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Putnam's Constructivization Argument

Abstract. We revisit Putnam's constructivization argument from his *Models and Reality*, part of his model-theoretic argument against metaphysical realism. We set out how it was initially put, the commentary and criticisms, and how it can be specifically seen and cast, respecting its underlying logic and in light of Putnam's contributions to mathematical logic.

Keywords: constructibility, $V = L$, model-theoretic argument, metaphysical realism.

Hilary Putnam's constructivization argument, involving the axiom of constructibility $V = L$ of set theory, is at the cusp of mathematics and philosophy, being the most mathematically pronounced argument that he has put in the service of philosophical advocacy. In his shift in the mid-1970's to his internal realism, the argument appeared in his 1980 *Models and Reality* [14], as a "digression". Nonetheless, with subsequent commentary and criticisms it became considered a substantive piece of what has come to be called Putnam's "model-theoretic argument against metaphysical realism". What follows is a mainly mathematical meditation on the constructivization argument: how it was initially put, the commentary and criticisms, and how it can be specifically seen and cast respecting its underlying logic and in light of Putnam's mathematical work.

Putnam's contributions to mathematical logic—his work in recursion theory, on Hilbert's 10th Problem, on constructible reals and the ramified analytic hierarchy—are mainly from his early years. Whether mathematical results can or should be deployed to support philosophical positions at all, Putnam's subsequent deployment of model-theoretic arguments against an uncompromising realism was a novel and remarkable move.

At the outset, it should be said that we will not directly illuminate how the constructivization argument integrates with Putnam's broad philosophical stance at the time. For one thing, it argues only against a realist concept of set. Rather, we will bring out the flow of Putnam's thinking as he put his mathematical experience to work and how in its byways the constructivization argument actually worked.

In what follows, §1 reviews the constructivization argument, as presented in [14]. §2 describes the to and fro of commentary and criticisms of it in the literature. §3 takes a deeper look at the constructivization argument—the mathematical context, the inner logic, and the specific ways in which it can be taken. §4 coordinates the various criticisms, and in the process, consolidates the mathematical issues about the constructivization argument.

1. The Constructivization Argument

Putnam began his [14] with introductory remarks and then paragraphs headlined "The philosophical problem". He briefly recalled the Skolem-paradox argument about having unintended interpretations of set theory in which nondenumerable sets are "in reality" denumerable. He then specifically recalled the Downward Löwenheim-Skolem Theorem, according to which models can have countable elementary submodels. He pointed out that by the Skolem-paradox argument, "even a *formalization of total science* (if one could construct such a thing), or even a *formalization of all our beliefs* (whether they count as 'science' or not), could not rule out denumerable interpretations." With this showing that "theoretical constraints" "cannot fix the interpretation of the notion *set* in the 'intended' way", he proceeded to argue that even "operational constraints" cannot either. With the Downward Löwenheim-Skolem Theorem, "we can find a countable submodel of the 'standard model' (if there is such a thing)" that also preserves all the information the operational constraints provide. The philosophical problem that then emerges is that if axiomatic set theory does not capture the intuitive notion of set, then "understanding" might; but "understanding" cannot come to more than "*the way we use our language*"; yet even "*the total use of the language* (operational plus theoretical constraints) does not fix a unique interpretation."

In the next paragraphs of [14], headlined "An epistemological/logical digression", Putnam presented his constructivization argument, which amplifies the above argument with respect to constructibility. He briefly discussed Gödel's axiom $V = L$, that the set-theoretic universe V coincides with Gödel's universe L of constructible sets, and soon continued:

> [Gödel's] later view was that '$V = L$' is *really* false, even though it is consistent with set theory, if set theory itself is consistent.
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> Gödel's intuition is widely shared among working set theorists. But does this 'intuition' make sense?
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> Let MAG be a countable set of physical magnitudes which includes all magnitudes that sentient beings in this physical universe can actually measure (it certainly seems plausible that we cannot

> hope to measure more than a countable number of physical magnitudes). Let OP be the 'correct' assignment of values; that is, the assignment which assigns to each member of MAG the value that that magnitude actually has at each rational space-time point. Then all the information 'operational constraints' might give us (and in fact, infinitely more) is coded into OP.
>
> One technical term: an ω-*model* for a set theory is a model in which the *natural numbers* are ordered as they are 'supposed to be'; that is, the sequence of 'natural' numbers of the model is an ω-sequence.
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> Now for a small theorem.[2] [[2] Barwise has proved the much stronger theorem that every countable model of ZF has a proper end extension which is a model of ZF + $V = L$ (in *Infinitary methods in the model theory of set theory*, published in ***Logic Colloquium '69***). The theorem in the text was proved by me before 1963.]
>
> THEOREM: *ZF plus* $V = L$ *has an* ω-*model which contains any given countable set of real numbers.*

Taking a countable set of reals as routinely coded by a single real, Putnam proceeded to provide an informal proof of his theorem using the Downward Löwenheim-Skolem Theorem to get a countable elementary submodel of L and then applying the Shoenfield Absoluteness Lemma. He noted in passing that "What makes [his] theorem startling" is that while a nonconstructible real cannot be in a β-model of $V = L$, it *can* be in an ω-model.

Putnam continued:

> Now, suppose we formalize *the entire language of science* within the set theory ZF + $V = L$. Any model for ZF which contains an abstract set isomorphic to OP can be extended to a model for this formalized language of science which is *standard with respect to* OP—hence, even if OP is nonconstructible 'in reality', we can find a model *for the entire language of science* which satisfies *everything is constructible* and which assigns the correct values to all the physical magnitudes in MAG at all rational space-time points.
>
> The claim Gödel makes is that '$V = L$' is false 'in reality'. But what on earth can this mean? It must mean, at the very least, that in the case just envisaged, the model we have described in which '$V = L$' holds would not be *the intended model*. But why not? It satisfies all theoretical constraints; and we have gone to great length to make sure it satisfies all operational constraints as well.

Putnam concluded this section:

> What the above argument shows is that if the ‘intended interpretation’ is fixed only by theoretical plus operational constraints, then if ‘$V \neq L$’ does not follow from those theoretical constraints—if we do not *decide* to make $V = L$ true or to make $V = L$ false—then there will be ‘intended models’ in which $V = L$ is *true*. If I am right, then ‘the relativity of set-theoretic notions’ extends to a *relativity of the truth value of ‘$V = L$’* (and, by similar augments, of the axiom of choice and the continuum hypothesis as well).

2. Commentary and Criticisms

Putnam’s advocacy of internal realism through articles starting in the later 1970s generated a philosophical literature both extensive and sundry. A focus was on his “model-theoretic argument against metaphysical realism”, and eventually the most mathematically pronounced argument, the constructivization argument, itself came under sustained scrutiny in the literature, starting in the later 1990s. What is of particular interest is the extent to which a mathematically-based argument for a philosophical stance has elicited a range of responses about the mathematics and its applicability. In what follows, we review in chronological order the to and fro of commentary and criticisms.

Shapiro [15], on second-order logic and mathematical practice, briefly attended (p.724) to the constructivization argument. From a standard fact that he cites, if the set isomorphic to OP is nonconstructible then Putnam’s final model of ZF + $V = L$ containing the set would not have a (really) well-founded membership relation. But for Shapiro, “one can surely claim that the well-foundedness of the membership relation is a ‘theoretical constraint’ on (intended) models of set theory.”

Levin [12] mounted a detailed critique of the constructivization argument, in terms of the semantics of first-order logic. On its face a response to Putnam on reference and constructibility, it seems a tissue of conflations about constants and terms and their interpretations model to model. The argument devolves to what OP is, its role, its coding as a real number, and whether that real is constructible—all this riddled with confusions and missing the thrust of Putnam’s argument.

Velleman [17] reviewed Levin [12] and vetted it along the lines above. At the beginning, Velleman pointed out that Putnam’s theorem (as stated in the first quoted passage of §1) cannot be provable in ZFC as it implies the consistency of ZFC. (By Gödel's Second Incompleteness Theorem, no theory, unless inconsistent, can establish its own consistency.) "[T]here must be a mistake in Putnam's proof"; the mistake is that "the Löwenheim-Skolem theorem is only applicable to sets, not to proper classes such as L"; and: "For example, the proof can be fixed by adding the hypothesis that there is an inaccessible cardinal κ, and then applying the Löwenheim-Skolem theorem to the set L_κ rather that to L."

Dümont [9] undertook a "detailed reconstruction" of Putnam's "model-theoretic argument(s)", and ultimately concluded that he "fails to give convincing arguments for rejecting mathematical or metaphysical realism". While mainly concentrating on Putnam's Skolemization argument, Dümont did attend, briefly, to the constructivization argument. Following his overall tack, he took Putnam to have failed to give a convincing answer to the realist who replies (p.348-9) to "the fact that V = L does not follow from the theoretical and operational constraints": "After all the theoretical and operational constraints have their source in *our* theoretical and empirical investigations and of course our faculties are limited. So our inability to fix one intended model only reflects our restricted access to the independently existing set-theoretical universe."

Bays [3] mounted a broad critique of the constructivization argument, both its mathematics and its philosophy. He argued first that "a key step in Putnam's argument rests on a mathematical mistake", discussing its philosophical ramifications; second, that "even if Putnam could get his mathematics to work, his argument would still fail on purely philosophical grounds", and third, that "Putnam's mathematical mistakes and his philosophical mistakes are surprisingly closely related".

As Velleman [17] had done, Bays indicated (p.366f) that Putnam's proof of his theorem (as stated in the first quoted passage of §1) is mistaken, as the Downward Löwenheim-Skolem Theorem cannot be applied to L, a proper class, and that the proof can be patched up e.g. by assuming that there is an inaccessible cardinal. Bays, however, argued (p.339f) that such patch-ups involving additional assumptions "do very little toward salvaging [Putnam's] overall philosophical argument". If in ZFC + XYZ one establishes that there is a model of ZFC + $V = L$, then XYZ would be part of the theoretical constraints yet would not hold in the model. The problem is "intrinsic" (because of Gödel's Second Incompleteness Theorem).

After criticizing Putnam's argument on philosophical grounds, Bays at the end made a connection between the mathematical and the philosophical. Putnam is not being fair to the realist, as (p.349):

> When the realist tries to 'stand back' from his set theory to talk about that theory's interpretation—to specify, for instance, that this interpretation must be transitive, or well founded, or satisfy second-order ZFC—Putnam accuses him of 'begging the question.' Although Putnam's own model-theoretic talk should be viewed as talk *about* set theory, the realist's talk must be viewed as talk *within* set theory.

Gaifman [10], on non-standard models in a broader perspective, brought up the constructivization argument, pointing out that "Putnam's proof contains a mathematical error" and that one needs an "additional assumption" to be believed by the realist. With this granted, Gaifman went on, favorably: "if s [coding OP] is not in L, the [final] model is not well-founded, but this makes no difference; we can carry out all our physical measurements, while assuming that $V = L$".

Gaifman proceeded to point out how a realist can appreciate the investigation of various structures, e.g. in which "false" propositions hold. He objected to Putnam's approach of treating "the problem as one that should be decided by appeal to general pragmatic criteria [operational constraints] and some blurry ideal of rationality [theoretical constraints]".

Bellotti [5] examined the constructivization argument and critiques thus far. Getting to Bays [3], Bellotti opined (p.404-5) that his charge that Putnam made a mathematical mistake "seems unfair, since Putnam is not clear about the theory in which he is working". *Contra* Bays, Bellotti argued that effecting Putnam's argument with an additional assumption, e.g. having an inaccessible cardinal, does not weaken Putnam's philosophical point. Such an additional assumption can be taken to be part of "our best theory of the world", and "Putnam *can* obtain a final model which satisfies the necessary assumption". On the other hand, Bellotti agreed with what Bays [3] had at the end (quoted above), that Putnam is not fair to his opponents, in that "he does not allow them what he allows himself", e.g. arbitrating what is an intended model. Following Shapiro [15], Bellotti focused on the ill-foundedness of the final model (p.408):

> . . . Putnam's models for nonconstructible reals are so definitely unintended (they are not well-founded, although they 'believe' themselves to be such) to lose much of their disquieting character for any philosophical reflection on unintended models of set theory.

In a reply to Bellotti [5], Bays [4] mainly reaffirmed his [3] position. Going on at considerable length, he nuanced and finessed, one specific point *contra* Bellotti: Putnam is working and needs to work in a fixed theory. Bays newly opined (p.133f), taking account of recent criticism, that "the issues involved in the other parts of [Putnam's] argument are more fundamental", with "the big conceptual questions " being "intended" vs. "unintended" models, "standard" vs. "non-standard" models, and the role of second-order logic. At the end, Bays concluded:

> ... I think it's still important to focus some of our attention on the purely mathematical problems in Putnam's argument. It's not that they are the *only* problems in this argument or even that they're the *deepest* problems in this argument; it's that they're the problems which are most closely connected to the things which make this argument philosophically interesting."

Finally, Button [7], in an account framed by Bays' criticisms, set out the details and imperatives of the "metamathematics of Putnam's model-theoretic arguments". Button discussed at length what he took to be some related mathematics, e.g. the Completeness Theorem and weak set theories, anticipating concerns and reactions of the metaphysical realist. He did forward a simple overall line of argument, that although "Bays' challenge poses considerable problems for the constructivization argument", "it has no impact at all on the Skolemization or the permutation arguments".* For these two arguments, only a conditional is needed: 'if there is a model at all, there is an unintended one".

3. Constructivization Revisited

Spurred by the commentary and criticisms, we here take a deeper look at Putnam's constructivization argument—the text, the mathematical context, the underlying logic, and the specific ways, in the end, in which it can be taken. Putnam's footnote just before his theorem (cf. in the first quoted passage of §1), though not discussed by any of the commentators, provides textual evidence, with its two items each serving as points of beginning in what follows.

How did Putnam actually conceive of and render his argument? In the footnote, he wrote that he had proved the theorem to be deployed before 1963. We look at the historical context here to get an appropriate construal of his theorem.

*The permutation argument is another model-theoretic argument from Putnam [14]; any theory with a model has multiple distinct yet isomorphic models given by permuting elements, and so there is a fundamental semantic indeterminacy.

Putnam's 1963 [13] was a short yet seminal paper on constructible sets of integers.[†] In it, Putnam established, with ω_1^L being the least uncountable ordinal in the sense of L:

(∗) There is an ordinal $\alpha < \omega_1^L$ such that
there is no set of integers in $L_{\alpha+1} - L_\alpha$.

Gödel, of course, had established that $L_{\omega_1^L} = \bigcup_{\alpha<\omega_1^L} L_\alpha$ contains every constructible set of integers, thereby establishing the relative consistency of the Continuum Hypothesis. Putnam's theorem revealed that sets of integers are not steadily constructed up the L_α hierarchy, with his proof of (∗) actually showing that there are arbitrarily large $\alpha < \omega_1^L$ such that there is no set of integers in $L_{\alpha+1} - L_\alpha$. Putnam [13] also showed that by the Shoenfield Absoluteness Lemma, which had just appeared in Shoenfield [16], for any Δ^1_2 ordinal γ, there is an $\alpha < \omega_1^L$ such that there is no set of integers in $L_{\alpha+\gamma} - L_\alpha$.[‡] This early and astute use of the Lemma is consonant with its use in Putnam's proof of his theorem for his constructivization argument.

Putnam's (∗) stimulated the dissertation work of his student Boolos on the recursion-theoretic analysis of the constructible sets of integers, this leading to their [6]. According to Jensen in his classic [11], p.230: "To my knowledge, the first to study the fine structure of L for its own sake was Hilary Put[nam] who, together with his pupil George Bool[o]s first proved some of the results in §3."

How Putnam [13] proceeded with the proof of (∗) anticipated his later constructivization argument. At the outset, he credited Cohen with the method; on his way to forcing, Cohen [8] had shown that there is a minimal $\in$-model of set theory, and to do this he closed off $\{0, 1, 2, \ldots, \omega\}$ under set-theoretic operations and most crucially, under the instances of the Replacement Schema, assuming that this is possible and appealing to the Löwenheim-Skolem Theorem to get the countability of the resulting model.

Recasting this, Putnam argued for (∗) by initially appealing to the Downward Löwenheim-Skolem Theorem to get a countable elementary submodel of L, and proceeding to its transitive isomorph $\langle M, \in\rangle$, so that $M = L_\gamma$ for some countable γ. He then pointed out that there is no set of integers in $L_{\omega_1+1} - L_{\omega_1}$ by Gödel, and hence by elementarity that there is an ordinal $\alpha \in M$ such that there is no set of integers in $L_{\alpha+1} - L_\alpha$.

[†] Sets of integers are routinely identifiable with, and called, reals, but we stick with the thematic trajectory here for a while.

[‡] In fact, this holds, by a straightforward modification of his argument, for *any* $\gamma < \omega_1^L$.

With VB (von Neumann-Bernays) being his working set theory, Putnam next pointed out: "...essentially the preceding argument can be formalized in VB. Of course, we cannot construct a model of *all* of VB in VB and also prove that it is a model." He then described formalizing argument in $\langle L_{\omega_1+2}, \in \rangle$ instead.

Finally, all this relativizes to L, and so there is an $\alpha < \omega_1^L$ such that there is no set of integers in $L_{\alpha+1} - L_\alpha$.

Now to Putnam's theorem (as stated in the first quoted passage of §1) for the constructivization argument, essentially:

(∗∗) For any real s, there is an ω-model of ZF + $V = L$ containing s.

The first point is that this cannot be a theorem of ZF, simply because of its asserting the existence of a model of ZF and hence the consistency of ZF. This would have been clear to anyone versed in mathematical logic as Putnam certainly was, and his [13] remarks on VB above bears this out. *Putnam did not make a mathematical mistake in stating the theorem*, for surely he did not intend to state a theorem of ZF.

Proceeding to Putnam's proof of (∗∗) as given in [14], one next sees the connection to his proof of (∗), described above. It is quite so, as commentators have observed, that using the Downward Löwenheim-Skolem Theorem on L requires additional resources beyond ZFC. This could be said also of Cohen's [8] proof and of Putnam's [13] argument. However, one sees in these argumentations from the early 1960s that they were proceeding informally to get at the fact of the matter. Putnam [13] understood that there is the Cohen minimal model conditionally "if there is any well-founded model" (p.269), and noted that his argument for (∗) with the Downward Löwenheim-Skolem Theorem can be carried out in $\langle L_{\omega_1+2}, \in \rangle$, as a model of full set theory is not required.

Similarly, *Putnam's* (∗∗) *is a theorem of informal mathematics*, stating a fact of the matter to be accepted by the metaphysical realist. His proof, getting quickly to the crucial use of Shoenfield Absoluteness, was meant, it would seem, to provide sufficient deductive ballast to usher the realist to the truth of (∗∗). If one does insist on a ZFC theorem, then the following is appropriate for an appeal to Shoenfield Absoluteness:

(1) If for any real s there is an $\in$-model of ZF containing s, then
for any real s there is an ω-model of ZF + $V = L$ containing s.

Given a constructible real s, there is by the hypothesis an $\in$-model M of ZF containing s and hence (by Cohen's argument!) there is such a model of form $\langle L_\gamma, \in\rangle$. Hence, the Π^1_2 statement formalizing "$\forall s \exists\, \omega$-model of ZF + $V = L$ containing s" is satisfied in L, and the result follows by Shoenfield Absoluteness.

That (1) is a *conditional* assertion in ZFC leads to a pivotal point about Putnam's model-theoretic arguments. Both his Skolemization and constructivization arguments are rhetorically in the form of a *reductio*, and the underlying logic can be carried by the conditional: if there is a model at all, then there is an unintended one. This being said, one can see what Putnam would have had in mind for the mathematics to be invoked by looking again at his footnote just before his theorem (cf. the first quoted passage of §1).

Putnam began the footnote with "Barwise has proved the much stronger theorem", that:

(2) Every countable model of ZF has a proper end extension
which is a model of ZF + $V = L$.

If $\langle M, E\rangle$ and $\langle N, E'\rangle$ are models of ZF, then the second is an *extension* of the first if $M \subseteq N$ and the membership relation E' extends the membership relation E; moreover, it is an *end extension* if for any $a \in M$ and $b \in N$, $b\, E'\, a$ implies that $b \in M$, i.e. elements of M have no new members in N. Barwise's theorem was a culmination of both the investigation of end extensions of models of set theory and the application of infinitary logic to the construction of models of set theory. His [1] proof can be described as a complex application of the Barwise Compactness Theorem and the Shoenfield Absoluteness Lemma; the proof, rendered in the elegant terms of "admissible covers", appears as the last in his book [2]. Barwise's theorem is evidently a strong "upward" Löweinheim-Skolem Theorem, in that one gets an end extension that also satisfies $V = L$. The analogy extends to having a sort of Skolem paradox for models of set theory, with any countable model of ZF being extendible to a canonically slimmest kind of model. This thematically suggests a role in the constructivization argument.

With Putnam in the footnote writing of Barwise's theorem as "much stronger" than his, we take the tack of deploying Barwise's theorem itself, rather than Putnam's, to effect a specification of the constructivization argument:

Both the Downward Löweinheim-Skolem Theorem and Barwise's theorem are conditional theorems of ZFC. With the former, having a (set) model of ZF that contains an abstract set isomorphic to OP amounts to having a countable such model. With the latter, having a countable model of ZF having an abstract set isomorphic to OP amounts to having such a model that also satisfies $V = L$. Thus, we have a ZFC rendition of Putnam's "Any model of ZF which contains an abstract set isomorphic to OP can be extended to a model for this formalized language of science which is *standard with respect to OP*" (cf. the second quoted passage of §1). Of course, the theorems used provide a close relationship between the resulting models and the initial one.

Putnam's Skolemization argument really turns on assuming that there is some model of set theory compatible with theoretical and operational constraints, and then showing that there is a countable one. Its logical structure analogous, his constructivization argument turns on assuming that there is some model of set theory compatible with theoretical and operational constraints, and then showing that there is one satisfying $V = L$. The first implication deployed the Downward Löweinheim-Skolem Theorem, and the second can be effected with Barwise's "upward" theorem.

Putnam [14] had deployed ω-models in order to preserve the sense of OP coded as a real. For a specification of his argument just turning on ω-models, one can argue as above with the following immediate corollary of Barwise's theorem:

(3) If there is a countable $\in$-model of ZF containing a real s, then there is an ω-model of ZF + $V = L$ containing s.

With (1) schematized as $\forall s\varphi \longrightarrow \forall s\psi$, (3) is seen as the stronger version $\forall x(\varphi \longrightarrow \psi)$.

In summary, Putnam's constructivization argument was directed against a realist concept of set. An "epistemological/logical digression" as he put it, it has the rhetorical form of his Skolemization argument, that if there is a model at all, then there is an unintended one. Putnam simply pointed to a mathematical fact of the matter $(**)$ for his argument, but if the realist insists, one can present a conditional ZFC theorem (1) to him. In fact, there are stronger ZFC results, e.g. (2) and (3), that can be invoked. The constructivization argument has various aspects, various ways of putting it and of taking it. However, its overall philosophical thrust and import would not seem to depend on its underlying mathematics. Several results and theorems can be cited or invoked, each perhaps toning the argument in different directions, but not affecting its overall philosophical arc.

4. Critical Coordination

In this final section, we coordinate various criticisms (§2) that have been made in the literature of the constructivization argument, and in the process, consolidate mathematical issues about the argument beyond what was brought out in §3. In the broad, Putnam famously attempted to use model theory, i.e. mathematics, to draw metaphysical conclusions. The particular, constructivization argument, depending on a mathematical contingency new at the time, became surprisingly pivotal in the philosophical literature decades later. Mathematics having a precision, there were specifics that could be aired and argued, and with some more confident than others about the mathematics, commentators generated a fine-grained mesh of interpretation and assessment. While adjudication is often not be the order of the day for philosophical arguments, those involving mathematical results can arguably be illuminated by seeing how they turn on or can be taken according to the mathematics. In what follows, we initially follow a simplified dialectical arc.

Bays [3, 4] has been the most persistent and uncompromising in his criticism of Putnam's constructivization argument. Caught up in the mathematics, Bays urged repeatedly that Putnam's proof of his theorem is "mistaken" and maintained that there is an "intrinsic" problem here because of the Second Incompleteness Theorem, and then that the overall argument, in rhetorically pursuing such paths, is compromised.

Taken to an extreme, if no theorems asserting the existence of models are to be allowed at all, then Putnam's argument would collapse through vacuity. This simple *reductio* could not be what Bays was pursuing; he did acknowledge, for both Putnam's Skolemization and constructivization arguments, that one is assuming first that there is a model and then getting an unintended one. However, there is an ostensible asymmetry in how Bays proceeded:

Putnam had buttressed his Skolemization argument with the Downward Löwenheim-Skolem Theorem applied to "the standard model (if there is one)" which could formally be the proper class V of all sets, to get at a fact of the matter for the realist. If one insists on a ZFC theorem, then one can appeal to the Downward Löwenheim-Skolem Theorem for sets, starting from a set model and getting a countable one. Bays acknowledged the Skolemization argument in passing, this conditional avenue to getting an unintended model from a given model presumably being operative in Putnam's argument.

For the constructivization argument, Putnam had used the Downward

Löwenheim-Skolem Theorem on L to get at a fact of the matter. Bays objected to this as such and did not pursue the underlying conditionality. However, if one insists on a ZFC theorem, (1) or (2), explicit in Putnam's footnote, could have been invoked, as discussed in §3.

Putnam's constructivization argument, it would seem, has a certain sense and an overall thrust. Its components can be addressed and debated variously, with its mathematical underpinnings renderable, e.g. with (2). Bays, in focusing on Putnam's "mathematical mistake"—and moreover treating it as symptomatic of Putnam's philosophical mistakes in general—seems to have fastened onto a relatively inconsequential eddy and distorted the overall flow of argumentation.

Bellotti [5], in arguing against Bays' [3] contention that Putnam had made a mathematical mistake in ZFC, first pointed out that Putnam had not specified his working theory. Belotti then became focused on possible extensions that could serve as that theory, to be part of "our best theory of the world". One additional assumption beyond ZFC sufficient for Putnam's theorem that Bellotti mentioned is akin to the antecedent of (1) (cf. §3). Taking the conditional (1), one can stay in ZFC as the working theory while advancing the constructivization argument rhetorically against the realist.

At the end, Bellotti [5] affirmed (p.407) his "most serious objection to Putnam", that the final models for nonconstructible reals are "definitely unintended", having an ill-founded membership relation. On this Bellotti followed Shapiro [15], who had approached the issue from the perspective of second-order logic. The set-theoretic Axiom of Foundation asserts that the membership relation is well-founded, and if one is working in second-order logic, the axiom would indeed require any model to have a (really) well-founded membership relation. *Contra* Shapiro, one sees, however, that Putnam was working in first-order logic. Indeed, his Skolemization argument would not even get off the ground in second-order logic, as the Löwenheim-Skolem Theorem would not hold. One can pursue this sort of *reductio* to vacuity of course, but it would be by changing the very ground of the argument. *Contra* Bellotti, if one stays in first-order logic but requires intended models to be well-founded, this imposition of a second-order condition still goes against the very tenor of the constructivization argument. Theoretical and operational constraints are to be seen at work inside the final model, and (real) well-foundedness is something one only sees from outside the model.

Button [7] did point out, *contra* Bays, that Putnam's Skolemization argument turned on the conditional: if there is a model at all, then there is an unintended one. Button also pointed out how the Completeness Theorem, provable in a weak set theory, can carry this conditional, while Putnam had appealed to the Downward Löwenheim-Skolem Theorem. As part of his extended analysis, he could have seen Putnam's footnote in his constructivization argument, from which it becomes evident how it too turns on the conditional, which can be carried by Barwise's theorem (2).

Separate from this to and fro, Gaifman [10] interestingly waded through the mathematics of the constructivization argument, landing on a different shore. After acknowledging that Putnam's theorem can be based on some additional assumption(s) to be granted by the realist, Gaifman also pointed out that whether the final model is well-founded or not makes no difference—only what holds in the model, like $V = L$, is substantive to the argument. After this however, Gaifman took basic issue with ever launching such an argument, in view of the viability of non-standard models for the realist.

Stepping back, one sees that the mathematics of Putnam's constructivization argument has been chewed over variously, with the §3 articulation very much possible to hold up as a conditional challenge to the realist. Looking past the mathematics, the commentators, including Bays, went on to address substantive issues about how further to put and take the constructivization argument and determine the extent to which it is philosophically effective. These turn mainly on possible skeptical responses and where the realist stands dialectically in relation to the argument's components and what and how its moves are to be accepted. Be that as it may, however, the mathematics does stand as an interesting and robust part of the argument that Putnam put into play.

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