5 Aspect-Perception and the History of Mathematics

Akihiro Kanamori

In broad strokes, mathematics is a vast yet multifarious edifice and mode of reasoning based on networks of *conceptual constructions*. With its richness, variety and complexity, any discussion of the nature of mathematics cannot but account for these networks through its evolution in history and practice. What is of most import is the emergence of knowledge, and the carriers of mathematical knowledge are *proofs*, more generally arguments and procedures, as embedded in larger contexts. One does not really get to *know* a proposition but, rather, a proof, the complex of argument taken together as a conceptual construction. Propositions, or rather their prose statements, gain or absorb their sense from the proofs made on their behalf, yet proofs can achieve an autonomous status beyond their initial contexts. Proofs are not merely stratagems or strategies; they, and thus their evolution, are what carry forth mathematical knowledge.

Especially with this emphasis on proofs, aspect-perception—*seeing* an aspect, seeing something *as* something, seeing something *in* something—emerges as a schematic for or approach to what and how we know, and this for quite substantial mathematics. There are sometimes many proofs for a single statement, and a proof argument can cover many statements. Proofs can have a commonality, itself a proof; proofs can be seen as the same under a new light; and disparate proofs can be correlated, this correlation itself amounting to a proof. Less malleably, statements themselves can be seen as the same in one way and different in another, this bolstered by their proofs.

With this, aspect-perception counsels the history of mathematics, taken in two neighboring senses. There is the history, the patient accounting of people and their mathematical accomplishments over time, and there is the mathematics, evolutionary analysis of results and proofs over various contexts. In both, there would seem to be the novel or the surprising. Whether or not there is creativity involved, according to one measure or another, analysis through aspects fosters understanding of the byways of mathematics.

In what follows, the first section briefly describes and elaborates aspect-perception with an eye to mathematics. Then in each of the succeeding two sections, substantial topics are presented that particularly draw out and show aspect-perception at work.

1. Aspect-Perception

Aspect-perception is a sort of meta-concept, one collecting a range of very different experiences mediating between seeing and thinking. Outwardly simple to instantiate, but inwardly of intrinsic difficulty, it defies easy reckoning, but, once seen, it invites extension, application, and articulation. For the discussion and scrutiny of mathematics, it serves to elaborate and to focus aspect-perception in certain directions and with certain emphases. And for this, it serves to proceed through a deliberate arrangement of some *loci classici* for aspect-perception in the writings of Ludwig Wittgenstein.

Aspect-perception was a recurring motif for Wittgenstein in his discussions of perception, language, and mathematics. His later writings especially are filled with remarks, some ambitious and others elliptical, gnawing on a variety of phenomena of aspect-perception. It was already at work in his *Notebooks 1914-16* and his 1921 *Tractatus*, with its *Bildhaftigkeit*, has at 5.5423 the Necker cube:

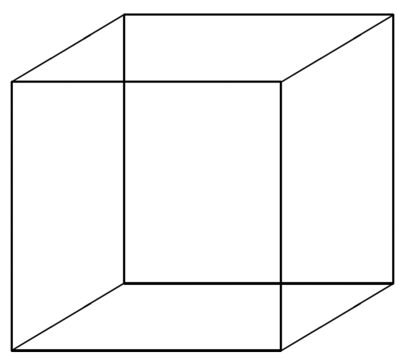


Figure 5.1

The figure can be seen in two ways as a cube, the left square in the front or the right square in the front. 'For we really see two different facts' (Wittgenstein 1921). Among the points that were made here: a symbol, serving to articulate a truth, involves a projection by us into a space of possibility, hence multifarious relationships, not merely the interpretation of a sign. This early juncture in Wittgenstein correlates with aspects of symbolization in mathematics: as explicit in succeeding sections, different modes of organization to be brought out for purposes of proof can be carried by one geometric diagram or one algebraic equation.

In his 1934 *Brown Book* (cf. 1958: II,§16), Wittgenstein discusses 'seeing it *as a face*' in the 'picture-face', a circular figure with four dashes inside, and in a picture puzzle, where 'what at first sight appears as "mere dashes" later appears as a face'. 'And in this way "seeing dashes as a face" does not involve a comparison between a group of dashes and a real human face.' We are taking or interpreting the dashes as a face. This middle juncture in Wittgenstein, at an aspect more conceptual than visual, correlates with aspects of contextual imposition in mathematics: as exhibited by the 'commonality' (**) in §2, there can be a structure or a proof, logical yet lean, whose 'physiognomy' can be newly seen by being placed in a rich conceptual or historical context.

In Wittgenstein's mature *Philosophical Investigations* (1953) aspect-perception comes to the fore in Part II, Section xi.[1] 'I contemplate a face, and then suddenly notice its likeness to another. I *see* that it has not changed; and yet I see it differently. I call this experience [*Erfahrung*] "noticing an aspect"' (p. 193c). Wittgenstein works through an investigation of various schematic figures, particularly the Jastrow duck–rabbit (p. 194e), a figure which can be seen either as a duck with its beak to the left or a rabbit with its ears to the left. With this, he draws out the distinction between 'the "continuous seeing" of an aspect' (seeing, with immediacy; later, 'regarding-as'), and 'the "dawning" of an aspect' (sudden recognition). With 'continuous seeing' he navigates a subtle middle road between being caused by the figure to perceive and the imposition of a subjective, private inner experience, undermining both as explanations of the phenomenon of seeing-as. Here follows a sequential arrangement of quotations, citing page and paragraph:

> 198c: The concept of a representation of what is seen, like that of a copy, is very elastic, and so *together with it* the concept of what is seen. The two are intimately connected. (Which is *not* to say that they are alike.)
>
> 199b: If you search in a figure (1) for another figure (2), and then find it, you see (1) in a new way. Not only can you give a new kind of description of it, but noticing the second figure was a new visual experience.
>
> 200f: When it looks as if there were no room for such a form between other ones you have to look for it in another dimension. If there is no room here, there *is* room in another dimension. [An example of imaginary numbers for the real numbers follows.]
>
> 201b: . . . the aspects in a change of aspects are those ones which the figure might sometimes have *permanently* in a picture.
>
> 203e: 'The phenomenon is at first surprising, but a physiological explanation of it will certainly be found.'—Our problem is not a causal one but a conceptual one.
>
> 204g: Here it is *difficult* to see that what is at issue is the fixing of concepts. A *concept* forces itself on one.
>
> 208d: One *kind* of aspect might be called 'aspects of organization'. When the aspect changes parts of the picture go together which before did not.

> 212a: . . . what I perceive in the dawning of an aspect is not a property of the object, but an internal relation between it and other objects.

For a visual context of some complexity pointing us towards the appreciation of aspects in mathematics, one can consider the Cubist paintings of Pablo Picasso and Georges Braque. These elicit aspects of aspect-perception emphasized by Wittgenstein like seeing faces and objects from various perspectives; continuous seeing; dawning of an aspect; aspects there to be seen; possible blindness to an aspect for a fully competent person and making sense of bringing such a person to see the aspect. Moreover, there is a shifting of aspects beyond complementary pairs, as several aspects can be kept in play at once, some superposing on others, some internal to others, some at an intersection of others, and, with the painters' willful obstructionism, some petering out at borders and some incompatible with others in varying degrees.

With an eye to mathematics, we can locate aspect-perception among broad philosophical abstractions in the following ways:

1. Aspect-perception is not (merely) *psychological* or *empirical,* but substantially *logical*, in working in spaces of logical possibility.
2. Aspect-perception is not in the service of *conventionalism* and is not (only) about *language.*
3. Aspect-perception, while having to do with *fact* and *truth*, is orthogonal to *value*.
4. Aspect-perception can figure as a mode of *analysis* of concepts and states of affairs.
5. Aspect-perception maintains *objectivity,* as aspects are there to be seen, but through a multifarious conception involving *modality.*

What, then, is the place and import of aspect-perception in mathematics and its history? The above points situate aspect-perception between seeing and thinking as logical—and so having to do with truth and objectivity—and not about convention or value and as possibly participating in analysis. In these various ways, aspect-perception can be seen to be fitting and indeed inherent in mathematical activity. At the very least, aspect-perception provides language, and so a way of thinking, for discussing and analyzing concepts, proofs, and procedures—how they are different or the same and how they can be compared or correlated. More substantially, since mathematics is a multifarious edifice of conceptual constructions, attention to aspects itself promotes seeing, seeing anew, and gaining insights. This is particularly so in connection with how we gauge *simplicity*, how we account for *surprise*, and how we come to *understand* mathematics.

On these last points, earlier remarks circa 1939 of Wittgenstein from *Remarks on the Foundations of Mathematics* have particular resonance. Part III starts with a discussion of proof, beginning: '"A mathematical proof must be surveyable [*übersichtlich*].” Only a structure whose reproduction is an easy task is called a “proof”.’ Aspect-perception casts light here, since seeing an argument organized in a specific way, for example, through the projection of another or as figuring in a larger context, can lead to (the dawning of) a perception of simplicity and thereby newly found perspicuity. In this way, aspect-perception shows the limits of logic conceived to be (merely) a sequence of local implications.

In Part I, Appendix II, Wittgenstein discusses the surprising in mathematics. The first specific situation he considered is when a long algebraic expression is seen shrunk into a compact form and where being surprised shows (§2) ‘a phenomenon of failure to command a clear view [*übersehen*] and of the change of aspect of a seen complex.’

> For one surely has this surprise only when one does not yet know the way. Not when one has the whole of it before one’s eyes. . . . The surprise and the interest, then come, so to speak, from the outside.

After the dawning of the aspect, there is no surprise, and what then remains of the surprise is the idea of seeing the logical space of possibilities. Wittgenstein subsequently wrote (§4),

> ‘There’s no mystery here!’—but then how can we have so much as believed that there was one?—Well, I have retraced the path over and over again and over and over again been surprised; and I never had the idea that here one can *understand* something.—So ‘There’s no mystery here!’ means ‘Just look about you!’

Though only elliptical, Wittgenstein here is suggesting that understanding, especially of novelty, as coming into play with the seeing and taking in of aspects.

Aspect-perception and mathematics have further useful involvements. Aspect-perception is an intrinsically difficult meta-concept in and through which to find one’s way, and by going into the precise, structured setting of mathematics one can better gauge and reflect on its shades and shadows. One can draw out aspects and deploy them to make deliberate conceptual arrangements for communicating mathematics. And aspect-perception provides an opportunity to bring in large historical and mathematical issues of context and method and to widen the interpretive portal to ancient mathematics.

In the succeeding sections, we show aspect-perception at work in mathematics by going successively through two topics, chosen in part for their differing features to illuminate the breadth of aspects. Section 2 takes up the classical and conceptual issue of the irrationality of square roots, bringing out aspects geometric and algebraic, ancient and modern. Section 3 sets out a circularity in the development of the calculus having to do with the derivative of the sine function, retraces features of the concept in ancient mathematics, and considers possible ways out of the circularity, thus drawing out new aspects. A substantial point to keep in mind is that these topics are conceptually complex; aspect-perception can work and, indeed, is of considerable interest at higher levels of mathematics.[2] In each of the sections is presented a 'new' result 'found' by the author, but one sees that creativity is belied to a substantial extent by context.

2. Irrationality of Square Roots

For a whole number n which is not a square number ($4 = 2^2$, $9 = 3^2$, $16 = 4^2, \ldots$), its square root $\sqrt{n}$ is irrational, that is, not a ratio of two whole numbers. This section attends to this irrationality; we shall see that its various aspects are far-ranging over time and mathematical context but, perhaps surprisingly, have a commonality. The irrationality can itself be viewed as an aspect of $\sqrt{n}$ separate from the seeing of it as or for its calculation as in Old Babylonian mathematics circa 1800 BCE, an aspect embedded in conceptualizations about the nature of number and of mathematical proof as first seen in Greek mathematics. In particular, the irrationality of $\sqrt{2}$ was a pivotal result of Greek geometry established in the later 5th century BCE. This result played an important role in expanding Greek concepts of quantity, and for contextually discussing $\sqrt{2}$ as well as the general $\sqrt{n}$, it is worth setting out, however briefly, aspects of quantity as then and now conceived.

For the Greeks, a *number* is a collection of *units,* what we denote today by 2, 3, 4, . . . with less a connotation of order than of cardinality. Numbers can be added and multiplied. A *magnitude* is a line, a (planar) region, a surface, a solid, or an angle. Magnitudes of the same kind can be added (e.g. region to region) and multiplied to get magnitudes of another kind (e.g. line to line to get a region). Respecting this understanding, we will deploy modern notation with its algebraic aspect, this itself partly to communicate further mathematical sense. For example, Proposition I 47 of Euclid's *Elements,* the Pythagorean Theorem, states rhetorically that '[i]n right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle' with a 'square' *qua* region. We will simply write the arithmetical $a^2 + b^2 = c^2$ where a and b are (the lengths of) the legs of a right triangle and c (the length of) the hypotenuse.

A *ratio* is a comparison between two numbers or two magnitudes of the same kind (e.g. region to region). Having ratio 2 to 3 we might today write as a relation 2:3 or a quantity $\frac{2}{3}$, with the first being closer to the Greek concept. There is a careful historiographical tradition promoting the first, but we will nonetheless deliberately deploy the latter in what follows. There are several aspects to be understood here: the fractional notation itself can be read as the Greek ratio; it can be read as part of a modern numerical-algebraic construal; and finally, the two faces are assertively to be seen as coherent.

A *proportion* is an equality of two ratios. We deploy the modern = as if for numerical quantities, this again having several aspects to be understood: it can be read as the Greek proportion, it can be read as an identity of two numerical quantities, and finally, the two faces are assertively to be seen as coherent. In what follows, the notation itself is thus to convey a breadth of aspect as well as a change of aspect, something not always made explicit.

Two magnitudes are *incommensurable* if there is no 'unit' magnitude of which both are multiples. While we today objectify $\sqrt{n}$ as a (real) number, that $\sqrt{n}$ is irrational is also to convey, in what follows, a Greek geometric sense: a square containing n square units has a side which is incommensurable with the unit. The pivotal result that $\sqrt{2}$ is irrational was for the Greeks that the side s and diagonal d of a square are incommensurable: $d^2 = s^2 + s^2 = 2s^2$ by the Pythagorean Theorem, and the ratio $\frac{d}{s}(= \sqrt{2})$ is not a ratio of numbers. To say that this result triggered a *Grundlagenkrise* would be an exaggeration, but it undoubtedly stimulated both the development of ratio and proportion for general magnitudes in geometry and a rigorization of the elements and means of proof.

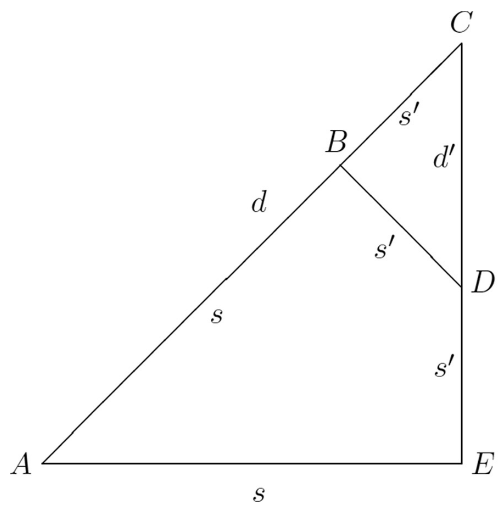
One of the compelling results of the broader context was just the generalization that $\sqrt{n}$ for non-square numbers n is irrational, sometimes called Theaetetus's Theorem. *Theaetetus* (ca. 369 BCE) is, of course, the great Platonic dialogue on epistemology. Socrates takes young Theaetetus (ca. 417–369 BCE) on a journey from knowledge as perception, to knowledge as true judgement, to knowledge as true judgement with *logos* (an account), and, in a remarkable circle, returns to perception: How can there even be knowledge of the first syllable *SO* of "Socrates"—is it a simple or a complex? Early in the dialogue (147c–148d), Theaetetus suggested conceptual clarification vis-à-vis square roots. He first noted that the elder geometer Theodorus (of Cyrene, ca. 465–398 BCE) had proved by diagrams the irrationality of $\sqrt{n}$ for non-square n up to 17. Then dividing the numbers into the 'square' and the 'oblong', he observed that they can be distinguished according to whether their square roots are numbers or irrational. In view of this and derivative commentary, Theaetetus has in varying degrees been credited with much of the content of the arithmetical Book VII of Euclid's *Elements* and of Book X, the meditation on incommensurability and by far the longest book.

The avenues and byways of the Theodorus result and the Theaetetus generalization have been much discussed from both the historical and mathematical perspectives.[3] In what follows we point out aspects there to be seen that coordinate across time and technique, and to this purpose we first review proofs for the irrationality of $\sqrt{2}$.

The argument most often given today is algebraic, about $\sqrt{2}$. Assume that $\sqrt{2} = \frac{a}{b}$ for (whole) numbers a and b so that, squaring, $a^2 = 2b^2 \cdot a^2$ is thus even and so, consequently, is a, say, $a = 2c$. But then, $4c^2 = 2b^2$, and so $2c^2 = b^2 \cdot b^2$ is thus even and so, consequently, is b, say, $b = 2d$. But then, $\frac{a}{b} = \frac{c}{d}$. Now this reduction to a ratio of smaller numbers cannot be repeated forever (infinite regress), or had we started with the least possibility for *b*, we would have a contradiction (*reductio ad absurdum*).

Cast geometrically in terms of the side and the diagonal of squares within squares, this is plausibly the earliest proof, found by the "Pythagoreans" in the first deductive theory, the even versus the odd (even times even is even, odd times odd is odd, and so forth). The proof is diagrammatically suggested in Plato's *Meno* 82b–85b, and, as an example of reasoning *per impossible,* in Aristotle's *Prior Analytics* I 23.

Another proof proceeds directly on a square, the features conveyable in the diagram showing half a square with side *s* and diagonal *d*.



$$s' = d - s$$

$$d' = s - s' = 2s - d$$

Figure 5.2

On the diagonal, a length *s* is laid off, getting *AB*, and then a perpendicular *BD* is constructed. The three *s*′s consequently indicate equal line segments, as can be seen using a series of what can be deduced to be isosceles triangles: the triangle *ABE* (formed by introducing line segment *BE*), the triangle *BDE,* and the triangle *BCD*. Now triangle *BCD* is also half of a square, with side s' and diagonal d', given in terms *s* and *d* as above. So, if *s* and *d* were commensurable, then so would be s' and d'. But this reduction cannot be repeated forever (infinite regress), or had we started out with commensurability with *s* and *d* being the smallest possible multiples of a unit, we would have a contradiction (*reductio*).

From the algebraic aspect, one sees that the components of a ratio have been made smaller:

$$\frac{d}{s} = \frac{d'}{s'} = \frac{2s-d}{d-s}, \text{ where } 0 < d-s < s.$$

This proof correlates with the Greek process of *anthyphairesis*, the Euclidean algorithm for line segments, whereby one works towards a common unit for two magnitudes by iteratively subtracting off the smaller from the larger. Because of this, the proof or something similar has been thought by some historians to be the earliest proof of incommensurability.[4] The proof appeared as a simple approach to irrationality in the secondary literature as early as in (Rademacher and Toeplitz 1930: 23) and, recently, with the simple diagram as shown above, in Apostol (2000).

Proceeding to $\sqrt{n}$, Knorr (1975, chap. VI) worked out various diagrammatic versions of the $\sqrt{2}$ even–odd proof as possible reconstructions for Theodorus's $\sqrt{n}$ result n up to 17, and Fowler (1999: 10.3) provided various anthyphairetic proofs up to 19. The following proof of the general Theaetetus result appears to be new; at least it does not seem to appear put just so in the historical and mathematical literature.

Assume that $\frac{a}{b} = \sqrt{n}$. Laying off copies of b on a, the anthyphairetic 'division algorithm', let $a = qb + r$ in algebraic terms with 'quotient' q and 'remainder' r, where $0 < r < b$ ($r = 0$ would imply that $\frac{a}{b}$ is a number and n a square). Consider the following diagram generalizing the previous one for $\sqrt{2}$.

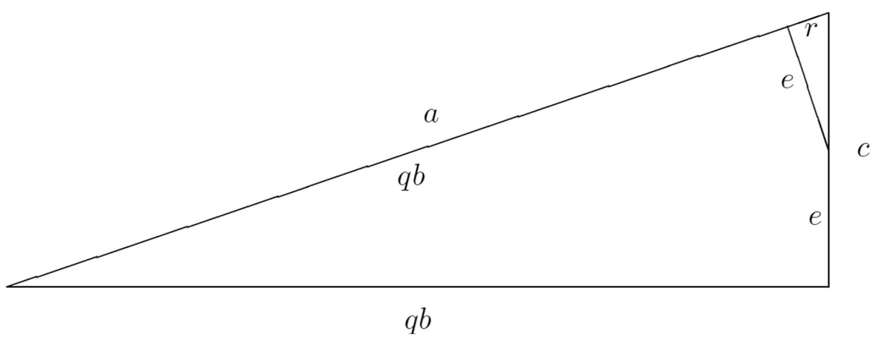


Figure 5.3

On the hypotenuse of length a, a length qb has been laid off, and so the two es are, in fact, the same as in the $\sqrt{2}$ case. This time, we appeal to *similarity*; the two triangles are seen to have pairwise the same angles, and so we have the proportion

$$\frac{a}{c} = \frac{c-e}{r}, \text{ or } a = \frac{c^2 - ce}{r}.$$

There is a factor of b on both sides of the latter: $a = b\sqrt{n}$; $c^2 = a^2 - q^2b^2 = b^2n - q^2b^2$ by the Pythagorean Theorem; and $ce = qbr$, since $\frac{qb}{c} = \frac{e}{r}$ again by similarity. Reducing by b,

$$\sqrt{n} = \frac{bn - q^2 b - qr}{r}, \text{ where } 0 < r < b,$$

so that the ratio $\frac{a}{b} = \sqrt{n}$ has been reduced to a ratio of smaller numbers. But as before, this reduction cannot be repeated forever (infinite regress) or had we started out with commensurability with smallest possible multiples, we would have a contradiction (*reductio*).

Since $r = a - qb$ and $qa = q^2 b + qr$, one sees again that from the algebraic aspect, the components of a ratio have been made smaller:

$$(*) \qquad \sqrt{n} = \frac{bn - qa}{a - qb}, \text{ where } 0 < a - qb < b.$$

The author found this proof, and there is an initial sense of surprise in that through all the centuries there seems no record of a proof put just so. Is this creative? Novel? One should be loath to speculate in general for mathematics, but this is not an atypical episode in its progress. Perhaps there is surprise at first, but there quickly comes understanding by viewing aspects that are really there to be seen. With the $\sqrt{2}$ anthyphairetic proof given above as precedent, one is led to such a proof in order to account for the *generality* of Theaetetus's Theorem and his having been alleged to have had a proof. It could, in fact, have been the original proof; its use of the division algorithm and ratios is within the resources that were presumably available already to the elder Theodorus. One notices an aspect of generality emerging in context, like a face out of a picture puzzle.

Be that as it may, historians in their ruminations have attributed proofs to Theaetetus that can be drawn out from propositions in the arithmetical Books VII and VIII of *Elements*, the first having been attributed to Theaetetus himself in his efforts to rigorize his theorem.[5] Scanning these books, there are several propositions that lead readily to Theaetetus's Theorem.

According to Book VII, 'a *number* is a multitude composed of units' (Definition 2); 'a number is *part* of [divides] a number, the less of the greater, when it measures the greater' (Definition 3); and 'numbers *relatively prime* are those which are measured by a unit alone as a common measure' (Definition 12). Assuming that $\frac{a}{b} = \sqrt{n}$ so that $a^2 = nb^2$, each of the following propositions about numbers readily implies that n is a square:

1. (VII 27) If r and s are relatively prime, then r^2 and s^2 are relatively prime. (Assume that a and b are the least possible so that they are relatively prime. As b^2 divides a^2, by the proposition b^2 must be the unit. Hence, b must be the unit, and so n is a square.)

2. (VIII 14) If r^2 divides s^2, then r divides s. (It follows that b divides a, and so n is a square. This proposition is not used elsewhere in the *Elements* and seems earmarked for Theaetetus's Theorem.)
3. (VIII 22) If r, s, t are in continued proportion (i.e. $\frac{r}{s} = \frac{s}{t}$) and r is a square, then t is a square. (Since $\frac{a^2}{nb} = \frac{nb}{n}$ and a^2 is a square, n is a square.)

Having allowed the positive conclusion that n, after all, could be a square, only the argument from VII 27 still depends on least choices for a and b. However, all the propositions depend on the much-used VII 20, which is *about* least choices:

$$\text{If } \frac{a}{b} = \frac{c}{d}$$

and a and b are *least* possibilities for this ratio, then a divides c and b divides d. The proof of VII 20 given in the *Elements* seems roundabout, and we give a sequentially direct proof, for example, that b must divide d: assume to the contrary that $d = qb + r$ with the division algorithm, where $0 < r < b$. $\frac{a}{b} = \frac{qa}{qb}$ (VII 17), and this together with $\frac{a}{b} = \frac{c}{d}$ implies $\frac{a}{b} = \frac{c-qa}{d-qb}$ (VII 12). But this contradicts the leastness assumption as $d - qb = r < b$.

VII 20 itself leads quickly to Theaetetus's Theorem:

$$\text{Assume } \frac{a}{b} = \sqrt{n}, \text{ so that } \frac{a}{b} = \frac{n}{\sqrt{n}} = \frac{nb}{a}.$$

Then if b is the least possibility for this ratio, then b divides a, and so n is square.

These proofs of Theaetetus's Theorem drawn from the *Elements* are arithmetical and veer towards *reductio* formulations, while the 'new' proof given earlier is diagrammatic and more suggestive of infinite regress. Is there, after all, a commonality of aspect? Yes, it is there to be seen but somewhat hidden. It is seen through a simple algebraic proof of Theaetetus's Theorem using a scaling factor, which is mysterious at first:

If $\frac{a}{b} = \sqrt{n}$ and there is a number q such that $q < \sqrt{n} < q + 1$ so that $qb < a < (q+1)b$, we then have the algebraic reduction

$$(**) \qquad \frac{a}{b} = \frac{a\left(\sqrt{n} - q\right)}{b\left(\sqrt{n} - q\right)} = \frac{bn - qa}{a - qb}, \text{ where } 0 < a - qb < b.$$

This q is just the q of the division algorithm $a = qb + r$ of the diagrammatic proof, and (**) is a rendering of the (*) after that proof. As for the arithmetical proof, the ratio reduction is there but only as part of the proof of VII 20 given above, at the use of VII 12. These aspects of various propositions and proofs are all there to be seen in a kind of whirl, the interconnections leading to understanding.

Today, prime numbers and the Fundamental Theorem of Arithmetic, that every number has a unique factorization into prime numbers, are basic to number theory, and it is a simple exercise in counting prime factors to establish Theaetetus's Theorem. However, for more than two millennia until Gauss the new simplicity afforded by the fundamental theorem was not readily attainable. There have recently been several accounts of the irrationality of $\sqrt[k]{n}$ *ab initio* that exhibit a minimum of resources though without conveying historical resonance, and these ultimately turn on (**), what was there to be seen.[6]

Richard Dedekind in his 1872 *Stetigkeit und irrationale Zahlen,* the foundational essay in which he formulated the real numbers in terms of Dedekind cuts, provided (IV), what has been considered a short and interesting proof of the irrationality of $\sqrt{n}$ for non-square n: assume that $\frac{a}{b} = \sqrt{n}$ with b the least possibility and $q < \sqrt{n} < q+1$. Then algebraically

$$(***) \qquad (bn-qa)^2 - n(a-qb)^2 = (q^2-n)(a^2-nb^2)$$

However, $a^2 - nb^2$ is zero by assumption and so is the left side, and hence,

$$\frac{bn-qa}{a-qb} = \sqrt{n} \quad \text{where } 0 < a-qb < b,$$

contrary to the leastness of b.

Again the scaling ratio of (**) has emerged, but how had Dedekind gotten to it? During this time, Dedekind was steeped in algebraic number theory, particularly with his introduction of ideals. The *ring* $Z\left[\sqrt{n}\right]$ consists of $x+y\sqrt{n}$, where x and y are integers and the ring has a *norm* given by $N\left(x+y\sqrt{n}\right) = \left(x+y\sqrt{n}\right)\left(x-y\sqrt{n}\right) = x^2-ny^2$. The norm of a product is the product of the norms—Brahmagupta's identity, first discovered by the 7th century CE Indian mathematician. In these terms, (***) above is just expressing

$$N\left(\left(-q+\sqrt{n}\right)\cdot\left(a+b\sqrt{n}\right)\right) = N\left(\left(-q+\sqrt{n}\right)\cdot N\left(a+b\sqrt{n}\right)\right).$$

The appearance of the scaling factor $\sqrt{n}-q$ of (**) is motivated here in terms of norm reconstruing distance. Also, in this wider context of algebraic structures, it is well known that unique factorization into 'prime' elements may not hold, and so there is a *logical* reason to favor the Dedekind approach to the irrationality.

That (**) emerges as a commonality in proofs of Theaetetus's Theorem is itself a notable aspect. Although indicating a proof on its own, (**) remains thin and mysterious in juxtaposition with the ostensible significance of the result, both historical and mathematical. One sees more sides and angles, whether about number, discovery or proof, in the other proofs—embedded as they are in larger ways of thinking—and these aspects garner a mathematical understanding of the proofs and related propositions.

3. Derivative of Sine

In this section, a basic circularity in textbook developments of calculus is brought to the fore, and this logical node is related to the ancient determination of the area of the circle. How to progressively get past this node is considered, the several ways bringing out different aspects of analysis, parametrization, and conceptualization. There is quite a lot of mathematical and historical complexity here, but this is requisite for bringing out the subtleties of aspect-perception in this case, especially of seeing something *as* something and *in* something.

The calculus of Newton and Leibniz revolutionized mathematics in the 17th century, with dramatically new methods and procedures that solved age-old problems and stimulated remarkable scientific advances. At the heart is a bifurcation into the differential calculus, which investigated instantaneous rates of change, like velocity and acceleration, and the integral calculus, which systematized total size or value, like areas and volumes. And the Fundamental Theorem of Calculus brought the two together as opposite sides of the same coin.

In modern standardized accounts, the differential calculus is developed with the notion of limit. Functions working on the real numbers are differentiated, that is, corresponding functions, their derivatives, are determined that are to characterize their rates of change. The differentiation of the trigonometric functions is a consequential part of elementary differential calculus. The process can be reduced to determining that the derivative of the sine function is the cosine function, and this devolves, fortunately, to the determination of the derivative of the sine function evaluated at 0. This amounts to showing

$$(*) \qquad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1,$$

that the limit as θ approaches 0 of the ratio of sin θ (the sine of θ) to θ is 1. This is the first interesting limit presented in calculus courses, bringing together angles and lengths. How is it proved?

In all textbooks of calculus save for a vanishing few, a geometric argument is invoked with the following accompanying diagram:

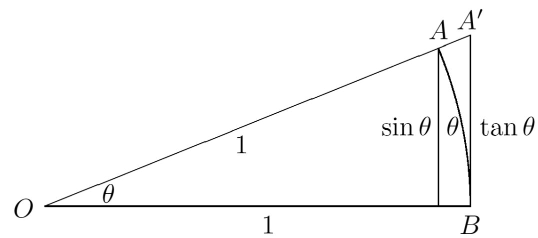


Figure 5.4

Consider the arc AB subtended by (a small) angle θ on the unit circle, the circle of radius 1, with center O. The altitude dropped from A has length sin θ, 'opposite over hypotenuse', for the angle θ; the length of the circular arc AB is θ, the (radian) measure of the angle (with 2π for one complete revolution); and the length of $A'B$ is tan θ (the tangent of θ), 'opposite over adjacent'. Once

$$(**) \qquad \sin\theta < \theta < \tan\theta$$

is established, pursuing the algebraic aspect and dividing through by sin θ and then taking reciprocals yields

$$1 > \frac{\sin\theta}{\theta} < \frac{\sin\theta}{\tan\theta} = \cos\theta,$$

and since cos θ (the cosine of θ), 'adjacent over hypotenuse', approaches 1 as θ approaches 0, (∗) follows.

In geometric aspect, the first inequality of (∗∗) as a comparison of lengths is visually evident, but the second is less so. One can, however, proceed with areas: The area of triangle OAB (formed by introducing line segment AB) is $\frac{1}{2}\sin\theta$, 'half the base times the height'; the area of circular sector OAB is $\frac{\theta}{2}$, since the ratio of this area to the area π of the unit circle is proportional to the ratio of θ to the circumference 2π; and $\frac{1}{2}\tan\theta$ is the area of triangle $OA'B$. With the figures subsumed one to the next, (∗∗) follows by comparison of areas.

But this is a circular argument! It relies on the area of the unit circle being π, where 2π is the circumference, but the proof of this would have to entail taking a limit like (∗), or at least the comparison of lengths (∗∗) as in the diagram. And underlying this, what after all *is* the length of a curve, like an arc? We can elaborate on, and better see, this issue by looking at the determination of the area of a circle in Greek mathematics.[7]

Archimedes in his treatise *Measurement of a Circle* famously established that the area of a circle of radius r is equal to the area of a right triangle with sides r and the circumference. With this latter area being $\frac{1}{2}\cdot r\cdot(2\pi r)$, we today pursue the algebraic aspect and state the area as πr^2. Archimedes briefly sketched the argument in *Measurement* in terms of the right triangle; it is more directly articulated through Propositions 3 through 6 of his *On the Sphere and Cylinder I*.

The method was to inscribe regular n-gons (polygons with n equal sides) in the circle of radius r, to circumscribe with such, and then to take a limit as n gets larger and larger via the Eudoxan method of exhaustion. The following diagram taken from *On the Sphere* pictures a sliver of the argument:

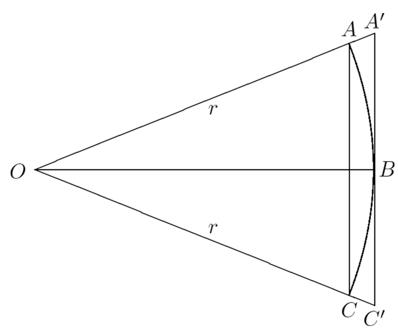


Figure 5.5

The triangle OAC is one of the n triangles making up an inscribing n-gon, the circular arc AC is $\frac{1}{n}$ of the circumference, and the triangle $OA'C'$ is one of the n triangles making up a circumscribing n-gon. This figure is just a coupling of the previous figure scaled to radius r with its mirror image, and so in (modern, radian) measure we would have:

The angle AOC is $\frac{2\pi}{n}$, and so with half of this as the angle $\theta = \frac{\pi}{n}$ of the previous figure, the line segment AC has length twice $r\sin\frac{\pi}{n}$, or $2r\sin\frac{\pi}{n}$. Similarly, the line segment $A'C'$ has length $2r\tan\frac{\pi}{n}$. Finally, the arc AC has length $\frac{2\pi r}{n}$, based on the circumference being $2\pi r$.

Archimedes in effect used the version

$$2r\sin\frac{\pi}{n} < \frac{2\pi r}{n} < 2r\tan\frac{\pi}{n}$$

of (**) to show, for the Eudoxan exhaustion—we would now say for the taking of a limit—that the perimeter of the inscribing and circumscribing polygons approximate the circumference from below and from above. This appeal to (**) is not itself justified in *Measure*, but the first two "assumptions" in the preface to *On the Sphere* serve: (1) the shortest distance between two points is that of the straight line connecting them (so $\sin\theta < \theta$), and (2) for two curves convex in the same direction and joining the same points, the one that contains the other has the greater length (so[8] $\theta < \tan\theta$).

With such assumptions, Archimedes had set out the conditions for how his early predecessors had constructed arc length. Archimedes' work evidently built on a pre-Euclidean tradition of geometric constructions,[9] in which an important motif had been how to *rectify* a curve, that is, render it as a straight line. Indeed, Archimedes stated and conceptualized his area theorem as one sublimating the circumference as a straight line, the side of a triangle; he *could not* in any case have stated the area as πr^2, the Greek geometric multiplication having to do with areas of rectangles and not generally allowed for magnitudes.

These aspects of area and length are illuminated by Euclid's *Elements* XII 2: *Circles are to one another as the squares on their diameters*. This had been applied over a century before by Hippocrates of Chios in his 'quadrature of the lunes', and Euclid managed a proof in his system with a paradigmatic use of the method of exhaustion that borders on Archimedes' later use. Commentators have pointed out how XII 2 falls short of getting to the actual ratio π, but in thinking through the aspects here, Euclid *could not* have gone further. In his rigorization over his predecessors, Euclid had famously restricted geometric constructions to straightedge and compass, and he had no way of rectifying a curve and so of formulating the ratio π. For Euclid, and Greek theoretical mathematics, area was an essentially simpler concept, from the point of view of proof, than length (for curves); area could be worked through congruent figures, and there were no beginning, 'common notions' for length. Comparison of areas with one figure subsumed in another is simpler than a comparison of length (of curves), and XII 2 epitomizes how far one can go with the first. Archimedes went further to the actual ratio π, for 20 centuries called 'Archimedes' constant', but this depended on his assumptions (1) and (2).

Especially with this historical background uncovering aspects of length and area for the circle, one can arguably take (**) as logically immediate as part of the concept of length. We today have a mathematical concept of rectifiable curves, a concept based on small straight chords approximating small arcs so that polygonal paths approximate curves. Seeing the concept of length from this aspect, Archimedes' assumptions (1) and (2) are more complicated in theory than (**) itself. Moreover, there is little explanatory value in proving (∗), as it is presupposed in the definition of arc length.

The logical question remains whether we can avoid the theft of assuming what we want and move forward to the derivative of the sine with honest toil. There are several ways, each illuminating further aspects of how we are to take analysis and definition, from arc length to the sine function itself.

(a) Taking area as basic, *define* the measure of an angle itself in terms of the area of the subtended sector. To scale for radian measure, *define the measure of an angle to be twice the area of the sector it subtends in the unit circle*. Then by comparison of areas, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$, and (∗∗) is immediate. With this, one could deduce à la Archimedes that the area of the unit circle is π, where 2π is the circumference, so the measure for one complete revolution is 2π, confirming that we do, indeed, have the standard radian measure. Defining angle measure in this way may be a pedagogical or curricular shortcoming, but the shift in logical aspect is quickly re-coordinated and, moreover, resonates with how the conceptualization of area is simpler than that of length.

G. H. Hardy's classic *A Course in Pure Mathematics* (1908: §§158, 217) and Tom Apostol's calculus textbook (1961: §1.38) are singular in pointing out the logical difficulty of defining the measure of an angle in terms of an unrigorized notion of arc length, and in advocating the definition of the measure of an angle in terms of area. Hardy (1908) advocated several approaches to the development of the trigonometric functions, two of which are conglomerated in (c) below. Apostol's 1961 work is the rare calculus textbook of recent memory that does not proceed circularly; it develops the integral calculus prior to the differential calculus, defines area as a definite integral, and only later defines length for rectifiable curves.[10]

(b) First define the length of a rectifiable curve in the usual way as an integral. Then, get the derivative of sine using methods of calculus:

Let $x = \sin\theta$ and $y = \cos\theta$ so that with Pythagorean relation $y = \sqrt{1-x^2}$,

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}.$$

Anticipating the use of the length integral, note that

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}.$$

Since, according to the parametrization, θ is the length of the arc from 0 to x,

$$\theta = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$$

By the Fundamental Theorem of Calculus,

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-x^2}},$$

so that by the Inverse Function (or Chain) Rule,

$$\frac{dx}{d\theta} = \frac{1}{\frac{d\theta}{dx}} = \sqrt{1-x^2},$$

or in terms of θ,

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

One can similarly get $\frac{d}{d\theta}\cos\theta = \sin\theta$.

This approach underscores how length can be readily comprehended with infinitesimal analysis and how the derivative of the sine being cosine can be rigorously established by a judicious ordering of the development of calculus. Importantly, the approach depends on the Fundamental Theorem, which, in turn, depends on the conceptualization of area as a definite integral. In this logical aspect, too, area is thus to be conceptually subsumed first. There would be a pedagogical or curricular shortcoming here as well, this time with the derivative of the sine popping out somewhat mysteriously.

The author found this non-circular proof that the derivative of the sine is the cosine and could not find this approach in the literature. Is this a new theorem? Is it creative? Novel? Here, a logical gap was filled with familiar methods. The proof can be given as a student exercise, once a direction is set and markers laid. The task set would be to outline, with an astute ordering of the topics, a logical development of the calculus through to the trigonometric functions. If the Fundamental Theorem and the length of a rectifiable curve as an integral are put first, then the above route becomes available to the derivatives of the trigonometric functions. This logical aspect of the derivative of sine was there to be found, emerging with enough structure.

(c) Taking seriously the study of the trigonometric functions as part of *mathematical analysis*—the rigorous investigation of functions on the real and complex numbers through limits, differentiation, integration, and infinite series—*define* the sine function as an infinite series. One can follow a historical track as in the following.

Let $f(x)$ by a function defined by the integral in (b):

$$f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt$$

Newton knew this to be, as in (b), the *inverse* of the sine function: $f(x) = \theta$ exactly when $\sin\theta = x$. He had come early on to the general binomial series, and so in his 1669 *De analysi* (cf. Newton 1968: 233ff.) he expanded the integrand $\frac{1}{\sqrt{1-t^2}}$ as an infinite series, integrated it term by term, and then, applying a key technique for inverting series term by term, determined the infinite series for sine:

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

One can, however, take this *ab initio* as simply a function to investigate. Term-by-term differentiation yields

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Hence, by series manipulation, $S^2(x) + C^2(x) = 1$, which sets the stage as the Pythagorean Theorem for the unit circle. Next, define π as a parameter, the least positive x, such that $S(x) = 0$.

With these, one works out that as x goes from 0 to π, the point $(S(x), C(x))$ in the coordinate plane traverses half the unit circle and, through the length formula, that the circumference of the unit circle is 2π. Hence, taking $\sin x = S(x)$ and $\cos x = C(x)$ retrieves the familiar trigonometric functions and their properties, as well as the derivative of the sine being cosine.[11]

This approach draws out how the trigonometric functions can be developed separately and autonomously in the framework of mathematical analysis. The coordination with the classical study of the circle and its measurement then has a considerable aspectual variance: one can take the geometry of the circle as the main motivation, one can bring out interactions between the geometric and the analytic, or one can even avoid geometric 'intuitions', a thematic feature of analysis into the 19th century.

Lest the analytic approach to the trigonometric functions still seems arcane or historically Whiggish, consider the function $g(x) = \int_0^x \frac{1}{\sqrt{1-t^4}}\, dt$, where the '2' of the previous integral has been replaced by '4'. This 'lemniscatic integral' arose as the length of the 'lemniscate of Bernoulli' at the end of the 17th century, and it was the first integral which defied the Leibnizian program of finding equivalent expressions in terms of 'known' functions (algebraic, trigonometric, or exponential functions and their inverses). At the end of the

18th century, the young Gauss focused on the *inverse* of the function $g(x)$ and found its crucial property of *double periodicity.* By 1827 the young Abel had also studied the inverse function, and in 1829 Jacobi wrote a treatise on the subject, from which such functions came to be known as *elliptic functions*, the integrals *elliptic integrals*, and the curves they parametrize *elliptic curves*. Thence, elliptic functions have played a large, unifying role in number theory, algebra, and geometry as they were extended into the complex plane. On that score, by 1857 Riemann had shown that the complex parametrizations are on a *torus*, a 'doughnut', with double periodicity an intrinsic feature. This is how the geometric figure of the torus came to be of central import in modern mathematics—the arc of discovery going in the opposite direction from the circle to the integral $f(x)=\int_0^x \frac{1}{\sqrt{1-t^2}}dt$.

There is a broad matter of aspect to be reckoned with here, and, as a matter of fact, throughout this topic, as well as the previous. Taking mathematics as based on networks of conceptual constructions, one sees through aspects various historical and logical progressions. Simply seeing a certain 'face' on a topic in mathematics is to make connections with familiar contexts and modes of thinking, and this leads to the sometimes sudden dawning of logical connections. That aspects are logical thus has a further dimension in mathematics, that there are webs of logical connections. We can be provoked to seeing logical sequencings of results and themes, all there to be found. As we look and see, we can develop and reconstruct mathematics. In this, aspect-perception counsels the history of mathematics and draws forth an understanding of it, that is, its truths as embedded in its proofs.

Stepping back further from the two topics presented in this chapter, one may surmise that many pieces of mathematics can be so presented as pictures at an exhibition of mathematics, with aspects and aspect-perception helping to get one about. Between seeing and thinking, aspect-perception is logical and can participate in the analysis of concepts. In the discussion of the irrationality of $\sqrt{n}$, we saw conceptions of number themselves at play, the diagrammatic geometry of the Greeks stimulating a remarkable advance, various arithmetical manipulations of number domesticating the irrationality, and the play of recent systematizations reinforcing a commonality. In the discussion of the derivative of the sine function, we saw a basic limit issue of calculus swirling with the ancient determination of the area of a circle, the involvement of Greeks conceptualizations of area and length, different approaches to establishing a rigorous progression, and how older concepts can be transmuted in a broad new context. Venturing some generalizing remarks, across mathematics there are many angles, faces, and views and the noticing, continuous seeing, and dawning of many aspects. Especially in mathematics, however, aspect-perception is not just about conventions or language. Rather, aspects get at objectivity from a range of perspectives and, thus, collectively track and convey necessity, generality, and truth.

Notes

Aspects of this chapter were presented at 2013 seminars at Carnegie Mellon University and at the University of Helsinki; many thanks to the organizers for having provided the opportunity. The chapter has greatly benefitted from discussions with Juliet Floyd.

1. Part II is renamed *Philosophy of Psychology—a Fragment* in the recent edition (2009) of *Philosophical Investigations*.
2. This belies objections at times lodged against Wittgenstein that he only raised philosophical issues of pertinence to very simple mathematics.
3. See Knorr (1975) and Fowler (1999) for extended historical reconstructions based on different approaches, and see Conway and Shipman (2013) for the most recent mathematical tour.
4. See Knorr (1975, chap. II, sect. II) for a critical analysis. Knorr (1998) late in his life maintained, however, that a specific diagrammatic rendition of the proof was the original one.
5. Cf. Knorr (1975, chap. VII).
6. See Beigel (1991) and references therein. Conway and Guy (1996: 185) conveys a proof in terms of fractional parts, which again amounts to (**).
7. This circularity was pointed by Richman (1993) and, in the context of ancient mathematics, by Seidenberg (1972). Both pursue the trail in ancient mathematics at some length.
8. For this, one imagines in the first diagram a mirror image of the figure put atop it, say with a new point C corresponding to the old B. Comparing the arc CAB to the path $CA'B$ with (2), one gets $2\theta < 2 \tan \theta$.
9. Cf. Knorr (1986).
10. The calculus textbook by Spivak (2008, III.15) defines the sine and cosine functions as does Apostol (1961) and is not circular, but, on the other hand, it also (problem 27) gives the circular approach to the derivative of the sine as 'traditional'.
11. See the classics Landau (1934, chap. 16) and Knopp (1921: §24) for details on this development.

References

Apostol, Tom M. (1961). *Calculus*, Vol. 1. New York: Blaisdell.

Apostol, Tom M. (2000). Irrationality of the square root of two—A geometric proof. *The American Mathematical Monthly* 107: 841–842.

Beigel, Richard. (1991). Irrationality without number theory. *The American Mathematical Monthly* 98: 332–335.

Conway, John H. and Guy, Richard K. (1996). *The Book of Numbers*. New York: Springer-Verlag.

Conway, John H. and Shipman, Joseph. (2013). Extreme proofs I: The irrationality of $\sqrt{2}$. *Mathematical Intelligencer* 35(3): 2–7.

Fowler, David H. (1999). *The Mathematics of Plato's Academy: A New Reconstruction*, 2nd edition. Oxford: Clarendon Press.

Hardy, G. H. (1908). *A Course in Pure Mathematics*. Cambridge: Cambridge at the University Press.

Knopp, Konrad. (1921). *Theorie und Awendung der unendlichen Reihen*. Berlin: Julius Springer. Translated as *The Theory and Applications of Infinite Series*. Dover Publications, 1990.

Knorr, Wilbur R. (1975). *The Evolution of the Euclidean Elements*. Dordrecht: D. Reidel Publishing Company.

Knorr, Wilbur R. (1986). *The Ancient Tradition of Geometric Problems*. Boston: Birkhäuser.

Knorr, Wilbur R. (1998). 'Rational diameters' and the discovery of incommensurability. *American Mathematical Monthly* 105(5): 421–429.

Landau, Edmund. (1934). *Einführung in die Differentialrechnung und Integralrechnung*. Groningen: Nordhoff. Translated as *Differential and Integral Calculus*. New York: Chelsea Publishing Company, 1951.

Newton, Isaac. (1968). De analysi per aequationes numero terminorum infinitas. In D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton, Volume II 1667–1670*. Cambridge: Cambridge University Press, 206–276.

Rademacher, Hans and Toeplitz, Otto. (1930). *Von Zahlen und Figuren: Proben mathematischen Denkens für Liebhaber der Mathematik*. Berlin: Julius Springer. Latest version in English translation *The Enjoyment of Mathematics: Selections From Mathematics for the Amateur*. Dover, 1990.

Richman, Fred. (1993). A circular argument. *The College Mathematics Journal* 24: 160–162.

Seidenberg, Abraham. (1972). On the area of a semi-circle. *Archive for History of Exact Sciences* 9: 171–211.

Spivak, Michael. (2008). *Calculus*, 4th edition. Houston: Publish-or-Perish Press.

Wittgenstein, Ludwig. (1953). *Philosophical Investigations* (trans. G. E. M. Anscombe). Oxford: Basil Blackwell.

Wittgenstein, Ludwig. (1958). *The Blue and Brown Books*. Oxford: Basil Blackwell.

Wittgenstein, Ludwig. (2009). *Philosophical Investigations* (trans. G. E. M. Anscombe, P. M. S. Hacker and Joachim Schulte). Oxford: Wiley-Blackwell.