Cantor and Continuity

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Georg Cantor (1845-1919), with his seminal work on sets and number, brought forth a new field of inquiry, set theory, and ushered in a way of proceeding in mathematics, one at base infinitary, topological, and combinatorial. While this was the thrust, his work at the beginning was embedded in issues and concerns of real analysis and contributed fundamentally to its 19th Century rigorization, a development turning on limits and continuity. And a continuing engagement with limits and continuity would be very much part of Cantor's mathematical journey, even as dramatically new conceptualizations emerged. Evolutionary accounts of Cantor's work mostly underscore his progressive ascent through set-theoretic constructs to transfinite number, this as the storied beginnings of set theory. In this article, we consider Cantor's work with a steady focus on continuity, putting it first into the context of rigorization and then pursuing the increasingly set-theoretic constructs leading to its further elucidations.

Beyond providing a narrative through the historical record about Cantor's progress, we will bring out three aspectual motifs bearing on the history and nature of mathematics. First, with Cantor the first mathematician to be engaged with limits and continuity through progressive activity over many years, one can see how incipiently metaphysical conceptualizations can become systematically transmuted through mathematical formulations and results so that one can chart progress on the understanding of concepts. Second, with counterweight put on Cantor's early career, one can see the drive of mathematical necessity pressing through Cantor's work toward extensional mathematics, the increasing objectification of concepts compelled, and compelled only by, his mathematical investigation of aspects of continuity and culminating in the transfinite numbers and set theory. And third, while Cantor's constructions and formulations may seem simple, even jejune, to us now with our familiarity with set theory and topology, one has to strive, for a hermeneutic interpretation, to see how difficult it once would have been to achieve basic, especially founding, conceptualizations and results.

This article has a pyramidal structure which exhibits first a mathematical and historical basis for Cantor's initial work on limits and continuity and then his narrowing ascent from early conceptualizations to new, from interactive research to solo advance. There is successive tapering since continuity has wide antecedence out of which Cantor proceeds to more and more specific results, just as he is developing more and more set theory.

§1 has as a central pivot Cantor's construction of the real numbers. Leading up to it, we draw in the relevant aspects of real analysis, much having to do with continuity and convergence, and following it, we set out aspects and consequences of constructions of the real numbers that establish a larger ground for mathematics in set theory and topology. §2 gets to Cantor's work on uncountability and dimension, seminal for set theory while also of broad significance—and this is the emphasis here—for the topological investigation of continuity and continua. Finally, §3 finishes up with point-sets and in particular perfect sets, which is a culmination of sorts for Cantor's work on the integrated front of continuity and set theory.

In several ways we follow well-trodden paths; it is through a particular arrangement and emphasis that we bring out the import and significance of Cantor's work on continuity. The books [Ferreirós, 2007] and [Dauben, 1979] proved to be particularly valuable for information and orientation.

1 The Real Numbers

In his earliest researches that anticipated his development of set theory and the transfinite, Cantor provided a construction—or theory—of the real numbers out of the rational numbers. Seen in terms of his overall accomplishments, this construction can be said to be basic and straightforward, an initial stone laid presaging remarkable advances. Nonetheless, it is worth dwelling on the construction and its role in Cantor's research, especially as they have a larger significance when set in a broad context of ponderings about continua and continuity. This section is much longer than the others, being given over to establishing and working that context.

Aristotle, in *Physics*, famously argued (III.5) that "infinity cannot exist as an actual thing" but only has a "potential existence", and, related to this, maintained (VI.1) that "anything continuous" cannot be made up of "indivisibles", e.g. "a line cannot be made up of points". Cantor, with others like Riemann and Dedekind who developed continua, decidedly opted for the actual infinite in mathematics, and, as he formulated the real numbers, he identified them with points, conceiving the continuum as consisting extensionally of points. Cantor's construction, together with Weierstrass' and Dedekind's, completed the "arithmetization" of real analysis. No longer would number be the account of quantity, reckoning and measuring; number becomes inherent and autonomous, given by arithmetic and order relations and completed by extension. This arithmetization was in the wake of an incisive 19th Century mathematical inquiry about continuity and convergence of series of functions. Of this, we give a brief, if necessarily potted, history:

What has been called the 19th Century "rigorization" of real analysis could fairly be said to have been initiated by Cauchy's classic 1821 text *Cours d'analyse*, in which he set out formulations of function, limit, and continuity and encouraged the careful investigation of series (infinite sums) and convergence. He established [note III] *inter alia* a "pure existence" proposition, The Intermediate

Value Theorem. Cauchy's initiative promoted norms and procedures for working with continuous functions, but also a Leibnizian "ideal of continuity" whereby properties are to persist through limits. Fourier's remarkable 1822 *Théorie analytique de la chaleur* brought "Fourier series"—certain series of sines and cosines—to the fore, and, with some leading to discontinuous functions, would exert conceptual pressure on the new initiative. Indeed, Abel, in an incisive 1826 paper on the binomial series, specified at one point that an appeal to the "Cauchy sum theorem"—that a (convergent, infinite) sum of continuous functions is again a continuous function—would have been unwarranted as it "suffers exceptions", one being a simple sine series. It was left to Dirichlet in a penetrating 1829 paper to provide broad sufficient conditions, the "Dirichlet conditions", for a (possibly discontinuous) function to be representable as a Fourier series. In the face of such developments, various mathematicians in the 1840's reaffirmed a new "ideal of continuity" by formulating the appropriately articulated concept of *uniform convergence*, so that e.g. a uniformly convergent sequence of continuous functions does converge to a continuous function. In 1861 lectures at Berlin, Weierstrass carefully set out continuity and convergence in terms of the now familiar ε–δ language, the "epsilontics".

A plateau was reached by Riemann in his 1854 *Habilitation.* In 1868, after his untimely death, his colleague Dedekind published the lecture [1868b] and the dissertation [1868a]. From the first emanated Riemann's far-reaching concept of a continuous manifold, cast in terms of extensional, set-theoretic conceptualizations and now fundamental to differential geometry. From the second we have the now familiar Riemann integral for the assimilation of arbitrary continuous functions, and, with it, Riemann's magisterial extension of Dirichlet's 1829 work to general trigonometric series—arbitrary series of sines and cosines—arriving at necessary and sufficient conditions for a function to be representable by such a series.

However one may impart significance to Cantor's construction of the real numbers, it is best seen in light of this past as prologue, from both mathematical and conceptual perspectives. Cantor too worked on trigonometric series, getting to the next stage, the uniqueness of representation, and, for the articulation, it became necessary to have a construction of the real numbers in hand for conceptual grounding. Historically, uniform convergence had been similarly worked in, for better articulation of results on representability by Fourier series.

A side question might be raised here as to why constructions of the real numbers, being conceptually simple to modern eyes, appeared so relatively late. First, there was still a tradition persisting, going back to the Greek notion of magnitude (*megelos*), that based number on quantity. Rational numbers were given by ratio, and proportion—the equality of ratios—led to further, piecemeal development of parameters. Occam's razor is a hallmark in the development of mathematics, with mathematicians proceeding steadily with the fewest ontological assumptions, and it seems that only by Cantor's time did having a construction of the real numbers as such became necessary, to have a ground for defining collections of real numbers based on taking arbitrary limits. Second, as for conceptual simplicity, basic conceptualizations simply formulated lend themselves to generalization, and we today in axiomatizations of complete metric spaces and the like tend to take what Cantor and others did with the constructions of the real numbers as jejune, underestimating the initial difficulty of a communal carrying out of a regressive analysis.

In what follows, we briefly describe in §1.1 Cantor's early work on trigonometric series, set in its historical context, and describe in §1.2 the [Cantor, 1872] construction of the real numbers and move into limit points and point-sets. In that year, there also appeared constructions of the real numbers in [Heine, 1872] and in [Dedekind, 1872]. The construction in [Heine, 1872] is Cantorian, with acknowledgement, and it is deployed there to establish governing results on functions and continuity. §1.3 sets this out and, embedding it into our narrative, serves to draw out the relevance of the Cantorian construction for functions and continuity. [Dedekind, 1872] provided a thematically different construction of the real numbers which would gain comparable standing. §1.4 makes comparisons, and serves to bring the Cantorian construction into sharper relief as well as raise issues about extensionalism.

1.1 Uniqueness of Trigonometric Series

After his studies and some teaching at Berlin, Cantor in 1869 took up a position to teach as *Privatdozent* at the university at Halle, and upon arrival presented the faculty with a *Habilitationsschrift* in number theory. Eduard Heine was an elder colleague there who decades before had been a student of Dirichlet at Berlin. Working with him, Cantor soon made a consequential change of research direction, to real analysis and the study of trigonometric series.

In his [1870], Heine had pointed out the role of uniform convergence in the work of Abel and Dirichlet and how the significance attributed to the representability of a function as a trigonometric series depended in large part on the uniqueness of the representation. He then built on Dirichlet's 1829 work to establish the following, where by "generally" he meant except at finitely many points.

Theorem 1 ([Heine, 1870, p.355]): "A generally continuous but not necessarily finite function $f(x)$ can be expanded as a trigonometric series of the form

$$f(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \sin nx + b_n \cos nx)$$

in at most one way, if the series is subject to the condition that it is generally uniformly convergent. The series generally represents the function from $-\pi$ to π."

Heine mentioned that it was not known that a trigonometric series representing a continuous function must be uniformly convergent. Cantor set out to eliminate the "uniformly" from the theorem, in itself a significant move since uniform convergence had become so woven into the representability by trigonometric series. Working as Heine had done with Riemann's [1868a] key function $F(x)$, the formal double integration of the trigonometric series, Cantor was able to establish:

Theorem 2 ([Cantor, 1870, p.142]): "When a function $f(x)$ of a real variable x is given by a trigonometric series convergent for every value of x, then there is no other series of the same form which likewise converges for every value of x and represents the function $f(x)$."

The next step forward, to a further generalization, would be momentous. Heine [1870, p.355] had acknowledged Cantor for proposing for uniqueness of representation that, as in Dirichlet's work on Fourier series, there could be a finite number of exceptional points at which the convergence of the trigonometric series fails—so the "generally" in Theorem 1. Cantor, having reduced uniform convergence to just convergence with Theorem 2, worked the nice properties of Riemann's $F(x)$ to allow finitely many exceptional points, effectively incorporating "generally" into Theorem 2. Cantor then realized that a further elaboration with $F(x)$ involving limits would allow *infinitely* many exceptional points in a systematic way. With this central insight about possibility, Cantor developed dramatically new conceptualizations to accommodate his arguments and results, particularly "point-sets" and "derived" such sets in a hierarchy of "kinds". In these terms, he generalized Theorem 2 to the following uniqueness theorem (the theorem is about the zero function, but of course, uniqueness of representation ensues by subtraction of two possible representations of the same function).

Theorem 3 ([Cantor, 1872, p.130]): "If an equation is of the form

$$0 = C_0 + C_1 + C_2 + \ldots + C_n + \ldots,$$

where $C_0 = \frac{1}{2}d_0$, $C_n = c_n \sin nx + d_n \cos nx$ for all values of x with the exception of those corresponding to the points of a given point-set P of the νth kind in the interval $(0, 2\pi)$, where ν denotes an integer, then $d_0 = 0$ and $c_n = d_n = 0$."

1.2 Cantor [1872]

It is at the beginning of his [1872] that Cantor presented his construction of the real numbers. The necessity for Cantor was that for Theorem 3 he had newly developed the topological notion "point-set of νth kind" with respect to the linear continuum, and, heading toward a rigorously presented proof about a real function, he had to be able to apprehend the real numbers "corresponding to the points" in such sets. On this, one can say, again, that the late construction of the real numbers had to do with its only becoming incumbent for proceeding further, and this, appropriately enough, at a next stage in the investigation of trigonometric series, a subject that was interwoven with conceptualizations of continuity and convergence.

Cantor's construction of the real numbers was not *sui generis*. Weierstrass, in Berlin lectures from the early 1860's, based his theory of analytic functions on a construction of the real numbers.[1] He began with the natural numbers as collections of units—much like the Greek *arithmos*—and fractions as collections

[1] A general reference here is [Dugac, 1973].

of aliquot parts $1/n$—as did the Egyptians—to develop real numbers as series (infinite sums)—his *Zahlengrößen*. Thus, in a rather prolix way, the actual infinite could be said to have surfaced here, as well as elemental set-theoretic conceptualizations. There is evidence that Cantor lectured on his own construction of the real numbers in 1870,[2] his second year at the university at Halle. What distinguishes Cantor's construction is its simplicity at a higher level and how it was deployed in the development of new mathematics.

Starting with the rational numbers as given, Cantor [1872] specified (p.123f):

> When I speak of a numerical magnitude in a further sense [*Zahlengröße im weiteren Sinne*], it happens above all in the case that there is present an infinite series [*Reihe*], given by means of a law, of rational numbers
>
> (1) $$a_1, a_2, a_3, \dots$$
>
> which has the property that the difference $a_{n+m} - a_n$ becomes infinitely small with increasing n, whatever the positive integer m may be; or in other words, that given an arbitrary (positive, rational) ε one can find an integer n_l such that $|a_{n+m} - a_n| < \varepsilon$, if $n \geq n_l$ and m is an arbitrary positive integer.
>
> This property of the series (1) I will express by means of the words *"The series (1) has a definite limit [bestimmte Grenze] b."*

"series" here is evidently in the sense we now specify by "sequence"; we refer to a sequence as above, as Cantor subsequently did, as a *fundamental sequence*.[3] Cantor went on to emphasize that "has a definite limit b" is to have no further sense than as set out, with b a "symbol [*Zeichen*]" and different symbols $b, b', b'', \dots$ to be associated with different sequences. He then defined, in terms of associated sequences, $b = b'$ if for any positive rational ε, $|a_n - a'_n| < \varepsilon$ for n sufficiently large, and similarly, $b > b'$ and $b < b'$. Finally, he stipulated that $b * b' = b''$ for $*$ any of $+, -, \times, /$ according to $\lim(a_n * a'_n - a''_n) = 0$ in the expected sense, that for any positive rational ε the value for sufficiently large n is within ε of 0. These definitions conform to the Leibnizian "ideal of continuity" of properties persisting through limits.

These order relations and arithmetical operations as defined can be regarded as extending those for the rational numbers, if e.g. we construe a rational number a as the definite limit of the constant sequence of a's. In particular, that a sequence $a_1, a_2, a_3, \dots$ "has a definite limit b" has an *a posteriori* justification in $\lim(b - a_n) = 0$. Cantor considered that the domain A of rational numbers has been extended, by the introduction of definite limits, to a domain B. However circumspect Cantor had initially been about "definite limit", he thence referred to the members of B as *Zahlengrößen*—and we have a construction of the real numbers.

[2] [Purkert and Ilgauds, 1987, p.37].

[3] See [Cantor, 1883a]. Such a sequence came to be known as a "Cauchy sequence" in the 20th Century, there being an an antecedence in [Cauchy, 1821, p.125].

Cantor then entertained the extension of the domain B to a domain C by analogously introducing as "definite limits" fundamental sequences of members of B. However, he pointed out that whereas there are members of B that do not correspond to any rational number, every member of C "can be set equal [*gleichgesetzt werden kann*]" to a member of B. Nonetheless (p.126)

> ... it is ... essential to maintain the conceptual distinction between the domains B and C, just as the identification of two numerical magnitudes b, b' from B does not include their identity, but only expresses a certain relation which takes place between the series to which they refer.

This remark is revealing about Cantor at this juncture vis-à-vis number and identity: With *process* paramount, C is to be regarded as conceptually different from B; B itself is not quite *the* domain of real numbers as $b = b'$ does not *entail* their identity; and yet, there is commitment to number as *given* by ratio and order relations. Cantor is interestingly at a cusp of the intensional vs. extensional distinction here; he insists on a meaningful distinction between members of C and B, yet subscribes to how they "can be set equal" on the way to identification.

Maintaining the conceptual distinction and regarding B as consisting of *Zahlengrößen* "of the first kind", Cantor proceeded to iterate the process of going from B to C to get from C to a domain D and so on, getting generally to *Zahlengrößen* of the νth kind. With respect to the intensional vs. the extensional distinction, this iteration as carried out with general collections of numbers (see below) would foster an increasingly extensional approach, at the very least because of the need to have simplicity through making identifications—and one has the naissance of Cantor's extensional set theory.

Cantor next went about correlating his *Zahlengrößen* with points on the straight line—so yes, he was inherently committed to the continuum as consisting extensionally of points. Once an origin o and a unit distance have been specified, the rational numbers correlate to points according to ratio. Then, any point is approached arbitrarily closely by a sequence of points corresponding to rational numbers in a fundamental sequence $a_1, a_2, a_3, \ldots$. So, (p.127) "The distance of the point to be determined from o is equal to b, where b is the numerical magnitude [*Zahlengröße*] corresponding to the sequence." How about the converse? Cantor astutely saw (p.128) the need to postulate an axiom to complete the correlation:

> ... to every numerical magnitude [*Zahlengröße*] there corresponds a definite point of the line, whose coordinate is equal to that numerical magnitude.
>
> I call this proposition an *axiom*, since it is in the nature of this statement that it cannot be proven.
>
> Through it the numerical magnitudes also gain a certain objectivity, from which they are, however, quite independent.

We today so readily *identify* real numbers with points on the straight line, that Cantor's initial identification may seem jejune or at best a dutiful correlation. However, one can try to approach hermeneutic interpretation by seeing how Cantor in his day is taking the straight line *qua* linear continuum in a prior sense, one through which his *Zahlengrößen* are to gain "a certain objectivity". A plausible way of thinking is that Cantor's axiom is analogous to Church's Thesis, correlating an informal notion with a formal one. It will be that Cantor would continue to be invested in the investigation of the continuum, enabled in this—through his identification—with what he would increasingly call "arithmetic" means.

With *Zahlengrößen* identified with points on the line and collections of points being "point-sets [*Punktmengen*]", Cantor formulated (p.129) some concepts that would become basic for topology as well as crucial for his uniqueness theorem:

> By a limit point of a point-set P I understand a point of the line whose position is such that in any neighborhood [*Umgebung*], infinitely many points of P are found, whereby it can happen that the same point itself also belongs to the set. By a neighborhood of a point one should understand here any interval which contains the point *in its interior*. Accordingly, it is easy to prove that a point-set consisting of an infinite number of points always has at least *one* limit point.

This last proposition, with the presumption of the point-set being bounded, is recognizably the Bolzano-Weierstrass theorem, and it is indeed easy to prove given Cantor's context of sets and real numbers. Weierstrass, in his lectures at Berlin, had the concepts of concepts of neighborhood and limit point more elementally put. With Cantor's synthetic approach involving actually infinite point-sets, there is a higher-order picture, one which will provide the basis for his development of set theory and topology.

For any infinite point-set P, considering that "limit point of P" is a well-determined concept Cantor took the limit points of P *collectively* to form a new point-set P'. Thus, for the first time, an *operation* on infinite sets was devised. P' is the *derived* set of P, and if P' is again infinite, it too has a (non-empty) derived set P'', and so on. Either this process can be iterated to get for each ν the *νth derived set* $P^{(\nu)}$ of P, or else there is a least ν when $P^{(\nu)}$ is finite. In the latter case Cantor stipulated P to be of the *νth kind*, and those P being of the νth kind for some (finite) ν as *derived sets of the first species*. With an evident correlation between *Zahlengrößen* of the νth kind and derived sets of the νth kind, Cantor pointed out that if one takes a single *Zahlengröße* of the νth kind and traces back through the fundamental sequences all the way back to the rational numbers, the resulting point-set Q of rational numbers is of νth kind—$Q^{(\nu)}$ in fact consists of a single point. While the correlation with *Zahlengrößen* may have stimulated such analysis, the coming to the fore of derived sets as a systemization of the construction of the real numbers promoted a picture of the νth kinds not as different types but as of the same extensional domain.

Finally, with all this structure in place, Cantor established (p.130f) the new uniqueness theorem, Theorem 3 above, which allows infinitely many exceptional points. One sees, through his proof idea of a further elaboration involving Riemann's $F(x)$ with attention to nested intervals, how he had come to entertain his derived point-sets of the first species.

1.3 Heine [1872]

With [Heine, 1870] having been a motivation for [Cantor, 1872], it will be à propos to discuss Heine's [1872], which followed up on an issue—uniform continuity—from his [1870] and relied on the Cantorian construction of the real numbers. Through the discussion one can bring out the operative efficacy of the construction for real analysis.

Heine began [1872] with a lament that function theory as promulgated by Weierstrass in his Berlin lectures had not appeared in "authentic" form, and suggested that in any case its truth rests on a "not fully definite [*nicht völlig feststehenden*] definition" of the irrational numbers. Thanking Cantor for his number conceptualization with sequences, Heine would rigorously set out in his paper the "elements of function theory" as per the title.

Heine began with "number series [*Zahlenreihe*]", fundamental sequences of rational numbers, like Cantor [1872]. He observed that if $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two such, then so are $a_1{*}b_1, a_2{*}b_2, a_3{*}b_3, \ldots$ for $*$ any of $+, -, \times, /$ (taking care not to divide by 0). Heine then associated to each sequence $a_1, a_2, a_3, \ldots$ a "number symbol [*Zahlzeichen*]", $[a_1, a_2, a_3, \ldots]$. Evidently writing "$A = [a_1, a_2, a_3, \ldots]$" etc. to express abbreviation, he then formulated relations between symbols $A = B, A > B, A + B = C, AB = C$ etc., each given in terms of associated sequences. Finally, Heine defined "limit" for symbols, first taking $a = [a, a, a, \ldots]$ for rational numbers a and, working with such, establishing criteria for general symbols. Thus, unlike Cantor [1872] who initially introduced "definite limit" as an expression and, developing symbols, later justified its use, Heine developed symbols first and only later brought in "limit" as concept.[4]

Like Cantor, Heine next considered fundamental sequences consisting of "number symbols" and so forth, getting to "irrationals of higher orders". Cantor had pointed out that such an irrational number "can be set equal" to one of first order, but insisted on maintaining the conceptual distinction for his account of point-sets of higher kind. Heine, on the other hand, merely sketched that "the irrationals of higher order are not new, agreeing with those of first order," and proceeded to use $[x_1, x_2, x_3, \ldots]$ where the x_i's could be irrational numbers. Thus taking an extensional view of the real numbers at the outset, Heine could be said to have *proved* the completeness of the real numbers under the taking of limits.

[4]Years later, making his only reference to [Heine, 1872], Cantor [1883a, §9.8] describes how with a fundamental sequence (a_ν) he correlates a number b, "for which one can expediently use the symbol (a_ν) itself (as Heine, after many conversations with me on the subject, has proposed)."

With the real numbers thus formulated, Heine forthwith developed his function theory *ab initio* (p.180f). "A single-valued function of a real variable x is an expression which is uniquely defined for every rational or irrational value of x." Proceeding to continuity (p.182f),

> A function $f(x)$ is called *continuous for a given individual value* $x = X$ if for any positive ε however small, there exists a positive η_0 such that for no positive η smaller than η_0 does the value of $f(X \pm \eta) - f(X)$ exceed ε.

This is essentially Weierstrass's "epsilontics" formulation of continuity at X. Heine then presented a characterization that he credited to Cantor: A function $f(x)$ is continuous at $x = X$ *if and only if* whenever $X = [x_1, x_2, x_3, \ldots]$, $f(x_1), f(x_2), f(x_3), \ldots$ is a fundamental sequence such that:
$f(X) = [f(x_1), f(x_2), f(x_3), \ldots]$.[5]

Heine next formulated pointwise and uniform continuity (p.184):

> A function $f(x)$ is called *continuous from* $x = a$ *to* $x = b$ if it is continuous for each individual value $x = X$ between $x = a$ and $x = b$, including the values a and b; it is called *uniformly continuous from* $x = a$ *to* $x = b$ if for each positive ε however small, there exists a positive η_0 such that for all positive η smaller than η_0, the value of $f(x \pm \eta) - f(x)$ remains below ε.

As Heine emphasized, this last is to be so for all values of x and $x \pm \eta$ between a and b.

Working with Cantor's characterization, Heine then, in quick order, established the following theorems, now seen as basic to continuity. Weierstrass had attended to these theorems in his lectures with his own, more involved construction of the real numbers.

- Intermediate Value Theorem (p.185f): a continuous function from $x = a$ to $x = b$, with values at a and b of opposite sign, achieves 0 in the interval.
- Greatest Lower Bound Theorem (p.186f): a continuous function from $x = a$ to $x = b$, never negative yet becomes arbitrarily small in the interval, achieves 0 in the interval.
- Extreme Value Theorem (p.188): a continuous function from $x = a$ to $x = b$ achieves both a maximum and a minimum in the interval.

In [1870, p.361], Heine had pointed out the importance of uniform continuity in work of Dirichlet and Abel. To conclude, he showed that, in the recurrent situation of intervals including the endpoints, uniform continuity and continuity coincide:

[5] It is notable that from a logical point of view, the only-if direction made the first, unavoidable use of the Countable Axiom of Choice in mathematics. See [Moore, 1982, p.15f].

Theorem 4 ([Heine, 1872, p.188]): "A continuous function $f(x)$ from $x = a$ to $x = b$ (for all individual values) is also uniformly continuous."

This theorem has come to be called the *Cantor-Heine Theorem*. It requires a higher level of argumentation than for the basic continuity theorems, and can be seen as the thematic climax of Heine's paper. Heine's proof, turning on Cantor's characterization, is sketched as follows:

Suppose that a positive 3ε is given. Let x_1 be the largest $y \leq b$ such that $a \leq x \leq y$ entails $|f(x) - f(a)| \leq \varepsilon$. ($x_1$ is the greatest lower bound of those x such that $|f(x) - f(a)| > \varepsilon$.) If $x_1 < b$, note that $|f(x_1) - f(a)| = \varepsilon$ (by a continuity argument). In that case, let x_2 be the largest $y \leq b$ such that $x_1 \leq x \leq y$ entails $|f(x) - f(x_1)| \leq \varepsilon$. If $x_2 < b$, note that $|f(x_2) - f(x_1)| = \varepsilon$. In this way, proceed for as long as possible to get $a < x_1 < x_2 < x_3 < \ldots < b$.

If this sequence is finite, then the result is established. (In detail, if M is taken to be half the minimum of the $|x_{n+1} - x_n|$'s, then a straightforward "triangle inequality" argument shows that for any z_1 and z_2 between a and b, $|z_2 - z_1| < M$ entails $|f(z_2) - f(z_1)| \leq 3\varepsilon$.)

If this sequence is infinite, then $x_1, x_2, x_3, \ldots$ is a fundamental sequence and f is continuous at $[x_1, x_2, x_3, \ldots]$, yet $f(x_1), f(x_2), f(x_3), \ldots$ is not a fundamental sequence—which contradicts Cantor's characterization.

With the Cantorian construction of the real numbers playing a significant role throughout [Heine, 1872], it is worth describing how its arguments sit among those in the historical "rigorization" of real analysis. In a historical context where the Intermediate Value Theorem *qua* principle was presupposed and applied as part of the sense of continuity, the Bohemian philosopher Bernard Bolzano [1817] in his conceptual approach and Cauchy in his expository text *Cours d'analyse* [1821, note III] enunciated and established it *qua* theorem by "purely analytic" means. Bolzano first "proved" (§7) that a fundamental sequence of partial sums converges; the argument is circular, in that the convergence cannot be proved except on some basis equivalent to it. Bolzano then proceeded (§12) to establish the Greatest Lower Bound Theorem, and with that, (§15) the Intermediate Value Theorem. Cauchy proved (p.460-462) the Intermediate Value Theorem by numerical approximation, constructing two fundamental sequences, one approaching the intermediate value from above and the other from below, their convergence then taken for granted. These argumentations are quite creditable as early gestures in the rigorization of real analysis. With them, one sees specifically how making explicit beforehand the Cantorian construction of the real numbers—so objectifying limits of fundamental sequences—renders argumentation for the Intermediate Value Theorem rigorous and routine, as in [Heine, 1872].

With respect to the Cantor-Heine Theorem on uniform continuity, it is first of all a notable historical happenstance that Bolzano in his *Functionenlehre*—written in the 1830s but only published a century later as [Bolzano, 1930]—had engaged with the concept of uniform continuity.[6] In improvements written for

[6]See [Rusnock and Kerr-Lawson, 2005] for this and what follows. Bolzano pointed out

his work, Bolzano stated (see [Russ, 2004, §6,p.575ff]) the Cantor-Heine Theorem and made an unsuccessful attempt at proof. In 1854 Berlin lectures on the definite integral, Heine's teacher Dirichlet (see [1854 1904, pp.3-8]), with uniform continuity on closed intervals a needed refinement, discursively established the Cantor-Heine Theorem. Dirichlet's argument and Heine's proof, given above, proceed along the same lines, and it can be specified exactly where the latter has the sufficient buttress: where the Greatest Lower Bound Theorem provides for the increasing sequence of x_i's and where Cantor's characterization is applied to deny their infinitude.

1.4 Dedekind [1872]

Richard Dedekind, in his essay *Stetigkeit und irrationale Zahlen* [1872], provided his now well-known construction of the real numbers. While Cantor was invested in his as part and parcel of his research, Dedekind had arrived at his as a matter of the conceptual analysis of continuity. In his preface, he recounts that he had done so already in 1858, and that it was his receipt of Heine [1872] that prompted publication. He also mentioned that he was just in receipt of Cantor [1872] going into press, and specifically pointed to Cantor's axiom as correlated with his own "essence of continuity".

Dedekind initially set out (§1) the rational numbers and their ordering and reviewed (§2) how they can be correlated with points on a straight line, once an origin o and a unit distance have been specified. Presupposing the line to consist extensionally of points, Dedekind recalled how any point partitions the line into two parts, those points to the left and those points to the right, and formulated (§3) the "essence of continuity" to be the converse, the following principle:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.

Dedekind thus fixed on the straight line for his analysis and principle—or axiom—whereas Cantor worked up his *Zahlengrößen* first and, latterly correlating with the straight line, posited his axiom, through which his *Zahlengrößen* gain "a certain objectivity".

Pursuant of his principle, Dedekind formulated (§4) his now well-known *cut* [*Schnitt*] as any pair (A_1, A_2) of non-empty classes that constitute a partition of the rational numbers, with any member of the first less than every member of the second. A rational number produces two cuts—one with that number as maximum of the left set and the other with it as the minimum of the right set—these cuts to be regarded as essentially the same. Those cuts (A_1, A_2) with

(see [Russ, 2004, §49,p.456]) that the function $f(x) = \frac{1}{1-x}$, while continuous, is not uniformly continuous in an open interval (i.e. excluding endpoints) around $x = 1$.

no maximum nor minimum "create a new, *irrational* number α which we regard as completely defined by this cut (A_1, A_2)". In this, α is like Cantor's "definite limit" in having no further sense than as given, though Dedekind's principle objectifies the number as corresponding to "one and only one point" on the straight line.

After setting out the order relations between his real numbers according to set-inclusion of corresponding cuts, Dedekind established (§5) the thematically central result, that the real numbers—autonomous and no longer correlated with the straight line—satisfy his principle of continuity: Given a cut of real numbers—a partition of the real numbers into two non-empty classes with any real number in the first less than every real number in the second—there is one and only one real number that produces the cut. This corresponds to Cantor's observation that any member of his domain C, the result of taking limits of fundamental sequences from his domain B of *Zahlengrößen*, "can be set equal" to a member of B.

Rhetorically, having focused on order and continuity, Dedekind latterly attended (§6) to the formulation of the arithmetical operations for real numbers. He detailed only the addition of cuts: For cuts (A_1, A_2) and (B_1, B_2), define C_1 to consist of those rational numbers c such that for some a in A_1 and b in B_1, $c \leq a + b$; then taking C_2 to be the complement of C_1, (C_1, C_2) is a cut that appropriately serves as the sum. Actually, the multiplication of cuts cannot be analogously defined because of the law of signs $(-a)(-b) = ab$, and a proper definition would have to involve intervals of rationals of differing signs. Dedekind suggested introducing the ideas of "*variable magnitudes, functions, limiting values*, and it would be best to base the definition of even the simplest arithmetic operations upon these ideas"—thus approaching Cantor's definitions of the arithmetical operations.

Dedekind concluded (§7) his essay by proving two "fundamental theorems of infinitesimal analysis", each of which he noted is equivalent to his principle of continuity: "If a magnitude x grows continually but not beyond all limits it approaches a limiting value", and "If in the variation of a magnitude x we can, for every given positive magnitude δ, assign a correspond position from and after which x changes by less than δ, then x approaches a limiting value." Were these stated in terms of sequences, then the first would assert that an increasing sequence bounded above has a limit, and the second, that a fundamental sequence has a limit.

Stepping back, one sees that, however Dedekind actually arrived at his construction of the real numbers, his account and Cantor's proceed in diametrically opposite directions. Dedekind began with the straight line and his principle of continuity and got to the existence of limits in real analysis; Cantor started with fundamental sequences having "definite limits" and made his way to an axiom for correlating numbers with the linear continuum. With its explicit, existence postulation about the straight line, Dedekind's principle has been deployed as an axiom to rigorize Euclidean geometry.[7] *Once* the real numbers have been

[7]See [Greenberg, 2008, p.134ff] and [Heath, 1956, p.236ff]. From it, one can prove that if a

defined and are in place, Dedekind's principle is seen to be equivalent to fundamental sequences having limits, as well as to each of the Intermediate Value Theorem, the Greatest Lower Bound Theorem, and the Extreme Value Theorem (cf. [Heine, 1872]).

As for constructions as *definitions* of the real numbers, it is informative to consider how Cantor himself in his later, 1883 *Grundlagen* (§9) saw and compared them. Cantor weighed three definitions of the real numbers, those of Weierstrass from his Berlin lectures, [Cantor, 1872], and [Dedekind, 1872]. After loosely describing the first, Cantor pointed out how Weierstrass was the first to avoid the "logical error" of assuming a finished number exists to which a defining process aspires. Cantor here was bringing out the motivating point of genetic construction for rigorization. Then briefly sketching Dedekind's definition, Cantor asserted that it "has the great disadvantage that the numbers of analysis *never* occur as 'cuts', but must be brought into this form with a great deal of artificiality and effort." Cantor here speaks as the researcher in real analysis who finds Dedekind's conceptualization distant from utilizability; recall Dedekind's last efforts in his §6 and §7. Cantor subsequently settled into an extensive account of his own definition, more detailed than in [Cantor, 1872] and more in the style of [Heine, 1872].

In the middle of this comparative account, Cantor wrote (para.7):

> The disadvantage in the [Weierstrass] and [Cantor] definitions is that the same (i.e. equal) numbers occur infinitely often, and that accordingly an unambiguous overview of all real numbers is not immediately obtainable. This disadvantage can be overcome with the greatest ease by a specialization of the underlying sets (a_ν) using one of the well-known unambiguous systems, such as, for instance, the decimal system or the simple development in continued fractions.

While Dedekind's cuts can themselves serve as (stand for, be identified with) the real numbers, with the Weierstrass and the Cantor definitions one real number corresponds to infinitely many *Zahlengrößen* that are pairwise equal according to a derivative notion of equality. This falls short of the ideal of extensionalism, in that real numbers are not fixed as well-defined by construction. There are two ways of rectifying this, one by way of *equivalence classes* and the other, as Cantor mentions, by way of specializing fundamental sequences to correspond to *decimal expansions* or to *continued fractions*.

The mode of equivalence classes actually has antecedence in the work of Dedekind. In [1857], Dedekind had proceeded in $\mathbb{Z}[x]$, the ring of polynomials in x with integer coefficients, by taking as unitary objects infinite collections of polynomials pairwise equivalent modulo a prime p. One can arguably date the entry of the actual infinite into mathematics here, in the sense of infinite totalities serving as unitary objects within an infinite mathematical system.

circle has one point inside and one point outside another circle, then the two circles intersect in two points; with this, one can fill a well-known lacuna in the proof of the very first proposition of Euclid's *Elements*.

Had Dedekind come to the genetic construction of the real numbers “from the other end”—Cantor’s approach—he might well have taken equivalence classes of fundamental sequences as the real numbers.

As for the mode of specializing fundamental sequences, Cantor had engaged with it in the course of his research in June 1877, as brought out in an exchange of letters then with Dedekind.[8] For specializing via *decimal expansions*, one would consider fundamental sequences only of the form: a_1 is an integer, and $a_{n+1} = a_n + d_n \cdot 10^{-n}$ for some integer d_n satisfying $0 \leq d_n \leq 9$. With the happenstance that e.g. $.3000\cdots = .2999\cdots$, one has to further restrict consideration to those sequences without a tail of 0’s. For specializing via *continued fractions*, one would first call on the known fact that every number r in the interval $(0,1)$ would have a *unique* representation as a continued fraction

$$r = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{\ddots}{\ddots + \cfrac{1}{\alpha_\nu + \dots}}}}$$

where each α_i is a positive integer. Exactly when r is a rational number, there is a last α_ν as denominator, and we denote this by $r = [\alpha_1, \alpha_2, \dots, \alpha_v]$. With this, one would consider fundamental sequences only of the form: a_1 is an integer, and a_{n+1} is either: a_n, or else $a_1 + [\alpha_1, \dots, \alpha_{n-1}, \alpha_n]$—this last possibility only in the case that a_n has the form $a_1 + [\alpha_1, \dots, \alpha_{n-1}]$.

Cantor’s “specializing” of fundamental sequences to those corresponding either to decimal expansions or to continued fractions does achieve the ideal of extensionalism in that there is a one-to-one correlation between sequences and real numbers. This however comes at the sacrifice of the ideals of simplicity and perspicuity as one incorporates *a posteriori* understandings, so much so that one might even say that one is looking at different constructions of the real numbers, not aspects of the same. Cantor’s [1872] construction of the real numbers was integrated with his research and has a basic relevance and applicability, as brought out in [Heine, 1872]. Continued fractions too were brought into his research, this for working the theme of one-to-one correlation, as he advanced into transfinite set theory.[9]

Starting in late 1873, Cantor and Dedekind began a correspondence that would last, on and off, for decades, a correspondence that was stimulating for Cantor and is informative to us about his thinking and progress. We mentioned an exchange of letters in June 1877 above, and in an exchange a month earlier, Cantor and Dedekind discussed aspects of [Dedekind, 1872]. It is through

[8] See [Ewald, 1996, p.853ff].
[9] See §2.2.

this correspondence that we learn much about Cantor's next advances, those squarely in transfinite set theory yet much having to do with continuity.

2 Uncountability and Dimension

With his formulation of the real numbers in play, Cantor, in the initiating correspondence with Dedekind in late 1873, pursued a question about one-to-one correlation and the real numbers, a question that he had apparently considered several years earlier. The result was a compelling new mathematical understanding of cardinality as concept applied to the real numbers, one that would stimulate Cantor to the development of transfinite numbers and set theory. Cantor then, in a letter to Dedekind of 5 January 1874, followed up with the question of whether there could be a one-to-one correlation between a line and a surface. In 1877, Cantor also compellingly settled this issue, stimulating the initial work on the invariance of dimension.

These two results on one-to-one correlation would be the bulwark for setting up the concept of infinite cardinality. Both proofs still being embedded in real analysis and talk of correlation, transfinite set theory would emerge with the consideration of arbitrary correspondence, in the form of the study of transfinite cardinality. In what follows, we describe one by one these developments, the main points of which are well-known in the history of set theory as Cantor's initial accomplishments, with attention put to the specificities of research activity and the underlying involvement of continuity.

2.1 Uncountability

Cantor in a letter of 29 November 1873 to Dedekind posed the question:[10]

> Take the totality [*Inbegriff*] of all positive whole-numbered individuals n and designate it by (n). And imagine say the totality of all positive real numerical magnitudes [*Zahlengrößen*] x and designate it by (x). The question is simply, can (n) be correlated to (x) in such a way that to each individual of the one totality there corresponds one and only one of the other?

("totality" here is a deliberate translation of "Inbegriff".[11]) Note the tentativeness of setting out in uncharted waters of totalities and correlation. Today, this primordial question is put: Are the real numbers countable? Cantor opined that the answer would be *no*, that the explanation may be "very easy". He did point out that it is not difficult to correlate one-to-one the totality of positive integers with the totality of rational numbers, and indeed with the totality of finite tuples of positive integers.

[10] See [Ewald, 1996, p.844].

[11] What may first come to mind today for "Inbegriff" may be "essence", "embodiment", or "paradigm". However, Cantor likely meant "totality", with a precedent for this in [Bolzano, 1851], who used "Inbegriff" in proximity to "Ganzes". "totality" conveys an appropriately incipient extensionalism, soon to become more substantive in Cantor's work.

Dedekind answered by return post that he could not answer the question. However, bringing in his algebraic experience, Dedekind included a full proof that even the totality of algebraic numbers, roots of polynomials, can be correlated one-to-one with the totality of positive integers.[12] Cantor in his responding letter of 2 December, encouraged, wrote that he had wondered about the question "already several years ago"; that he agreed with Dedekind that it has "no special practical interest" and "for this reason does not deserve much effort"; but that it would be good to answer the question—a negative answer would provide a new proof, in light of the algebraic number correlation, of Liouville's theorem that there are transcendental (i.e. non-algebraic) real numbers.[13]

Presumably stimulated to success, Cantor in his letter of 7 December wrote that "only today do I believe myself to have finished" and included for appraisal an argument that the totality of real numbers *cannot* be correlated one-to-one with the totality of positive integers.[14] On that day, transfinite set theory was born. Again Dedekind answered by return post, with "congratulations for the fine success" and a much simplified version of the proof.[15] Cantor in his letter of 9 December announced that he had already simplified his proof, that it shows that for any sequence and any interval of real numbers, there is a real number in the interval not in the sequence.[16] In his notes, Dedekind remarked that their letters must have crossed.[17]

In a letter of 25 December, Cantor wrote that, with the encouragement of Weierstrass at Berlin, he had written and submitted a short paper, this to become "On a property of the totality of real algebraic numbers" [1874].[18] Although the paper is where one points to for the naissance of transfinite set theory, it is restricted in purpose, as Cantor pointed out in a letter of 27 December, because of "local circumstances"—these presumably being Weierstrass' restrictive focus on the algebraic numbers.[19] In the paper, Cantor established that the totality of the algebraic numbers is countable and that the totality the real numbers is not. In his notes, Dedekind recorded that both proofs were taken "almost word-for-word" from his letters.[20] From the correspondence, it can fairly be said that the first result is actually Dedekind's.

As for the uncountability of the totality of the real numbers, the [1874] proof is schematically as follows: Suppose that a sequence and an interval of real numbers is given. The goal is to find a real number in the interval but not in the sequence. Let α_1 and β_1 be the least two members of the sequence, if any, in the interval, say with $\alpha_1 < \beta_1$. Generally, given α_n, and β_n, let α_{n+1} and β_{n+1} be the next two least members of the sequence, if any, in the interval

[12]See [Ewald, 1996, p.848].
[13]See [Ewald, 1996, p.844f].
[14]See [Ewald, 1996, p.845f].
[15]See [Ewald, 1996, p.849].
[16]See [Ewald, 1996, p.846f].
[17]See [Ewald, 1996, p.849].
[18]See [Ewald, 1996, p.847].
[19]See [Ewald, 1996, p.847f].
[20]See [Ewald, 1996, p.848f]. This may have contributed to Dedekind not responding to Cantor's letters for quite some time.

(α_n, β_n), say with $\alpha_{n+1} < \beta_{n+1}$. If ever this process terminates at a finite stage, then we are done, as the interval at that stage will have a real number not in the sequence. Assume then that this process is infinite, and let α^∞ be the upper limit of the α_n's, and let β^∞ be the lower limit of the β_n's. Then any real number η such that $\alpha^\infty \leq \eta \leq \beta^\infty$ cannot be in the sequence.

This proof has an evident involvement of continuity, viz. the fundamental sequences of Cantor's construction of the real numbers. But also, as Cantor pointed out in a prescient footnote, the rth member of the sequence is not in the particular interval (α_r, β_r), thereby correlating the *indexing* of the sequence with the *enumeration* of the nested intervals. Almost two decades later, Cantor in [1891] would present his famous *diagonal* proof, abstract and no longer involving continuity, of a vast generalization of uncountability: For any set whatsoever, there is no one-to-one correlation of that set and the collection of functions from that set into a fixed two-element set.

2.2 Dimension

Cantor already in a letter of 5 January 1874 to Dedekind raised a new question pursuant of the motif of one-to-one correlation:[21]

> Can a surface (say a square including its boundary) be one-to-one correlated to a line (say a straight line including its endpoints) so that to every point of the surface there corresponds a point of the line, and conversely to every point of the line there corresponds a point of the surface?

He opined that the answer is "very difficult", that as with the previous question "one is so impelled to say *no* that one would like to hold the proof to be almost superfluous." Again a primordial question, one for which an answer of *no* does look difficult to establish, but this time the pathways of proof would lead the other way, working against the initial surmise. Note, importantly, that Cantor could only hope to answer such a question—indeed, even pose it—with his construction of the real numbers in place to work combinatorial possibilities for one-to-one correlation.

Dedekind did not respond, nor when Cantor brought up the question again in a letter of 15 May 1874. Fully three years would pass before there was again an exchange of letters, this initially about aspects of Dedekind's [1872]. During this time, Cantor had evidently developed a new conceptualization.

In a letter of 20 June 1877 to Dedekind, Cantor now set out his question:[22]

> The problem is to show that surfaces, bodies, indeed even continuous structures of ρ dimensions can be correlated one-to-one with continuous lines, i.e. with structures of only *one* dimension—so that surfaces, bodies, indeed even continuous structures of ρ dimensions have the same power [*Mächtigkeit*] as curves.

[21] See [Ewald, 1996, p.850].
[22] See [Ewald, 1996, p.853f].

Note the “continuous”, “dimension”, and especially “power”. Two totalities have the same *power* if there is a one-to-one correlation between them, and with this notion of cardinality Cantor had begun his ascent into transfinite set theory. In the letter, specifically addressing whether the ρ-tuples of real numbers in the closed unit interval [0,1] can be correlated one-to-one with the real numbers in $[0,1]$, Cantor now answered *yes*. Writing each such real number in infinite decimal expansion, he simply interlaced ρ expansions into one. In the $\rho = 2$ case, given $.\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\cdots$ and $.\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\cdots$ the result would be $.\alpha_{1,1}\alpha_{2,1}\alpha_{1,2}\alpha_{2,2}\alpha_{1,3}\alpha_{2,3}\cdots$.

By return post Dedekind pointed out the problem that, presuming in the cases e.g. of $.3000\cdots = .2999\cdots$ one would choose the latter representation, those real numbers with expansion consisting of a tail of alternating 0’s would never be the result of an interlacing.[23] By postcard, Cantor acknowledged this, and noted the reduction of the issue to finding a one-to-one correlation between the interlaced real numbers with *all* the real numbers in $[0,1]$.[24]

In a pivotal, long letter to Dedekind of 25 June 1877,[25] Cantor accepted that the “subject demands more complicated treatment” and set out to prove a theorem that he now stated in Riemannian terms, (A): “A continuous manifold [*Mannigfaltigkeit*] extended in e dimensions can be correlated one-to-one with a continuous manifold in one dimension.” His proof, “found even earlier than the other”, established that the ρ-tuples of real numbers in the closed unit interval $[0,1]$ can be correlated one-to-one with the real numbers in $[0,1]$.

Cantor first took an irrational number in $[0,1]$ to be represented as an infinite continued fraction,[26] and correlated one-to-one the ρ-tuples of irrational numbers in $[0,1]$ with the irrational numbers in $[0,1]$ by interlacing the fraction entries—in analogy to what he had tried with decimal expansions. This time, the known fact that continued fraction representations are unique ensures one-to-one correlation. What must now be established is (B): The irrational numbers in $[0,1]$ can be correlated one-to-one with *all* the numbers in $[0,1]$. For this, he first enumerated the rational numbers in $[0,1]$ (recall his first, 1873 letter to Dedekind!) and then correlated them one-to-one with certain $0 < \varepsilon_i < 1$ with $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots$ having limit 1. The proof then devolves to proving (C): The numbers in $[0,1]$ except for the ε_i’s can be correlated one-to-one with *all* the numbers in $[0,1]$. Since this in turn would follow if each half-open interval $(\varepsilon_i, \varepsilon_{i+1}]$ can be correlated one-to-one with the closed interval $[\varepsilon_i, \varepsilon_{i+1}]$, one is left to proving the paradigmatic (D): The half-open interval $(0,1]$ can be correlated one-to-one with the closed interval $[0,1]$. And this he establishes with a step function of line segments, providing a detailed diagram.

Note how this proof proceeds by successive reduction of the problem, each step having to do with composing one-to-one correlations of various domains. One can say that Cantor was driven, almost by necessity, from analytic thinking

[23]See [Ewald, 1996, p.855f]. See, several paragraphs below, how with an adjustment this problem can be avoided.
[24]See [Ewald, 1996, p.856].
[25]See [Ewald, 1996, p.856ff].
[26]cf. end of 1.4.

about correlations to set-theoretic, combinatorial thinking about manipulating them. In modern terms, the initial continued-fraction correlation of the irrationals is continuous, while the rest is gradually worked down combinatorially to (D), which cannot be carried out with a continuous function. As here and generally in his work, as one-to-one correspondence comes to the fore, continuity recedes, this seen in the new ways that one must entertain arbitrary functions.

Today, (D) is a simple exercise in set theory texts. Cantor's overall result, as he later pointed out in his mature *Beiträge* [1895, §4], could be derived with "a few strokes of a pen" in his cardinal arithmetic:

$$2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} \,.$$

Finally, if one insists on working with infinite expansions, then it is straightforward, just working with infinite decimal expansions, to take all non-zero real numbers as successive blocks each consisting of a sequence of zeros followed by a non-zero digit—e.g. $.|08|1|3|2|001|03|1|2|\cdots$—and, through interlacing according to blocks, one-to-one correlate pairs of real numbers in $(0,1]$ with *all* the real numbers in $(0,1]$. What then remains, if $[0,1]$ is desired, is to apply (D).

Cantor concluded the 25 June 1877 letter with remarks that positioned his result in a larger tradition. For years he had followed with interest the efforts of Gauss, Riemann, Helmholtz and others at clarification of the foundations of geometry. The important investigations in this field proceed from a presupposition that Cantor too had held to be correct, though he alone had thought that it was a theorem in need of a proof. Attending the *Gauss-Jubiläum*,[27] he had aired (A) above as a question. There was acknowledgement that a proof was needed, to show the answer to be *no*. But "very recently" he had arrived at the conviction that the answer is an unqualified *yes*, and thus he had found the proof presented in the letter. All deductions that depend on the erroneous presupposition are now inadmissible. "Rather, the difference that obtains between structures of *different* dimension-number must be sought in aspects completely different from the number of independent coordinates, which is taken to be characteristic."

Dedekind in a substantive reply of 2 July 1877 first off avouched that Cantor's proof is correct and congratulated him on the result.[28] Dedekind however took issue with Cantor's last remarks on the dissolution of dimension. He maintained that the "dimension-number of a continuous manifold remains its first and most important invariant", though he would gladly concede that this invariance is in need of a proof. All authors have made the "completely natural presupposition" that transformations of continuous manifolds via coordinates should also be via *continuous* functions. Thus, he believes the following theorem:

> If it is possible to establish a reciprocal, one-to-one, and complete correspondence between the points of a continuous manifold A of a

[27] 30 April 1877 at Göttingen, on the centenary of his birth.
[28] See [Ewald, 1996, p.863f].

> dimensions and the points of a continuous manifold B of b dimensions, then this *correspondence itself*, if a and b are *unequal*, must be *utterly discontinuous*.

Dedekind here, with his acute sense of mathematical structure and consequent emphasis on structure-preserving mappings, had swung the pendulum back toward continuity. Riemann, and others, had put forward an informal theory of continuous manifolds with an implied concept of dimension based on the number of coordinates. Dedekind forthwith asserted the *invariance of dimension* of continuous manifolds under *homeomorphisms*, i.e. one-to-one correspondences of their points which are continuous in both directions. Framed as a question, his proposition can be fairly said to have stimulated the study of topological invariants.

Cantor in his letter of 4 July 1877 responded that he had not intended to give the appearance of opposing the concept of ρ-fold continuous manifold, but rather "to clarify it and to put it on the correct footing."[29] He agreed with Dedekind that if the correspondence is to be continuous, then only structures of the same dimension can be correlated one-to-one. He suggested that, if so, difficulties might arise in "limiting the concept of *continuous correspondence in general*." Indeed, in the decades to come, how to frame continuity for mappings between continuous manifolds so as to establish Dedekind's proposition would itself become a substantial issue.

Cantor published his one-to-one correlation result in his "A contribution to the theory of manifolds" [1878], a paper he pitched be to promulgating his concept of power [*Mächtigkeit*]. Having made the initial breach in [1874] with a negative result about the *lack* of a one-to-one correlation, he worked to secure the new ground by setting out the *possibilities* for having such correlations. With "manifold" evidently meant in a broad sense, two manifolds have the *same power* if there is a one-to-one correlation between their elements. "If two manifolds M and N are not of the same power, then M either with a part [*Bestandteile*] of N or N with a part of M has the same power; in the first case we call the power of M *smaller*, and in the second we call it *greater*, than the power of N."[30] The class [*Klasse*] of manifolds of the power of the positive whole numbers is "particularly rich and extensive", consisting of the algebraic numbers, the point-sets of the νth kind from [1872], the n-tuples of rational numbers, and so forth. If M is in this class, so also is any infinite part [*Bestandteil*] of M; and if $M', M'', M''' \ldots$ are all in this class, so is their union [*Zusammenfassung*].[31]

Proceeding to n-fold continuous manifolds, Cantor first elaborated on how invariance of dimension under continuous correspondence had always been pre-

[29] See [Ewald, 1996, p.864f].

[30] Note the locution "power of M". Already here, at the incipience of Cantor's theory of cardinality, we have the assertion of the trichotomy of cardinals. As set theory became axiomatized, it was seen that the trichotomy of cardinals is equivalent to the Axion of Choice; see [Moore, 1982, p.10].

[31] Note the set-theoretic delving. Logically speaking, this last assertion, put in axiomatic set theory as "the countable union of countable sets is countable", requires the Countable Axiom of Choice. See [Moore, 1982, pp.9,32].

supposed but should be demonstrated, and then offered up his theorem as to what becomes possible when no assumptions are made about the kind of correspondence.[32] Most of the paper is given over to a proof of his one-to-one correlation result, almost verbatim as given in his 25 June 1877 letter to Dedekind save for some notational refinements made in succeeding letters.

At the end of the paper, having reduced considerations of power to linear manifolds, Cantor opined:

> . . . the question arises how the different parts of a continuous straight line, i.e. the different infinite manifolds of points that can be conceived in it, are related with respect to their powers. Let us divest this problem of its geometric guise, and understand (as has already been explained in §3) by a *linear* manifold of real numbers any conceivable totality [*Inbegriff*] of infinitely many, distinct real numbers. Then the question arises, into *how many* and which classes [*Klassen*] do the linear manifolds fall, if manifolds of the same power are placed in the same class, and manifolds of different power into different classes? By an inductive procedure, whose more exact presentation will not be given here, the theorem is suggested that the number of classes of linear manifolds that this principle of sorting gives rise to is finite, and indeed, equals *two*.
>
> Thus the linear manifolds would consist of two classes, of which the first includes all manifolds that can be given the form of a function of ν (where ν ranges over the positive whole numbers), while the second class takes on all those manifolds that are reducible to the form of a function of x (where x can assume all real values ≥ 0 and ≤ 1). Corresponding to these two classes, therefore, would be only two powers of infinite linear manifolds; the exact study of this question we put off for another occasion.

Note that the "geometric guise" *can* be divested through Cantor's construction of the real numbers; how a linear manifold is any "conceivable totality" of real numbers; how "classes" consisting of these are being entertained; and how the initial "having the same power" has become being "of a power"—equivalence relation has become equivalence classes. We see entering line-by-line the more set-theoretic posing and thinking. The "inductive procedures" are presumably what evolved into the transfinite numbers in his coming papers. In suggesting the existence of only two power classes of linear manifolds, this passage has Cantor's first statement of the Continuum Hypothesis, a primordial, dichotomous assertion that he would wrestle with "on another occasion"—the rest of his life—and set theory still wrestles with to the present day.

Cantor's ascent into set theory would be by himself, but the issue raised of the invariance of dimension, with its foregrounding of continuity, elicited quick reaction. Within a year, five publications appeared that offered proofs

[32] Thus, Cantor set out the sequential thinking on the topic in reverse order relative to how it had been in his correspondence with Dedekind.

of the invariance of dimension in various formulations, by Jacob Lüroth, Johannes Thomae, Enno Jurgens, Eugen Netto, and soon after, by Cantor [1879a] himself.[33] The arguments were complex and would latterly be deemed as only partially successful, revealing a lack of command of the relevant topological notions at the time. In a letter of 29 December 1878 to Dedekind, Cantor wrote that he had seen the papers of the four others on the invariance of dimension, but that "the matter does not seem to me to be fully resolved."[34] In a letter of 17 January 1879, Cantor claimed to have settled the question, sketching an argument that turned on contradicting the Intermediate Value Theorem were there a continuous one-to-one correlation between continuous manifolds of different dimensions.[35] As on previous occasions, Dedekind in a reply of 19 January helpfully responded with issues, though this time he saw a "real difficulty".[36] In a postcard of 21 January, Cantor wrote, acknowledging the difficulty, that he would only consider publishing "*only in case* I should succeed in settling the point."[37] Cantor must have done so at least to his own satisfaction, for his [1879a] appeared shortly after with his proof.

Cantor's solution, his last work dealing directly with continuous correspondences, was thought to have settled the matter for decades. However, Guissepe Peano's [1890] "space filling curve", a continuous mapping from the unit interval $[0,1]$ onto the unit square $[0,1] \times [0,1]$, was latterly seen to be a counterexample to Cantor's formulation. As topological notions were developed, the stress brought on by the lack of firm ground led the young L.E.J. Brouwer to definitively establish the invariance of dimension in a paper [1911] that was seminal for algebraic topology.

In retrospect, it is to his considerable credit that Dedekind made explicit the invariance of dimension as an issue in his 1877 correspondence with Cantor. Cantor [1878] publicized it, and he [1879a] pursued this new angle on continuity for a while, but he soon reverted to earlier conceptualizations to be followed up into the transfinite. The renewed pursuit of invariance of dimension in the 20th Century led to the new field of algebraic topology.

3 Point-Sets and Perfect Sets

Through the early 1880s Cantor carried out what would be his major work, work which would be of basic significance for the subject he created, transfinite set theory. It featured continuing engagement with immanent topological notions, and through them, what would become his mature results having to do continuity, involving continua and perfect sets. Through sustained effort, in large part driven by the urge to establish the Continuum Hypothesis, Cantor vindicated his early construction of the real numbers and his derived sets by iterating the

[33]See [Dauben, 1979, p.70ff] for details and references.
[34]See [Ewald, 1996, p866f].
[35]See [Ewald, 1996, p.867ff].
[36]See [Ewald, 1996, p.869f].
[37]See [Ewald, 1996, p.870f].

derived set operation through limits and establishing a hierarchical structure of continuity.

During this period, Cantor published a series of six papers under the title "On infinite, linear point-manifolds" which documents his progress. Pursuing them in sequence one by one, we can see an overall forward logic in the progress of discovery. §3.1 describes the progress through the first four papers in the series, through (in modern terms): dense sets, derived sets of infinite order, the countable chain condition, and countability along the iteratively defined sequences of derived sets. §3.2 then describes the plateau reached, derived sets of uncountable sets, continua, and closed and perfect sets.

These sections, describing Cantor's major advances having to do with continuity, are comparatively short for several reasons. First, they describe work by one individual working on his own with novel conceptualizations and methods. Second, the new set-theoretic context thus established transcends continuity, which was to command our main focus. And third, the work builds on earlier, seminal formulations and results about continuity for which we have already provided comparatively elaborate historical and mathematical detail.

3.1 Point-Sets

The first paper [1879b] in Cantor's "linear point-manifolds" series established a base camp for his further ascent through infinitary processes. Containing no new results, it framed *ab initio* Cantor's earlier, basic work on limit points and power and cast it anew systematically. In doing so, it brought out aspects of simplicity and directness to constructs and results that had initially emerged in an encumbered way out of Cantor's construction of the real numbers and considerations of one-to-one correlation.

In the grip of his recent initiatives Cantor had titled his intended series with the ponderous "linear point-manifolds [*Punktmannigfaltigkeiten*]", but he quickly reverted in [1879b] to his earlier "point-set [*Punktmenge*]" for a collection of real numbers. Recalling the [1872] operation of taking for an infinite point-set P the derived set P' consisting of its limit points, he again set out its iteration $P^{(\nu)}$ through ν stages, and deemed P to be of the first species if $P^{(\nu)}$ is finite for some ν. He then officially stipulated that P is *of the second species* if the series of $P^{(\nu)}$ continues *ad infinitum*. Proceeding, Cantor brought to the fore what his [1874] uncountability result had turned on: A point-set P is *everywhere-dense*—for us, just *dense*—in an interval $[\alpha, \beta]$ if every sub-interval contains a point of P.[38] He observed that if P is everywhere-dense in $[\alpha, \beta]$, then $[\alpha, \beta]$ is included in P' and P is of the second species. Lastly, bringing in the concept of power Cantor reviewed its basics and focused on point-sets of two powers. Point-sets of the power of the natural numbers are now simply the *countable* [*abzählbaren*] point-sets. Any infinite point-set of the first species is countable, and so also are the rational and the algebraic numbers, which are

[38]We write the now-standard $[\alpha, \beta]$ for Cantor's (α, β). Cantor specifies that his "intervals" contain their endpoints.

of the second species. Point-sets of the power of the real numbers include any interval, and any interval from which a countable set is excluded—this recalling the [1878] dimension proof. Cantor concluded the paper by giving his [1874] uncountability result in the new terms, separating the two powers.

[Cantor, 1880], though quite brief, is remarkable for a palpable extensionalism as set out in set-theoretic notation and terminology and indexed by symbols of infinity. Cantor deployed $P \equiv Q$ for extensional equality; $P \equiv O$ for P "does not at all exist [*nicht vorhanden ist*]"; $\{P_1, P_2, P_3 \ldots\}$ for disjoint union; and, for inclusion $P \subseteq Q$, P is a "divisor" of Q or Q is a "multiple" of P. $\mathfrak{M}(P_1, P_2, P_3, \ldots)$ is union, "the least common multiple", and $\mathfrak{D}(P_1, P_2, P_3, \ldots)$ is intersection, "the greatest common divisor".[39] In these terms, Cantor considered point-sets of the second species. He observed, explicitly for the very first time, that the successive $P', P'', P''', \ldots$ satisfy the set-theoretic $P' \supseteq P'' \supseteq P''' \ldots$. Cantor could then move to the overall intersection $\mathfrak{D}(P', P'', P''', \ldots)$, which he denoted by the "symbol [*Zeichen*]" $P^{(\infty)}$. With type distinctions collapsed, Cantor could further pursue the uniformity of construction through set inclusion and intersection. As long as one has infinite point-sets, one can continue with $P^{(\infty+n)}$, the nth derived set of $P^{(\infty)}$, and get to their intersection $P^{(2\infty)}$. The intersection of $P^{(\infty)} \supseteq P^{(2\infty)} \supseteq P^{(3\infty)} \ldots$ would be denoted $P^{(\infty^2)}$. The intersection of $P^{(\infty)} \supseteq P^{(\infty^2)} \supseteq P^{(\infty^3)} \ldots$ would be denoted $P^{(\infty^\infty)}$. One gets to $P^{(\infty^{\infty^\infty})}$, etc., and only the notation is running out. "We see here a dialectical generation of concepts, which leads on and on, free from any arbitrariness, in itself necessary and consequent." In a footnote, he mentioned that "ten years ago" he had come to such concepts as a proper extension of the concept of number. Whatever is the case, unlike Cantor's [1872] *Zahlengrößen* which gained a certain objectivity according to historical antecedence and their relevance to the straight line, these symbols of infinity emerged *sui generis* in the study of point-sets and the iteration of the derived set operation, as necessary instruments for indexing.

The third paper [Cantor, 1882] broadened the context with new domains and issues, and in so doing established two new, significant results that mark the initial ascent into the elucidation of concepts of power and continuity. Recalling his [1878] work with manifolds consisting of n-tuples of real numbers, Cantor set out "limit point", "derived" and "dense" for these, now simply called "n-dimensional domains", or again, "point-sets". With this generality attained, Cantor emphasized the necessity of having "well-defined" "manifolds (totalities, sets)" for deploying the concept of power, "internally determined" on the basis of definition and according to the law of the excluded middle. (With set-theoretic constructs in place, the next stipulation is the *logical definability* of prospective

[39][Ferreirós, 2007, p.204] pointed out that this number-theoretic terminology agrees with Dedekind's in his [1871]. Dedekind there was famously developing his theory of ideals as a generalization of divisibility in number theory, and the terminology, used for ideals, is analogously appropriate. These set-theoretic relations and operations are now commonplace, but one must remember that at the time, working with them, especially with actually infinite totalities, was still quite novel. Both Cantor and Dedekind should be credited with domesticating set-theoretic operations on infinite totalities in the course of their work.

sets.[40]) With this, "the theory of manifolds as conceived embraces arithmetic, function theory and geometry." (Nascent set theory is beginning to be seen as *foundational.*)

Bringing to the fore power as "unifying concept", Cantor established:

> In an n-dimensional, infinite, continuous space A let an infinite number of n-dimensional, continuous sub-domains (a) be [well-]defined, disjoint and contiguous at most on their boundaries; then the manifold (a) of such sub-domains is always countable.

In modern terms, Cantor had affirmed that $\mathbb{R}^n$, the space of n-tuples of real numbers, satisfies the *countable chain condition*: any collection of pairwise-disjoint open sets is countable. This follows directly from $\mathbb{R}^n$ being *separable*—another modern topological notion—that is, having a countable dense subset; as Cantor had noted, the n-tuples of rational numbers are countable and are dense in $\mathbb{R}^n$. However, Cantor's proof was much more roundabout, this indicative of his forging a new path through new, basic topological notions. That the real numbers satisfy the countable chain condition would soon become of crucial import.

Cantor subsequently observed that if A is a continuous domain and M is a countable point-set consisting of points in A, then for any two points N, N' in the domain $\mathfrak{A}$ consisting of the points of A except those in M, there is a continuous, "analytically defined" line connecting N and N' and lying entirely in $\mathfrak{A}$.[41] His argument draws out his assumptions: Let L be a line in A connecting N and N'. Then L can be segmented into finitely many lines $NN_1, N_1N_2, \ldots, N_nN$ with $N_1, N_2, \ldots N_n$ not in M. Each line can now be replaced by a circular arc avoiding M completely. The result is an "analytically defined" line connecting N and N' and avoiding M. With this observation, Cantor speculated about the possibilities of continuous motion in a discontinuous space; about how the hypothesis of the continuity of space may not actually conform to the reality of phenomenological space; and about how a revised mechanics might be investigated for spaces like $\mathfrak{A}$.

With the broader context established by the first three, Cantor in his fourth paper paper [Cantor, 1883b] made headway by incorporating countability into the sequences of derived sets indexed with symbols of infinity. Notably, Cantor introduced further set-theoretic notation to new purpose: $P + Q$ for disjoint union; $P - Q$ for set difference when $Q \subseteq P$; and $P_1 + P_2 + P_3 \ldots$ for infinite disjoint union. With this, he stipulated that a point-set Q is *isolated* if it contains none of its limit points, i.e. $\mathfrak{D}(Q, Q') \equiv O$. For any point-set P, $P - \mathfrak{D}(P, P')$ is isolated. Crucially, countability enters the fray here: *Every isolated point-set is countable.* (Every point in an isolated point-set has a neighborhood disjoint from the point-set, and by the [1882] countable chain condition, there are only

[40] Zermelo, in his axiomatization of set theory, famously included the *Aussonderungsaxiom* for just this purpose.

[41] Cantor initially made this observation in a letter of 7 April 1882 to Dedekind. See [Ewald, 1996, p.871f].

countably many such neighborhoods.) With this, one has: *For any point-set P, if P' is countable, then so is P.* ($P = (P - \mathfrak{D}(P, P')) + \mathfrak{D}(P, P')$, and both point-sets are countable.)

Through a series of extensions Cantor proceeded to establish, with α any of the symbols loosely indicated in [1880]: *If P is a point-set such that $P^{(\alpha)}$ is countable, so is P.* Recalling that $P' \supseteq P'' \supseteq P''' \ldots$,

$$P' \equiv (P' - P'') + (P'' - P''') + \ldots + P^{(\alpha)}$$

is a disjoint union of countable sets. So P' is countable, and hence P is as well.

3.2 Perfect Sets

The fifth paper [1883c] in Cantor's "linear point-manifolds" series was conspicuously longer and magisterial, and Cantor published it separately, with a preface and footnotes, as an essay [1883a], his *Grundlagen.* In it Cantor presented his new conceptualization of number, the transfinite numbers [*Anzahlen*] couched in a carefully wrought philosophy of the infinite. Cantor's *Grundlagen*, together with his mature presentation *Beiträge* [1895, 1897] of his theory of sets and number, would become the definitive publications from which transfinite set theory would emanate. In what follows, we persist with our scheme of going through Cantor's publications, but now in a decidedly skewed fashion. We assume some familiarity with the transfinite (ordinal) numbers, and with continued our focus on continuity, emphasize Cantor's historical progress, setting out the details of his culminating work on limit points and derived sets, the analysis with perfect sets.

In keeping with the expository thrust of the *Grundlagen* Cantor briefly described (end of §3) his work on the iterations $P^{(\alpha)}$ of the derived set operation, indexed by the transfinite numbers, and (§9) his construction of the real numbers, making comparison with other treatments.[42] In the midst of §10, Cantor formulated a concept central to the last paper of the "linear point-manifolds" series: a *perfect* set is a (non-empty) set P such that $P' = P$. With this, Cantor proceeded to a resolution, in the *Grundlagen* spirit of coming to terms with number, of what is, or ought to be, a continuum.

Cantor began his §10 discussion of the concept of the "continuum" by recalling an age-old debate between partisans of Aristotle and of Epicurus, this leading to the regrettable impasse that the continuum is not analyzable. He then opined that the "concept of time" or the "intuition of time" is not the way to proceed, nor is any appeal to the "form of intuition of space". What is left then is to take the continuum to consist of points and to start with the concept of real number, this as arithmetically and mostly felicitously given by his limit construction. Taking as a foundation the "n-dimensional *arithmetical* space", essentially $\mathbb{R}^n$, endowed with the usual distance,

$$\left| \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + \ldots + (x_n' - x_n)^2} \right| ,$$

[42] We discussed Cantor's comparisons already in §1.4.

Cantor specified that a point-set $P \subseteq \mathbb{R}^n$, to be a continuum, ought to be *perfect*, i.e. satisfy $P' = P$. However, perfect sets are not necessarily dense. (To emphasize this, Cantor gave in an endnote the now famous "Cantor ternary set", the totality of all real numbers given by

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \ldots + \frac{c_\nu}{3^\nu} + \ldots$$

where the c_ν can be 0 or 2 and the series can be finite or infinite. This set is perfect yet nowhere dense, i.e. not dense in any interval, and serves today as a paradigmatic example of a set of real numbers with many distinctive properties, e.g. it is the prototype of fractal.) So, Cantor came up with a second condition for a continuum: A point-set T is *connected* "if, for any two of its points t and t' and for any arbitrarily small number ε there always exist [finitely many] points $t_1, t_2, \ldots, t_\nu$, such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \ldots, \overline{t_\nu t'}$ are all less than ε."

Putting these concepts together, Cantor defined a "point-continuum inside [$\mathbb{R}^n$] to be a *perfect-connected set*." "Here 'perfect' and 'connected' are not merely words but completely general predicates of the *continuum*; they have been conceptually characterized in the sharpest way by the foregoing definitions." Thus, as with his construction of the real numbers, Cantor has formulated, through mathematical precisification in topological terms, the concept of continuum.

Cantor concluded §10 forthwith with rhetorical remarks vis-à-vis Bolzano and Dedekind. *Contra* Bolzano, Cantor pointed out that Bolzano in his definition of the continuum in *Paradoxien des Unendlichen* [1851][43] had only managed to express *one* property of the continuum, connectedness. By way of counterexample, Cantor pointed out how with e.g. "sets which result from [$\mathbb{R}^n$] when one imagines an 'isolated' point-set at a distance from [$\mathbb{R}^n$]", there is no continuum, while Bolzano's definition would still be satisfied. This brings out how Cantor was not characterizing some one categorical continuum but rather entertaining a range of possibilities, marshaling them through his point-set theory.

As things would go, the concept of manifold would gain ascendancy following Riemann's articulation of n-dimensional manifolds, and connectedness would be built into modern formulations. In his classic *Topology* [1968, vol.II,chap.5], Kuratowski investigated connected spaces according to a general topological definition and defined a continuum to be a compact, connected Hausdorff space. He then observed (p.167) that a compact metric space is a continuum if and only if it is connected in Cantor's sense, acknowledging [Cantor, 1883c].

As for Dedekind, Cantor opined that in [Dedekind, 1872], "only another property of the continuum has been one-sidedly emphasized, namely, that property which is in common with all '*perfect*' sets." This remark is somewhat opaque, but some light is cast on it by their correspondence. In a letter of 15 September 1882 to Dedekind, Cantor initially raised the question of "what we are to understand by a *continuum*," and wrote: "An attempt to generalize your

[43] This, though the only reference to Bolzano in Cantor's works, shows that he was aware of it. *Paradoxien* is the first text that explicitly espoused the actual infinite.

concept of cut and to use of it for the general definition of the continuum did not succeed."

The sixth and last paper [1884] in Cantor's "linear point-manifolds" series was of comparable length to the previous, the *Grundlagen*, and continued its paragraph numbering with §§$15-19$. As counterpart to the *Grundlagen*, which was expansive in conceptualizations and philosophical underpinnings, Cantor in [1884] set out his mature mathematical work integrating continuity and cardinality, centering on perfect sets.

After advancing some involved technicalities in §15 to serve as lemmas, in the major paragraph §16 Cantor detailed his characterizing results about the transfinite iterations $P^{(\alpha)}$ and perfect sets. In §17, he re-articulated his results in terms of a now basic topological notion, one to which he thus gave rise. A point-set P is *closed* [*abgeschlossen*] if it contains its limit points, i.e. $P' \subseteq P$. Cantor observed that a point-set is closed exactly when it is of the form Q' for some point-set Q. As set out in the *Grundlagen*, a set is *of the first power* if it is countable; a transfinite (ordinal) number is *of the first number class* if it is finite; and it is *of the second number class* if it is infinite and countable. The second number class is uncountable, and in fact its power is the least after that of the first number class. Systematically, Cantor now allowed $P^{(\alpha)} \equiv O$. If $P^{(\alpha)}$ is finite for some α, then $P^{(\alpha+1)} \equiv O$, and if $P^{(\beta)} \equiv O$ for some β, then for $\gamma \geq \beta$, $P^{(\gamma)} \equiv O$.

Theorem 5 ([Cantor, 1884, p.471]): "If P is a closed point-set of the first power, there is always a smallest number of the first or second number classes, say α, so that $P^{(\alpha)} \equiv O$, or what is said, such sets are reducible."

Theorem 6 ([Cantor, 1884, p.471]): "If P is a closed point-set of higher than the first power, then P is divided into a perfect set S and a set of the first power R, so that $P \equiv R + S$."

Theorem 5 is a forward-direction version of the prominent [1883b] result for countable derived sets, afforded by the focal *Grundlagen* result that the second number class is uncountable. For closed P, one has $P \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \ldots$, and so

$$P \equiv (P - P^{(1)}) + (P^{(1)} - P^{(2)}) + \ldots.$$

The isolated sets $P^{(\beta)} - P^{(\beta+1)}$ are each countable, so if P itself were countable, then there must be a countable α such that $P^{(\alpha)} \equiv O$—else there would be the *contradiction* that the second number class is countable.

Theorem 6 is the crucial structure result for *un*countable point-sets. With the §15 lemmas, Cantor established the theorem through an involved argument that was indicative of his forging a new conceptual path. In terms of now standard topological notions, we can render his argument perspicuously. A point-set $Q \subseteq \mathbb{R}^n$ is *open* if $\mathbb{R}^n - Q$ is closed. A collection $\mathcal{B}$ of open sets of $\mathbb{R}^n$ is a *basis* if every open set is a union of members of $\mathcal{B}$. Cantor in effect devised a *countable* basis for $\mathbb{R}^n$ by taking the collection of n-spheres with rational radius and center an n-tuple of rational numbers.

To establish the theorem for an uncountable closed point-set $P \subseteq \mathbb{R}^n$, first fix a countable basis $\mathcal{B}$ for $\mathbb{R}^n$. The successive $P \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \ldots \supseteq P^{(\alpha)} \supseteq \ldots$ are all closed, so for each α let $B_\alpha \subseteq \mathcal{B}$ be such that $\mathbb{R}^n - P^{(\alpha)} = \bigcup B_\alpha$. Then if $P^{(\alpha+1)}$ is a proper subset of $P^{(\alpha)}$, B_α is a proper subset of $B_{\alpha+1}$. Consequently, there must be a countable α such that $P^{(\alpha+1)} \equiv P^{(\alpha)}$—else there would be the *contradiction* that $\mathcal{B}$ is uncountable. Specifying α to be the least such and setting $S \equiv P^{(\alpha)}$, note that

$$P \equiv (P - P^{(1)}) + (P^{(1)} - P^{(2)}) + \ldots + S\,,$$

where each $P^{(\alpha)} - P^{(\alpha+1)}$ is isolated and hence countable. Taking R to be their union, one has $P \equiv R + S$, where R is countable and S is perfect.

Cantor had asserted, in summarizing remarks in the *Grundlagen* (§10), that the set R of Theorem 6 is reducible in the sense of Theorem 5, i.e. there is a countable α such that $R^{(\alpha)} \equiv O$. The Swedish mathematician Ivar Bendixson, in a letter to Cantor of May 1883, pointed out that R is not necessarily reducible. In a careful analysis that Cantor included in [1884, Theorem G], Bendixson showed that there is a countable α such that, instead, $R \cap R^{(\alpha)} \equiv O$. Theorems 5 and 6 are nowadays often called the *Cantor-Bendixson analysis*, and the least α such that $P^{(\alpha)} \equiv O$ in the first and $P^{(\alpha+1)} \equiv P^{(\alpha)}$ in the second the *Cantor-Bendixson rank.* However, this eponymy hardly does justice to Cantor to whom the entire development of the iterations $P^{(\alpha)}$ leading to perfect sets ought to be credited.

In §18, Cantor with his perfect sets in hand took the time to develop a theory of "content [*Inhalt*]"—his word—pursuing a subject with which he had been dialectically engaged in the earlier [1883b]. He essentially showed that, according to his formulation, the content of an arbitrary set is equal to that of a perfect set.

In the last §19, Cantor affirmed the central role of perfect sets, as incipiently seen in his definition of the continuum (e.g. [1884, §10]) and later in his theory of content, and focusing on the subsets of the real numbers, proceeded to establish:

Theorem 7 ([Cantor, 1884, p.485]): Linear perfect sets have the power of the linear continuum $[0, 1]$.

The proof, indicative of how far he had journeyed, marshaled the accumulated store of topological concepts and results to establish the requisite one-to-one correlation. Hence, "closed sets satisfy the Continuum Hypothesis", in that either they are countable or have the power (cf. Theorem 6) of the real numbers. Cantor concluded optimistically,

> In future paragraphs it will be proven that this remarkable theorem has a further validity even for linear point-sets which are not closed, and just as much validity for all n-dimensional point-sets.
>
> From these future paragraphs, ... it will be concluded that *the linear continuum has the power of the second number class (II).*

That is, Cantor would establish the Continuum Hypothesis.

Here at last is revealed what must have been a driving motivation for Cantor's research. Having suggested that there are only two classes of infinite sets of real numbers according to power (cf. end of [1878]), Cantor had persisted with an analysis of sets through the iteration of the derived set operation—this soon proceeding into the new terrain of transfinite numbers. Along the way, he had developed now basic topological notions to handle limits and continuity, this resulting in the perfect "kernel" of uncountable sets. Perfect sets have the power of the continuum, as seen through accumulated experience, and the next stage would have been to extend the closed set result to all sets of real numbers, thereby establishing the Continuum Hypothesis. Cantor would fail to do this, and, as we now know, it could not be done in the setting that he was working in.

Stepping back, we see that Cantor in his remarkable ascent developed, in the mathematical articulation of continuity, the basic *topology* of point-sets. And we also become aware, here at the end, that as a matter of mathematical practice—as with his construction of the real numbers—Cantor thus made lasting conceptual advances concerning continuity in the course of establishing necessary ground for the resolution of a *problem*. These aspects of Cantor's work bring into sharper relief what we have focused on as Cantor's steady engagement with continuity as he ascended into set theory.

References

[Bolzano, 1817] Bernard Bolzano. *Rein analytischer Beweis der Lehrsatzes, daß zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege.* Gottliebe Hasse, Prague, 1817. A translation by Steve Russ in [Ewald, 1996], pages 225-248 and in [Russ, 2004], pages 253-277.

[Bolzano, 1851] Bernard Bolzano. *Paradoxien des Unendlichen herausgegeben aus dem schriftlichen Nachlasse des Verfassers von Dr. Fr. Příhonský.* Reclam, Leipzig, 1851. Translated in [Russ, 2004], pages 595-678.

[Bolzano, 1930] Bernard Bolzano. *Functionenlehre. Herausgegenben und mit Anmerkungen versehen von Dr. Karel Rychlík.* Königlich böhmische Gesellschaft der Wissenschaften, Prague, 1930.

[Brouwer, 1911] L. E. J. Brouwer. Beweis der Invarianz der Dimensionenzahl. *Mathematische Annalen*, 70:161–165, 1911.

[Cantor, 1870] Georg Cantor. Beweis, dass eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen lässt. *Journal für die reine und angewandte Mathematik*, 72:139–142, 1870. Reprinted in [Cantor, 1932], pages 80-83.

[Cantor, 1872] Georg Cantor. Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. *Mathematische Annalen*, 5:123–132, 1872. Reprinted in [Cantor, 1932], pages 92-102.

[Cantor, 1874] Georg Cantor. Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. *Journal für die reine und angewandte Mathematik*, 77:258–262, 1874. Reprinted in [Cantor, 1932], pages 115-118.

[Cantor, 1878] Georg Cantor. Ein beitrag zur mannigfaltigkeitslehre. *Journal für die reine und angewandte Mathematik*, 84:242–258, 1878. Reprinted in [Cantor, 1932], pages 119-133. A translation in [Ewald, 1996], pages 839-843.

[Cantor, 1879a] Georg Cantor. Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, pages 127–135, 1879. Reprinted in [Cantor, 1932], pages 134-138.

[Cantor, 1879b] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 1. *Mathematische Annalen*, 15:139–145, 1879. Reprinted in [Cantor, 1932], pages 139-145.

[Cantor, 1880] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 2. *Mathematische Annalen*, 17:355–358, 1880. Reprinted in [Cantor, 1932], pages 145-148.

[Cantor, 1882] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 3. *Mathematische Annalen*, 19:113–121, 1882. Reprinted in [Cantor, 1932], pages 149-157.

[Cantor, 1883a] Georg Cantor. *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*. B.G. Teubner, Leipzig, 1883. Separate printing of [Cantor, 1883c]. A translation in [Ewald, 1996], pages 878-920.

[Cantor, 1883b] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 4. *Mathematische Annalen*, 21:51–58, 1883. Reprinted in [Cantor, 1932], pages 157-164.

[Cantor, 1883c] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 5. *Mathematische Annalen*, 21:545–586, 1883. Reprinted in [Cantor, 1932], pages 165-208. Separately printed as [Cantor, 1883a].

[Cantor, 1884] Georg Cantor. Ueber unendliche, lineare Punktmannigfaltigkeiten, 6. *Mathematische Annalen*, 23:453–488, 1884. Reprinted in [Cantor, 1932], pages 210-244.

[Cantor, 1891] Georg Cantor. Über eine elementare Frage der Mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1:75–78, 1891. Reprinted in [Cantor, 1932], pages 278-280. A translation in [Ewald, 1996], pages 920-922.

[Cantor, 1895] Georg Cantor. Beträge zur Begründung der transfiniten Mengenlehre, I. *Mathematische Annalen*, 46:481–512, 1895. Reprinted in [Cantor, 1932], pages 282-311.

[Cantor, 1897] Georg Cantor. Beträge zur Begründung der transfiniten Mengenlehre, II. *Mathematische Annalen*, 49:207–246, 1897. Reprinted in [Cantor, 1932], pages 312-351.

[Cantor, 1932] Georg Cantor. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*. Springer, Berlin, 1932. Ernst Zermelo, editor. Reprinted Hildesheim, Olms, 1966.

[Cauchy, 1821] Augustin-Louis Cauchy. *Cours d'analyse de l'École royale polytechnique*. Imprimérie royale, Paris, 1821.

[Dauben, 1979] Joseph W. Dauben. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Harvard University Press, Cambridge, 1979.

[Dedekind, 1857] Richard Dedekind. Abriss einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus. *Journal für die reine und angewandte Mathematik*, 54:1–26, 1857.

[Dedekind, 1871] Richard Dedekind. Über die Komposition der binären quadratischen Formen. Supplement X of P.G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, second edition. Vieweg, Braunschweig, 1871.

[Dedekind, 1872] Richard Dedekind. *Stetigkeit und irrationale Zahlen*. F. Vieweg, Braunschweig, 1872. Translation in [Ewald, 1996], pages 765-779.

[Dirichlet, 1854 1904] Peter G. Lejeune Dirichlet. *G. Lejeune-Dirichlets Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen*. Vieweg, Braunschweig, 1854 / 1904. 1854 Lectures at the University of Berlin. Edited by Gustav Arendt.

[Dugac, 1973] Pierre Dugac. Eléments d'analyse de Karl Weierstrass. *Archive for History of Exact Sciences*, 10:41–176, 1973.

[Ewald, 1996] William Ewald. *From Kant to Hilbert*. Clarendon Press, Oxford, 1996.

[Ferreirós, 2007] José Ferreirós. *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics*. Birkhäuser, Basel, 2007. Second, revised edition.

[Greenberg, 2008] Marvin J. Greenberg. *Euclidean and non-Euclidean Geometries*. Freeman, New York, 2008. Fourth edition.

[Heath, 1956] Thomas L. Heath. *The Thirteen Books of the Elements, volume I*. Dover Publications, New York, 1956.

[Heine, 1870] Eduard Heine. Über trigonometrische Reihen. *Journal für die reine und angewandte Mathematik*, 71:353–365, 1870.

[Heine, 1872] Eduard Heine. Die Elemente der Functionenlehre. *Journal für die reine und angewandte Mathematik*, 74:172–188, 1872.

[Kuratowski, 1968] Kazimierz Kuratowski. *Topology*. PWN—Polish Scientific Publishers, Warszawa, 1968. Second edition, English translation of the first, 1958 edition.

[Moore, 1982] Gregory H. Moore. *Zermelo's Axiom of Choice. Its Origins, Development, and Influence*. Springer-Verlag, New York, 1982.

[Peano, 1890] Giuseppe Peano. Sur une courbe, qui remplit toute une aire plane. *Mathematische Annalen*, 36:157–160, 1890.

[Purkert and Ilgauds, 1987] Walter Purkert and Hans J. Ilgauds. *Georg Cantor, 1845-1918*. Birkhäuser, Basel, 1987.

[Riemann, 1868a] Bernhard Riemann. Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13:87–131, 1868.

[Riemann, 1868b] Bernhard Riemann. Über die Hypothesen, welche der Geometrie zu Grunde liegen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13:133–152, 1868. A translation in [Ewald, 1996], pages 652-661.

[Rusnock and Kerr-Lawson, 2005] Paul Rusnock and Angus Kerr-Lawson. Bolzano and uniform continuity. *Historia Mathematica*, 32:303–311, 2005.

[Russ, 2004] Steve Russ. *The Mathematical Works of Bernard Bolzano*. Oxford University Press, Oxford, 2004.