorov, and Lyapunov pushed the regularity properties to a stage intermediate between the first and second levels of the projective hierarchy with their study of the C-sets and especially the R-sets. The Poles, who had redeveloped the basic theory through various shorter papers in *Fundamenta Mathematicae* through the 1920's, emphasized topological generalization, lifting the theory to what are now known as *Polish spaces*.[12] Baire (1909) had already stressed the economy of presentation in switching from the reals to what is now known as *Baire space*, $\{f \mid f\colon \omega \to \omega\}$, essentially the "fundamental domain" of Luzin (1930). Soon the development of the theory in axiomatically presented topological spaces became popular. As for Luzin himself, he returned to the problem of the perfect set property for Π^1_1 sets, first broached in his (1917); in a lecture in 1935, anticipating Gödel's delimitative result Luzin stated several *constituent problems*, each of which would establish the existence of a Π^1_1 set without the perfect set property.[13]

The next advances were to be made through the infusion of metamathematical techniques. Kuratowski-Tarski (1931) and Kuratowski (1931) observed that in the study of the projective sets, the set-

theoretic operations correspond to the logical connectives, and projection to the existential quantifier, and consequently, the basic manipulations with projective sets can be recast in terms of logical operations. This move may seem like a simple one, but one must recall that it was just during this period that first-order logic was being established as the canonical language for foundational studies by the great papers of Skolem (1923) and Gödel (1930, 1931). This sacred tradition established a precise notion of "definable", and so in retrospect, prudent was the profane choice of the term "descriptive".

The total impasse in descriptive set theory with respect to the regularity properties was to be explained by Gödel's work on the consistency of the Axiom of Choice (AC) and the Continuum Hypothesis (CH). This work can be viewed as a steady intellectual development from his celebrated Incompleteness Theorem, and with respect to our theme of the mathematical necessity of the transfinite, the prescient footnote 48a of Gödel (1931) is worth quoting:

> The true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite . . . while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added. . . . An analogous situation prevails for the axiom system of set theory.

Gödel of course established his consistency results by formulating the inner model L, still one of the most beautiful constructions in set theory, and showing that if $V = L$, then AC and CH holds. His main breakthrough can be loosely described as taking the extent of the ordinals as *a priori* and carrying on a kind of Gödel numbering of definable sets through the transfinite. Here we see in an ultimate form how having enough length turns negative or paradoxical assertions to positive ones. Russell's paradox became the proposition that the universe is not a set; Burali-Forti's paradox became Hartog's proposition about the existence of the next aleph, and Gödel's Incompleteness Theorem became a rectification of Russell's ill-fated Axiom of Reducibility with the proof, making an ironic use of Skolem's paradox argument, of the consistency of the Continuum Hypothesis.

In his initial article (1938) on L, Gödel announced:

(12) If $V = L$, then there is a Δ^1_2 set of reals which is not Lebesgue measurable and a Π^1_1 set of reals which does not have the perfect set property.

Thus, the classical descriptive set theorists were up against an essential obstacle of ZFC. The importance that Gödel attached to these results can be evinced from his listing of each of them on equal footing with his AC and CH results. Gödel did not publish proofs, and more than a decade was to pass before proofs first appeared in Novikov (1951). In the meantime, Gödel in the second edition (1951, p. 67) of his monograph on L had sketched a more basic result:

(13) If $V = L$, then there is a Σ^1_2 well-ordering of the reals.

(According to Kreisel (1980, p. 197), ". . . according to Gödel's notes, not he, but S. Ulam, steeped in the Polish tradition of descriptive set theory, noticed that the definition of the well-ordering . . . of subsets of ω was so simple that it supplied a non-measurable [Σ^1_2] set of real numbers . . .".) Mostowski had also established the result in a manuscript destroyed during the war, but it is not apparent in Novikov (1951). Details were eventually provided by John Addison (1959) who showed that in L every projective set can be uniformized by a projective set:

(14) If $V = L$, then for $n > 1$ every Σ^1_n subset of R^2 can be uniformized by a Σ^1_n set.

Gödel's incisive metamathematical analysis not only provided an explanation for the descriptive set theorist in terms of the limits of formal systems, but also provided explicit counterexamples at the next level, once logical operations were correlated with the classical concepts. Perhaps Wittgenstein would have found congenial the theme of the mathematical necessity of the transfinite ordinals through their increasing use, but no friend of set theory, in his railings against metamathematics he would have frowned at its inversion into mathematics *par excellence*, owing ultimately to the coding possibilities afforded by infinite sets.

Looking ahead, just a year after Paul Cohen's invention of forcing, Robert Solovay (1965, 1970) established the following relative consistency result, showing what level of argument is possible with the method.

(15) Suppose that in ZFC there is an inaccessible cardinal. Then there is a ZFC forcing extension in which every projective set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.

The existence of an inaccessible cardinal is the weakest in the hierarchy of "large cardinal" axioms adding consistency strength to ZFC that have been extensively studied. Solovay himself noted that that consistency strength is necessary for the perfect set property, and rather unexpectedly, it was eventually shown by Saharon Shelah (1984) that it was also necessary for Lebesgue measurability, but not for the Baire property. These beautiful results in terms of relative consistency provide a mathematically satisfying resolution of the universal possibilities for the projective sets. Solovay was actually able to get a model of ZF in which every set of reals whatsoever has the regularity properties; this is well-known to contradict AC, but on the other hand Solovay's model satisfied the Axiom of Dependent Choices, a weak form of AC adequate for carrying out all of the arguments of descriptive set theory.

As for forcing and uniformization, Levy (1965) observed that in the original Cohen model ("adding a Cohen real over L") there is a Π^1_2 set that cannot be uniformized by an projective set, in contradistinction to (14) and establishing that (11) is the best possible (cf. the paragraph after).

Scott's result that if there is a measurable cardinal, then $V \neq L$ was already mentioned. The existence of a measurable cardinal is the paradigmatic large cardinal hypothesis, much stronger in consistency strength than the existence of an inaccessible cardinal. In 1965, building on (15) Solovay reactivated the classical program of investigating the extent of the regularity properties by providing characterizations at the level of Gödel's delimitative results (see Solovay (1969) for the perfect set property), and establishing the following direct implication:

(16) If there is a measurable cardinal, then every Σ^1_2 set is Lebesgue measurable, has the Baire property, and has the perfect set property.

Natural inductive arguments were later to establish that, under hypotheses about the determinateness of certain infinitary games, every projective set possesses the regularity properties. These results focused attention on the Axiom of Determinacy and its weak versions, and led in the latter 1980's to remarkable advances in the investigation of strong hypothesis and relative consistency.[14] But an adequate discussion of these matters would go far beyond the scope of this paper.

Returning to much earlier developments based on Gödel's work, after his fundamental work on recursive function theory in the 1930's Stephen Kleene expanded his investigations of effectiveness and developed a general theory of definability for relations on the integers. In (1943) he studied the *arithmetical relations*, those relations obtainable from the recursive relations by application of *number* quantifiers. Developing canonical representations, he classified these relations into a hierarchy according to quantifier complexity and showed that the hierarchy is proper. In (1955, 1955a, 1955b) he studied the *analytical relations*, those relations obtainable from the arithmetical relations by applications of *function* quantifiers. Again, he worked out representation and hierarchy results, and moreover, established an elegant theorem that turned out to be an effective analogue of Suslin's characterization (5) of the Borel sets.

Kleene was developing what amounted to the effective content of the classical theory, unaware that his techniques had direct antecedents in the papers of Lebesgue, Luzin, and Sierpiński. Certainly, he had very different motivations: with the arithmetical relations he wanted to extend the Incompleteness Theorem, and analytical relations grew out of his investigations of notations for recursive ordinals. On the other hand, already in (1943, p. 50) he did make elliptic remarks about possible analogies with the classical theory. Once the conceptual move was made to the consideration of relations on *functions* of integers and with the classical switch to Baire space already in place, it was Kleene's student Addison who established the exact analogies: the analytical relations are analogous to the projective sets, and the arithmetical relations are analogous to the sets in the first ω levels of the Borel hierarchy.

Another mathematical eternal return: Toward the end of his life, Gödel regarded the question of whether there is a linear hierarchy for the recursive sets as one of the big open problems of mathematical logic. Intuitively, given two decision procedures, one can often be seen to be simpler than the other. Now a set of integers is recursive *iff* both it and its complement are recursively enumerable. The pivotal result of classical descriptive set theory is Suslin's, that a set is Borel *iff* both it and its complement are analytic. But before that, a hierarchy for the Borel sets was in place. In an ultimate inversion, as we look back into the recursive sets, there is no known hierarchy.

Boston University

NOTES

[1] Moore (1982) also provided some historical guidance.

[2] It turns out that every real function is of Baire class at most 3 if $\aleph_1$ is a countable

union of countable sets, and this proposition is consistent with ZF by forcing as first observed by Feferman-Levy (1963).

[3] G_δ sets are the countable intersections of open sets in the cumulative hierarchy for Borel sets from Hausdorff (1914).

[4] See Phillips (1978). Uspenskii (1985) and Kanovei (1985) are recent, detailed surveys of the work of Luzin and his school in descriptive set theory. Uspenskii (1985, p. 98) wrote: "... in his days the descriptive theory, distinguished by his work and that of Suslin, Aleksandrov, Kantorovich, Keldysh, Kolmogorov, Lavrent'ev, Lyapunov, and Novikov, was the fame of mathematics in our country ...".

[5] See Kuzawa (1968) and Kuratowski (1980).

[6] The Borel subsets of the plane, R^2, and generally those of R^k, are defined analogously to those of R.

[7] This attribution is actually a faint echo of a question of priority. According to the memoirs of Aleksandrov (1979, pp. 284–286) it was he who had defined the A-sets, and Suslin proposed the name, as well as "Operation (A)" for the corresponding operation, in Aleksandrov's honor. This eponymy is not mentioned in Suslin (1917), but is supported by recollections of Lavrent'ev (1974, p. 175) and Keldysh (1974, p. 180) as well as Kuratowski (1980, p. 69). Aleksandrov recalled that it was he who had shown that every Borel set is an A-set and that every A-set has the perfect set property, although this s not explicit in his (1916). He then tried hard in 1916 to show that every A-set is Borel, only ceasing his efforts when it became known that in the summer Suslin had found a non-Borel A-set. According to Aleksandrov: "Many years later Luzin started to call A-sets analytic sets and began, contrary to the facts, which he knew well, to assert that the term 'A-set' is only an abbreviation for 'analytic set'. But by this time my personal relations with Luzin, at one time close and sincere, were estranged." Luzin (1925, 1927) did go to some pains to trace the term "analytic" back to Lebesgue (1905) and pointed out that the original example there of a non-Borel Lebesgue measurable set is in fact the first example of an non-Borel analytic set. See also the text Luzin (1930, pp. 186–187), in which the Operation (A) is conspicuous by its absence. Aleksandrov also wrote: "This question of my priority in this case never made much difference to me; it was just my first result and (maybe just because of that) the one dearest to me".

[8] Although there may have been growing acceptance of the denumerable ordinals, it was still considered hygienic during this transitional period to eliminate the use of the transfinite where possible. The emphasis of Suslin (1917) was on how (5) does this for the definition of the Borel sets. Sierpiński (1924) featured a "new" proof without transfinite ordinals of the perfect set property for Borel sets. And earlier his younger compatriot Kuratowski (1922) had offered what is now known as Zorn's Lemma primarily to avoid the use of transfinite ordinals.

[9] If X and Y are disjoint analytic sets, there is a Borel set B such that $X \subseteq B$ and $Y \cap B = \emptyset$.

[10] But see endnote 7.

[11] For details and references, see Kanovei (1985).

[12] Complete separable metric spaces.

[13] See Uspenskii (1985, p. 126ff), Kanovei (1985, p. 162ff), and especially Uspenskii-Kanovei (1983).

[14] See Martin-Steel (1988, 1989) and Woodin (1988).

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