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HILBERT AND SET THEORY*

David Hilbert (1862–1943) was the preeminent mathematician of the early decades of the 20th Century,[1] a mathematician whose pivotal and penetrating results, emphasis on central problems and conjectures, and advocacy of programmatic approaches greatly expanded mathematics with new procedures, initiatives, and contexts. With the emerging extensional construal of mathematical objects and the development of abstract structures, set-theoretic formulations and operations became more and more embedded into the basic framework of mathematics. And Hilbert specifically championed Cantorian set theory, declaring (1926, 170): "From the paradise that Cantor has created for us no one will cast us out".

On the other hand, Hilbert did not make direct mathematical contributions toward the development of set theory. Although he liberally used non-constructive arguments, his were still the concerns of mainstream mathematics, and he stressed concrete approaches and the eventual solvability of every mathematical problem. After its beginnings as the study of the transfinite numbers and definable collections of reals, set theory was becoming an open-ended, axiomatic investigation of *arbitrary* collections and functions. For Hilbert this was never to be a major concern, but he nonetheless exerted a strong influence on this development both through his broader mathematical approaches and through his specific attempt to establish the Continuum Hypothesis.

What follows is a historical and episodic account of Hilbert's results and initiatives and their ramifications and extensions, in so far as they bear on set theory and its development.[2] The emphasis on set theory presents a tangential view of Hilbert's main mathematical endeavors, but one that illuminates their larger themes and motivations. Because of its basic interplay with set theory, we deal at length with Hilbert's program for establishing the consistency of mathematics by "finitary reasoning". Section 1 discusses Hilbert's use of non-constructive existence proofs, with the focus on his first major result; Section 2 discusses his axiomatization of Euclidean geometry, with the focus on his Completeness Axiom; and then Section 3 discusses questions about the real numbers and their arithmetic that Hilbert would later approach through his proof theory. With this as a backdrop, Section 4 considers Hilbert's involvement in the early development of set theory, and Section 5 considers both his mathematical logic as a reaction to Russell's and the two crucial new questions that Hilbert raised. Section 6 describes Hilbert's approach to establishing the consistency of mathematics, and Section 7 its application to the Continuum Hypothesis. Then Section 8 discusses Gödel's work, particularly on the consistency of the Continuum Hypothesis, in relation to Hilbert's. In the appendix, Hilbert's consistency program is reconsidered in light of recent developments in "reverse" mathematics.

1. BASIS THEOREM

Hilbert's work from the beginning greatly accelerated the move away from the traditional constructive moorings, being driven by strong impulses: the solution of focal problems by intuitively clear though not necessarily constructive means, and the drive for systematization with its emerging concern with consistency. When Hilbert was in his late twenties, he (1890) established his first major result, *Hilbert's basis theorem*, which cast in current terms is the assertion:

> Suppose that F is a field and $F[x_1, \ldots, x_n]$ the ring of polynomials over F in $x_1, \ldots, x_n$. Then every ideal in $F[x_1, \ldots, x_n]$ is finitely generated.

Invariant theory, the subject of Hilbert's doctoral dissertation and Habilitationsschrift, was the bridge between geometry and algebra in 19th Century mathematics, and the basis theorem was the key ingredient in his solution (1890) of invariant theory's then central problem.[3] Moreover, Hilbert's (1890) with its new structural approach can be considered the first paper of modern algebra: In straightforward generalizations in terms of algebraic varieties the basis theorem serves as a foundation for algebraic geometry.

The basis theorem caused a sensation since it argued for a finite number of generators, yet provided no explicit construction. Moreover, in the form that it was actually established by Hilbert, that for any appropriate sequence of polynomials every polynomial in the sequence is a linear combination of the first few, it was a widely applicable result. Paul Gordan (1868) had solved the central invariant theory problem for the special case of two variables by an ingenious but tedious construction which was a culmination of what came to be called the "symbolic method". After seeing Hilbert's basis theorem Gordan quipped (Max Noether (1914, 18), Felix Klein (1926, 330)): "This is not mathematics; this is theology!" Hilbert had carried out a streamlining double induction (or rather, finite descent), first putting the case of n variables into a simple form, and then effecting a reduction to $n - 1$. He had established a startling result by a convincing argument, one that was soon accepted by the mathematical community. Not only was the proof reasonably surveyable, but it made a large array of algebraic constructions manageable and introduced simplicity where there had been none.

Nonetheless, Hilbert (1893) soon provided an even more informative proof of his invariant theory result. It was for this purpose that he established his well-known Nullstellensatz,[4] which like the basis theorem had a non-constructive proof and has become fundamental in modern algebra. Applying the non-constructive Nullstellensatz Hilbert provided an otherwise constructive algorithm for computing complete systems of invariants, building on a technique due to Arthur Cayley. This was a striking instance of what was becoming a major trend in mathematics: the development of contextually appropriate proofs for results established by apparently less informative means, leading to a further enrichment of mathematics. In particular, Gordan (Klein 1926, 331) conceded that "even theology has its merits", and soon provided his own proofs (Gordan 1893, 1899) of Hilbert's basis theorem.[5]

Both Hilbert's basis theorem as well as his Nullstellensatz would be precisely analyzed in terms of formal systems. In particular, the double induction in Hilbert's proof of the basis theorem would turn out to be a remarkable foreshadowing of how a close variant of the theorem would be shown equivalent to a proposition (the provable totality of Ackermann's function) that just transcends one common characterization of Hilbert's later finitistic viewpoint. (See in the appendix Theorem 3 and remarks following.)

In addition to non-constructive existence proofs Hilbert championed the use of "ideal elements". Well-established were the imaginary i and the points at infinity for projective geometry, and emerging into prominence were the ideals of algebraic number fields, to the theory of which Hilbert made fundamental contributions. The imaginary i had stimulated the inaugural use of non-constructive existence proofs in algebra: The fundamental theorem of algebra, that every polynomial in complex coefficients has a root, was first established by Gauss in his doctoral dissertation (1799) by a proof that provided no means of algebraically calculating a root. Weierstrass and Dedekind carried out involved constructive extensions of Gauss's work in the 1880's; but significantly, Hilbert (1896) considerably streamlined this work by applying his Nullstellensatz, later claiming (1928) (see van Heijenoort 1967, 474) that its (non-constructive) proof "uncovers the inner reason for the validity of the assertions adumbrated by Gauss and formulated by Weierstrass and Dedekind".

A remarkable example of the use of non-constructive existence proofs is Hilbert's ingenious solution to Waring's Problem. Broached by Edward Waring in 1770, it asks of natural numbers whether for every positive k there is a fixed r such that for every n,

$$n = n_1^k + \cdots + n_r^k \text{ for some } n_1, \ldots, n_r.$$

In that same year Lagrange had established the result for $k = 2$ with $r = 4$, but for no other $k > 2$ was the result known until Hilbert (1909) completely solved the problem by establishing the existence for every k of a corresponding r. However, taking $g(k)$ to be the least possible such r, Hilbert's proof provided no way of calculating $g(k)$. Hilbert's result spurred extensive activity in analytic number theory, in part to determine the values $g(k)$, and they are "almost" completely known today.[6]

However non-constructive Hilbert's approach, he himself never seemed to have entertained sets of arbitrary choices as formalized by the Axiom of Choice, an axiom first made explicit by Ernst Zermelo (1904). The expansion of mathematics to this level of abstraction was initiated by Felix Hausdorff in his classic *Grundzüge der Mengenlehre* (1914) which broke the ground for a generation of mathematicians in both set theory and topology. Of particular interest was Hausdorff's use of the Axiom of Choice (in 1914, 469ff) and also in (1914a)) to get what is now known as Hausdorff's Paradox, an implausible decomposition of the sphere; this was a dramatic synthesis of classical mathematics and the new set-theoretic view.

Of those directly influenced by Hilbert, Georg Hamel, whose doctoral work was supervised by Hilbert, made (1905) an early and explicit use of the Axiom of Choice to provide what is now known as a Hamel basis, a basis for the real numbers as a vector space over the rational numbers. The full exercise of the Axiom of Choice in ongoing mathematics first occurred in the pioneering work of Ernst Steinitz (1910), who made systematic use of well-orderings to establish the abstract theory of fields, their algebraic and transcendental extensions, and algebraic closures. Zermelo (1914) modified Hamel's basis to get one for the complex numbers and with a further use of the Axiom of Choice answered a question about the existence of a collection of complex numbers with special closure and basis conditions. Presaging her later work Emmy Noether (1916) axiomatically characterized those integral domains satisfying Zermelo's conditions.

Noether's mathematical roots were in invariant theory and in (1915) had brought together Hilbert's basis theorem arguments with those of Steinitz's field theory. Going to Hilbert's Göttingen, Noether became the leading figure in algebra there through her work on the theory of ideals in commutative rings. In her incisive (1921) she lifted the finiteness properties emanating from Hilbert's basis theorem to a general axiomatic setting by introducing the *ascending chain condition*, and rings satisfying this condition are now known as *Noetherian rings*. Similarly abstracting another finiteness property, Noether (1927) extended Dedekind's unique factorization theory for ideals of rings of algebraic numbers to the general setting. She (1927, 45ff) applied the Axiom of Choice without much ado, but only a weak version, the so-called Axiom of Dependent Choices, is needed for the general formulations of her basic results. The full exercise of the Axiom of Choice entered Noether's axiomatic ring theory when Wolfgang Krull (1929) investigated rings not necessarily satisfying the ascending chain condition, specifically in the general assertion that every ideal in a ring can be extended to a maximal ideal. Ring theory today is often presented at this level of generality, but Hilbert's basis theorem remains a palliative in the crucial cases for algebraic geometry, where the theorem's applicability renders any appeal to the Axiom of Choice unnecessary.

In terms of his later consistency program Hilbert's advocacy of non-constructive existence proofs and the use of ideal elements necessarily raised the stakes involved. Not only did the issue of consistency become more critical when explicit constructions were not available or ideal elements seamlessly introduced, but the weight was shifted from algebraic calculations to logical deductions, which, however, increasingly took on the spirit of calculations not unlike those in the "symbolic method" used by Gordan. The existential quantifier assumed a pivotal role, both in its interplay with the Law of Excluded Middle and the extent to which it could be construed as instrumental in the generation of terms through instantiation. Such issues became central for Hilbert in his mathematical investigation of formalized proofs (see Section 6), and his early work, which assumed an increasingly abstract and logical form from invariant theory to algebraic number theory, undoubtedly predisposed him to this later development.

2. GEOMETRY

Hilbert's new conception of the role of axiomatization as not reflecting an antecedently given subject matter and his resulting concern for consistency first took substantial shape in his *Grundlagen der Geometrie* (1899), based on lectures given in the 1890's and especially on those in the winter of 1898–9. In the introduction to the *Grundlagen* Hilbert wrote of his investigation as "a new attempt to establish for geometry a *simple* and *complete* [*vollständiges*] system of axioms *independent* of one another". What *vollständiges* was to mean would become a central concern of mathematical logic in later decades. He proceeded to provide a rigorous axiomatization of Euclidean geometry with five groups I–V of axioms, for *incidence, order, congruence, parallelism*, and *continuity* respectively. Previous and venerable work had already established the consistency of non-Euclidean geometries via models in Euclidean geometry. Hilbert in a groundbreaking move raised the question of the consistency of Euclidean geometry itself as given by his axioms, and proceeded to establish it via a *countable* arithmetical model. Then, as with the work on the Parallel Axiom, Hilbert went on to use various models of subcollections of his axioms to establish the independence of axioms and theorems.

Hilbert's model for the consistency of his full list of axioms took as its "points" the countable collection of ordered pairs of real numbers generated from 1 by the arithmetical operations and the taking of square roots of positive numbers. While fitting into the development of algebraic number fields, this model is notable as arguably the first instance of the Löwenheim-Skolem phenomenon, a "Skolem's Paradox" for the continuum. Hilbert had accentuated the reliance on arithmetic by reducing geometry to a countable domain of ordered pairs of algebraic real numbers: Skolem's (1923) argument for generating a countable model using Skolem terms would give a countable model for any (countable first-order) theory. To distinguish a countable substructure of the continuum as Hilbert had done was the most informative type of "application" that the Löwenheim-Skolem Theorem could have had before Skolem's own application in (1923) to get his "paradox" in set theory. However, despite his professed indifference to whether his axioms were about points or tables.[7] Hilbert did not dwell on this model and soon moved to secure Euclidean space.

Hilbert's axiom group V for continuity initially consisted of a single axiom, the Archimedean Axiom[8], but he soon added another, the Completeness [Vollständigkeit] Axiom V.2:

> It is impossible to adjoin further elements to the system of points, lines, and planes in such a way that the system thus extended forms a new geometry satisfying all the axioms in groups I–V; in other words, the elements of the geometry form a system which is not susceptible to extension, if all of the stated axioms are to be maintained.

An arithmetical version of this axiom first appeared in *Über den Zahlbegriff* (1900b), of which more in Section 3. The axiom itself is mentioned first in the French translation (1900a, 25) of the *Grundlagen* and then in the English translation (1902, 25), prior to its incorporation into the second edition (1903, 16). In the original *Grundlagen* (1899, 39) (see also 1971, 58ff.) Hilbert had shown that every "geometry" satisfying I–IV and the

Archimedean Axiom is faithfully embeddable into the "ordinary analytic geometry", i.e. Euclidean space.[9] The Completeness Axiom amounted to making this maximal geometry the unique geometry.

A set of axioms is *categorical* if it has a unique model up to isomorphism. Having investigated his axioms for geometry with models, Hilbert with his Completeness Axiom simply posited categoricity with the maximal geometry. Hilbert's professed aim in the introduction to the *Grundlagen* had been to get "a *simple* and *complete* system of axioms", yet today his axiom would be considered neither simple nor immediately related to notions of completeness later studied by Hilbert. With the Completeness Axiom Hilbert had come to an axiom about models of axioms and thereby raised the sort of issues that would become amenable to mathematical investigation only decades later. (See Section 8, especially footnote 51.)

The Completeness Axiom had specific antecedents in the tradition leading to the development of set theory. In the well-known formulations of the real numbers by Georg Cantor (1872) as fundamental sequences and by Richard Dedekind (1872) as cuts, the correlation with "the straight line" was not regarded as automatic. Cantor (1872, 128) wrote:

> In order to complete the connection ... with the geometry of the straight line, one must only add an *axiom* which simply says that conversely every numerical quantity also has a determined point on the straight line, whose coordinate is equal to that quantity .... I call this proposition an *axiom* because by its nature it cannot be universally proved. A certain objectivity is then subsequently gained thereby for the quantities although they are quite independent of this.

Dedekind (1872, III) wrote:

> If all points on the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point that produces this division .... The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line.

At such an interface one finds what one seeks: Henri Poincaré (1902, 40) commended Dedekind's cuts as reflecting the "intuitive truth that if a straight line is cut into two rays their common border is a point". On the other hand, Bertrand Russell (1919, 71) decried Dedekind's approach of postulating what one wants as having the same advantages as "theft over honest toil". Russell's genetic approach of building up from the natural numbers to the rational numbers and then *defining* the real numbers as the cuts is congenial to his logicist reductionism,[10] but obscures the antecedent sense of the continuum that both Cantor and Dedekind were trying to accommodate. They both had recognized the need for a sort of Church's Thesis, a thesis of adequacy for their new construals of the continuum.

Dedekind (1872, II) wrote of the "connection [Zusammenhang]" between the rational numbers and points on the straight line when an origin and a unit of length have been selected. This "connection" is accomplished in Hilbert's axiomatization through the Archimedean Axiom. Hilbert's Completeness Axiom then ensures through maximality that Dedekind's cuts actually correspond to points. Conversely, Dedekind's postulation of points corresponding to cuts entails the Completeness Axiom by an argument given in Section 9 of later editions of the *Grundlagen*: If to the contrary a new point could be added, it would induce a Dedekind cut of old points which would then have an old dividing point; but then, a simple argument using the Archimedean Axiom implies that there would be another old dividing point, which is a contradiction.

Although the Completeness Axiom would stir interest as an axiom about (models of) axioms, it could thus have been replaced by a continuity axiom along the lines of its antecedents. In remarks accompanying the first appearance of the Completeness Axiom, Hilbert (1902, 25ff) opined that "the value of [the Completeness Axiom] is that it leads indirectly to the introduction of limiting points". Today the view would be opposite: securing limit points directly through some axiom like Cantor's or Dedekind's would be considered more simple than to introduce an axiom about axioms. Not only does formalizing continuity axioms require second-order quantification over the real numbers, the Completeness Axiom has the added complication of having to formalize the second-order satisfaction relation. But with the central role that he accorded axiomatization, Hilbert thought that he had readily positioned continuity into the heart of his axioms with his Completeness Axiom, and upon its incorporation into the second edition of the *Grundlagen* he (1903, 17) wrote that it "*forms the cornerstone of the entire system of axioms*". Nevertheless, years later in a popular book on geometry, Hilbert and Stephan Cohn-Vossen (1932, 34) noted that the ways in which the axioms of continuity are formulated varies a great deal, and the Completeness Axiom is simply replaced by "Cantor's axiom", that every infinite sequence of nested segments has a common point.

3. ARITHMETIC

Before the Completeness Axiom appeared in any version of his *Grundlagen*, Hilbert in his *Über den Zahlbegriff* (1900b), dated 12 October 1899, provided an axiomatization of the real numbers as an ordered field satisfying arithmetical versions of the Archimedean Axiom and the Completeness Axiom.[11] Just as for geometry, Hilbert had in effect posited categoricity through maximality, for it must have been immediately seen that any system

satisfying the (1900b) axioms, except possibly that arithmetical version of Completeness, is faithfully embeddable into the real numbers.[12]

Although Hilbert (1900b) acknowledged the pedagogical value of the "genetic method" by which one builds up from the natural numbers through the rational numbers to the real numbers, he contended that only an axiomatic presentation of the real numbers all at once can be logically secure. Just as for geometry, Hilbert in (1900b) reduced arithmetic to the workings of a few axioms. Today "arithmetic" most often refers to number theory, i.e. the structure of addition and multiplication for the natural numbers, but for Hilbert "arithmetic" would remain what he would also refer to as analysis, i.e. the structure of addition, multiplication, and continuity for the real numbers. He initially expressed confidence that he could easily establish the consistency of his axioms.[13] However, this was to become a major and lifelong concern for him, and he was soon to promulgate it as the second of his famous problems.

Hilbert's main program for mathematics was launched by his famous declaration (1900) of 23 central problems for the 20th Century at the 1900 International Congress of Mathematicians at Paris.[14] Not only did he advance the basic picture of mathematical practice as driven by the force of problems and conjectures, but he inspired progress with his firm belief that every problem can ultimately be solved, that "in mathematics there is no *ignorabimus*". Fermat's last theorem, although unsolved, had already stimulated great developments in mathematics, and now the gauntlet was thrown to the coming generations, one that would gradually result in the development of new fields of mathematics.

Hilbert made the first of his (1900) problems the problem of establishing Cantor's Continuum Hypothesis, and the second, the problem of establishing "the consistency of the arithmetical axioms", referring to the axioms of his *Über den Zahlbegriff* (1900b). Both of these problems dealt with basic questions about numerical construals of the traditional continuum: the first about the possibility of enumerating the real numbers using the countable ordinal numbers, and the second about the consistency of an arithmetical axiomatization. It is quite remarkable that over two decades later Hilbert himself would use a specific strategy in his proof theory to attack *both* problems (see Sections 6 and 7).

In his (1900) discussion of his second problem, Hilbert remarked that the consistency of the geometrical axioms had been reduced to that of the arithmetical axioms, but that "a direct method is needed for the proof of the consistency of the arithmetical axioms". In the *Grundlagen* his axiomatically presented geometry can be shown consistent by taking as the "points" ordered pairs of real numbers and relying on their arithmetic. However, no such model-theoretic recourse is available for arithmetic itself, and what is left is a direct investigation of its axioms and their consequences. Hilbert argued (as translated in 1902a, 446):

> The totality of real numbers, i.e. the continuum ... is not the totality of all possible series of decimal fractions, or of all possible laws according to which elements of a fundamental sequence may proceed. It is rather a system of things whose mutual relations are governed by the axioms set up and for which all propositions, and only those, are true which can be derived from the axioms by a finite number of logical processes.

This view of the continuum as axiomatically given would later be reflected in Hilbert's own attempt to solve his first problem, the Continuum Hypothesis, through the use of definable functions, and the emphasis on deductive consequences of axioms would later animate his metamathematics. With an arithmetical axiomatization of the continuum whose consequences are exactly the true propositions of arithmetic consistency may be established through the finiteness of proofs without any reference to an antecedent geometric continuum, increasingly the bugbear of 19th Century mathematics.

Upon incorporating his Completeness Axiom into his *Grundlagen* Hilbert himself (1903,17) observed that it presupposes the Archimedean Axiom, in the sense that "it can be shown" that there are geometries satisfying I–IV, and not that axiom, that can be properly extended.[15] In the tradition of Hilbert (1900b), Hans Hahn (1907) introduced into the theory of ordered fields a completeness condition analogous to the Completeness Axiom, which however did not presuppose the Archimedean condition, and provided an incisive analysis of the resulting structures.[16] The connection to be made here is with Kurt Gödel who as a student and friend of Hahn's much admired his work.[17] It would only be through Gödel's epochal results, themselves responses to questions later raised by Hilbert, that the concepts of categoricity and completeness would become clarified (see Section 8).

4. SET THEORY

Although Hilbert did not himself pursue axiomatic set theory, he fostered its development through his encouragement of Ernst Zermelo.[18] Zermelo began his investigations of Cantorian set theory at Göttingen under Hilbert's influence. Zermelo soon found Russell's Paradox independently of Russell and communicated it to Hilbert. Zermelo then established the Well-Ordering Theorem in a letter to Hilbert, the relevant part of which soon appeared as Zermelo (1904). This seminal paper introduced

the Axiom of Choice and stirred considerable controversy. In the tradition of Hilbert's axiomatization of geometry, Zermelo (1908) subsequently provided the first substantial axiomatization of set theory, partly to establish set theory as a discipline free of paradoxes, and particularly to put his Well-Ordering Theorem on a firm footing. Zermelo's axiomatization shifted the emphasis from Cantor's transfinite numbers to an abstract view of sets as structured solely by $\in$ and simple operations. In addition to generative axioms corresponding to these operations and the Axiom of Choice, Zermelo with his Separation [Aussonderung] Axiom incorporated a means of generating sets corresponding to properties that seemed to avoid paradoxes. The Separation Axiom asserted that given a set M, for each *definite* property [definite Eigenschaft] a set can be formed of those elements of M having that property. The vagueness of definite property would invite Skolem's (1923) proposal to base it on first-order logic, and this would tie in with Hilbert's later development of mathematical logic (see Section 5).

For Hilbert himself much of what today would be regarded as the subject matter of set theory would remain largely embedded in mainstream mathematics or be intermixed with the emerging mathematical logic. In what was to be his only publication on logic when he was still in his mathematical prime, Hilbert (1905) addressed the recent paradoxes of logic and set theory with remarks that prefigured his later work in metamathematics and his finitistic viewpoint. Hilbert (1905) advocated an axiomatic approach, observing that (as translated in van Heijenoort 1967, 131)

> in the traditional exposition of the laws of logic certain fundamental arithmetical notions are already used, for example, the notion of set, and to some extent, also that of number. Thus we find ourselves turning in a circle, and that is why a partly simultaneous development of the laws of logic and of arithmetic is required if paradoxes are to be avoided.

Significantly, "the notion of set" for Hilbert here is an "arithmetical notion", and this is connected with his second (1900) problem, to establish the consistency of the "arithmetical axioms". As mentioned earlier, these axioms were to be those of *Über den Zahlbegriff* (1900b) including its version of the Completeness Axiom.

Hilbert (1905) provided only a tentative sketch of how he would carry out such a simultaneous development", but intriguingly it has some anticipation of Zermelo's (1908) generative view of sets. Schematizing a process proceeding by stages, Hilbert (1905) stated five principles, the first three of which are (see van Heijenoort (1967, 135ff.): (I) "a further proposition is true as soon as we recognize that no contradiction results if it is added as an axiom to the propositions previously found true"; (II) at any stage the "all" in the axioms is to range over only those "thought-objects" then taken to be primitive; and (III) a set is a "thought-object" and "the notion of element of a set appears only as a subsequent product of the notion of set itself".

Hilbert would become associated with the "consistency implies truth and existence" view behind principle I. First set out by him in correspondence with Frege about the axiomatization of geometry, the view is similar to that of Cantor but opposite to Frege's "truth implies consistency" view.[19] Principle II foreshadowed Hilbert's later advocacy of Russell's theory of types. As for the somewhat cryptic principle III Hilbert went on to develop its sense by deducing what amounts to a version of Zermelo's (1908) Separation Axiom: From the thought-objects taken to be primitive at a given stage, propositions determine subcollections that are then further thought-objects.

Despite Zermelo's association with Hilbert, it is notable that Hilbert's later lectures (1917) on set theory were imbued with the Cantorian initiatives on number and relatively unaffected by the Zermelian emphasis on abstract set-theoretic operations and axiomatization. Hilbert first discussed the real numbers, giving a detailed account of the transcendental numbers and his (1900b) axiomatization for an ordered field. He then developed Cantor's cardinal numbers, and after discussing well-orderings, Cantor's ordinal numbers. Without much ado Zermelo's Axiom of Choice is stated and his Well-Ordering Theorem proved. The approach is reminiscent of Hausdorff's *Grundzüge der Mengenlehre* (1914), with set theory presented as a new initiative *within* mathematical practice, one providing a new number context and new approaches to mathematical problems. Hilbert's lectures concluded with a discussion of the paradoxes, both set-theoretic and so-called semantic, and the Dedekind-Peano axioms for the natural numbers.

Given his own axiomatization of geometry and with Zermelo in his circle, one might have thought that Hilbert would have jumped at the issue of specific axiomatizations of set theory. Zermelo's axiomatization had for example been the setting for the incisive work of Friedrich Hartogs (1915) on Cardinal Comparability, cited by Hilbert in the (1917) lectures. However, not Hilbert but Abraham Fraenkel (1922) would investigate the independence of Zermelo's axioms, particularly the Axiom of Choice, in the style of Hilbert's *Grundlagen* with the liberal use of various models. Hilbert (1918, 411) did point out how the paradoxes were avoided by Zermelo's axiomatization. But significantly Hilbert (1918, 412) continued:

> the question of the consistency of the axiom system for the *real numbers* is reduced, through the use of set-theoretic concepts, to the same question for the natural numbers: This is the merit of the theories of irrational numbers of Weierstrass and Dedekind.
>
> Only in two cases, namely when it is a question of the axioms for the *natural numbers* themselves, and when it is a question of the foundations of *set theory*, is the method of reduction to another specific field of knowledge obviously unavailable, since beyond logic there is no further discipline to which an appeal is possible.
>
> Since however the proof of consistency is a task that cannot be dismissed, it seems necessary to axiomatize logic itself and then to demonstrate that number theory as well as set theory are only parts of logic.

This attitude would presumably have precluded any model-theoretic analysis of axioms for set theory, or indeed any detailed investigation of axiomatizations of set theory separate from axiomatizations of logic. The passage is consistent with the previously displayed passage from (1905). However, it does suggest a softening of both the *Über den Zahlbegriff* (1900b) attitude that a direct axiomatic presentation of the real numbers is more logically secure than the genetic method of set-theoretic building up from the natural numbers, and the attitude from his discussion of his second (1900) problem that a "direct method is needed" to establish the consistency of the axioms for the real numbers, in that Hilbert now acknowledges a reduction to number theory and set theory.[20]

Subsequently, Hilbert (1929, 136) did come to appreciate the importance of firmly establishing the underlying assumptions of Zermelo's axioms. But as with Gödel later, Hilbert would be more influenced by Russell than by Zermelo, and whatever the affinity of Hilbert's (1905) picture to Zermelo's (1908), Hilbert's investigation of purely set-theoretic notions would largely remain part of his investigations of the underlying logic. The Axiom of Choice would be positioned in logic (see Section 6), and the Continuum Hypothesis would be approached through a hierarchy of definable functions (see Section 7).

5. LOGIC

Hilbert only began to carry out a systematic investigation of mathematical logic over a decade after his precursory (1905) and after the appearance of the three tomes of Whitehead and Russell's *Principia Mathematica*.[21] This work was, in the words of Gödel (1944, 126), the "first comprehensive and thorough going presentation of a mathematical logic and derivation of Mathematics". Much of the further development of logic would turn on reactions to and simplifications for this system, but its two basic interlocking hierarchical features, *types* and *orders*, would be crucial to the development of set theory.

To the modern eye there are two main sources for the great complexity – and even greater obscurity – of the *Principia*. First, it is, as Gödel (1944, 126) went on to write, "greatly lacking in formal precision in [its] foundations .... What is missing, above all, is a precise statement of the syntax of the formalism". This lack of formal precision is exacerbated by Russell's elucidatory accounts of his key logical notions, especially of propositional function", which when taken literally are peculiarly opaque.[22]

The second source of difficulty, not unrelated to the first, is the complexity of Russell's "theory of logical types", his way of avoiding (Whitehead-Russell 1910, vii) "the contradictions and paradoxes which have infected logic and the theory of aggregates [sets]". Russell first diagnosed the paradoxes as resulting from the "vicious circle" of "supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole", and then adopted as a remedy the *vicious-circle principle*, "Whatever involves *all* of a collection must not be one of the collection" (Whitehead-Russell 1910, 39–40). Morever, he recognized that his own concept of propositional function represents "perhaps the most fundamental case" of the principle.[23]

Adhering to the vicious-circle principle Russell insisted that the universe of *Principia* be viewed as ramified into *orders*. Speaking anachronistically, we may say that this universe consists of *objects*, where those of the lowest order are the *individuals*, and both the objects and the (formalized) language of *Principia* are to satisfy at least the following three conditions:

(i) each object S "consists" of objects of some one fixed order, an order lower than the order of S;

(ii) all values of each variable are of some one fixed order, called the *order* of the variable; and

(iii) the *order* of any notational specification N of an object S is the least order (number) greater than the orders of all the bound variables in N and not exceeded by the orders of any free variables in N.

These are the essential features of what came to be called the *ramified theory of types*, and guided by them a full formalization up to modern standards can be carried out.[24]

In the ramified theory, objects of different orders can have constituents of the same order. The collection of such constituents (objects) Russell also called a *type*. In particular, by conditions (i) and (iii), there could be objects consisting of individuals but of orders differing according to definitional complexity. But then, by condition (ii), it is impossible to quantify over all objects having individuals as constituents. Analogous situations will occur for objects whose consitutents are of higher types, and this makes the formulation of numerous mathematical propositions at best cumbersome

and at worst impossible. Consequently, Russell was led to introduce the *Axiom of Reducibility*:

> For each object there is a predicative object consisting of exactly the same objects,

where Russell called an object *predicative* if its order is the least greater than that of its constituents. Clearly, Russell did not think that objects having exactly the same constituents need be identical; in his jargon, they were *intensional* and not *extensional*.

The order hierarchy becomes greatly simplified if it were restricted to just predicative objects. There would only be individuals, predicative objects consisting of individuals, predicative objects consisting of predicative objects consisting of individuals, and so on. In this simplified hierarchy, *the simple theory of types*, the orders are just the types.[25] For Russell, it was obvious that there could only be finite orders and types, that is, only natural numbers could index orders and types.

The subsequent simplifications introduced into the system of *Principia* have mostly amounted to adopting a purely extensional version of the simple theory of types in which polyadic relations are reduced to sets through the Wiener-Kuratowski definition of ordered pair.[26] The Axiom of Reducibility, only germane for the ramified theory, would become moot. However, for Gödel the axiom would be considered both the basis of comprehension axioms in set theory as well as the antecedent to his argument for the relative consistency of the Continuum Hypothesis (see Section 8).

Hilbert enthusiastically espoused the *Principia*, saying (Hilbert 1918, 412) "should Russell's impressive undertaking to *axiomatize logic* be carried to fruition it would be the crowning achievement of axiomatization". But by "fruition [Vollendung]" Hilbert meant something utterly unlike what Russell would have meant. At a minimum, Hilbert meant showing "the consistency of the arithmetical axioms", i.e. solving his second (1900) problem.

The book (1928) by Hilbert and Wilhelm Ackermann, originating in Hilbert's (1917a) lectures, reads remarkably like a recent text. In marked contrast to the formidable works of Frege and Russell with their forbidding notation and all-inclusive approach, it proceeded pragmatically and upward to probe the extent of structure, making those moves emphasizing syntactic forms and axiomatics typical of modern mathematics. After a thorough analysis of sentential logic, it distinguished and focused on first-order logic as already the source of significant problems. While Frege and Russell never separated out first-order logic, Hilbert would establish it as a subject in its own right. Nevertheless, for the formalization required to investigate the foundations of mathematical theories Hilbert thought that an "extended calculus is essential" (Hilbert-Ackermann 1928, 86). In the (1917a) lectures on logic, this extended calculus is evidently Russell's ramified theory of types, and in it Hilbert constructed the real numbers as the Dedekind cuts using an extensional version of Russell's Axiom of Reducibility. The book Hilbert-Ackermann (1928) continued to use Russell's ramified theory of types and the Axiom of Reducibility. However, in the course of his development of mathematical logic Hilbert, like Ramsey,[27] would come to regard Russell's ramifying orders and the Axiom of Reducibility as unnecessary, as is stated on the last two pages of Hilbert-Ackermann (1928).

While Hilbert was lecturing on set theory and logic his former student Weyl brought out a notable monograph, *Das Kontinuum* (1918). Waxing philosophical, Weyl railed against the "vicious circle" involved in even such basic concepts as the least upper bound for a bounded set of real numbers. That its definition presupposes its existence among the possible upper bounds would become the standard example of an *impredicative definition*, definitions that Weyl would banish (as did of course Russell through his ramified theory). Reasoning that he could not avoid presupposing the natural numbers, Weyl took these as the individuals and considered what is essentially a version of that part of the ramified theory of types in which quantification is restricted to variables ranging over the individuals.[28] The consequences of Weyl's system for the real numbers is the same as the system ACA_0, formulated in the appendix below. Weyl went on to show that the basic theory of continuous functions could be adequately developed in his system. This was a remarkable accomplishment at such an early stage, both in the formulation of a parsimonious formal system to mirror mathematical practice and in the use of coding procedures to adequately develop a surprisingly large part of analysis. The key ingredient was to revert from continuity in terms of *sets* as given by Dedekind cuts to continuity in terms of *sequences* in the spirit of Cantor's fundamental sequences, where however real functions and sequences of real numbers are simulated by just sets of natural numbers.

Hilbert to be sure was to inspire the development of subsystems of number theory and of analysis. However, he reacted vigorously against what he regarded as Weyl's emasculation of mathematics. The difference between the two is that Weyl was advocating his system as what mathematical analysis ought to be, whereas Hilbert was investigating formal systems for specific purposes, primarily to carry out proofs of consistency.

In spirited response to Weyl's constructivism and also to Brouwer's intuitionism, which would banish the Law of Excluded Middle and non-constructive existence proofs, Hilbert (1922; 1923) developed *metamathematics* and proposed, most fully in (1926), his program of establishing the consistency of ongoing mathematics by *finitary reasoning* [das finite Schliessen]. Metamathematics would grow to be a broad, ultimately mathematical, investigation of the content and procedures of ongoing mathematics through its formalization; for Hilbert, metamathematics was primarily his *proof theory*, the investigation of formalized proofs as objects of study. Elaborating on two motifs, the primacy of logical deduction and the finiteness of formal proof, Hilbert argued that the mathematical investigation of proofs would secure the reduction of the consistency of mathematics to a bedrock of finitary and incontrovertible means.

Hilbert-Ackermann (1928, 65ff. & 72ff) raised two crucial questions with respect to first-order logic: the semantic completeness of its axioms, that is, whether a formula holding in every model of the axioms is provable from the axioms; and its decision problem (Entscheidungsproblem], that is, whether there is an algorithm for deciding whether any formula has a model or not. The first figured in the last of the five problems raised in Hilbert's lecture (1929) at the 1928 International Congress of Mathematicians at Bologna, the main theme of which however was still his program for establishing the consistency of mathematics. Hilbert thus generated all the major problems of mathematical logic that would be decisively informed by Gödel's work (see Section 8). As with his (1900) problems, Hilbert was again to stimulate major developments through the formulation of pivotal questions, questions that are contextually specific yet set a new frontier. Such questions, especially weighted as conjectures, became increasingly significant for the progress of modern mathematics, and it is Hilbert whom one acknowledges as pioneer and exemplar for this new development.

6. METAMATHEMATICS

Much has been written about Hilbert's metamathematics. Here we restrict ourselves to describing his specific strategy for settling his own second problem from his (1900), namely the problem of showing the "consistency of the arithmetical axioms". In Section 7 we show how this strategy was a starting point for his attempt to solve his first problem from (1900), that of establishing the Continuum Hypothesis.

Pursuing the analogy with the introduction of ideal elements in mainstream mathematics Hilbert (1926) distinguished between numerical formulas communicating *contentual* [inhaltlich] propositions and those communicating *ideal* propositions. Quantifiers are contentual as long as they range over specified finite domains, in which case they can be replaced by finite disjunctions or conjunctions. Hilbert (1923, 154) had noted that the first time "something beyond the concretely intuitive and finitary" enters logic is in (unrestricted) quantification and this he (1926) took to be characteristic of ideal propositions, undertaking his metamathematics as an investigation toward establishing the consistency of their use. That investigation itself would be conducted in contentual mathematics with formalized proofs as objects of study, and indeed Hilbert (1926) wrote of metamathematics as "the contentual theory of formalized proofs".

Hilbert (1926b) (see van Heijenoort 1967, 382) stated several axioms for quantifiers, and then asserted that they can be derived from a single axiom, one that "contains the core" of the Axiom of Choice:

$$A(a) \rightarrow A(\epsilon(A)),$$

"where ϵ is the transfinite logical choice function". The symbol ϵ serves as a logical operator, taking formulas A as arguments and producing terms $\epsilon(A)$; the more specific $\epsilon_x A(x)$ was soon deployed to handle A's with several free variables. The ϵ-terms had an engaging indeterminism: they serve as syntactic witnesses to A if $\exists x A(x)$, but are *bona fide* terms even if $\neg\exists x A(x)$.[29] Like the ideal points at infinity of projective geometry, Hilbert had in effect introduced new ideal elements into first-order logic.

Hilbert (1928) (see van Heijenoort (1967, 466) spelled out how the quantifiers can be defined in terms of ϵ-terms:

$$\forall a A(a) \iff A(\epsilon(\neg A)), \quad \text{and} \quad \exists a A(a) \iff A(\epsilon(A)).$$

The usual quantifier rules follow immediately, e.g.

$$\neg\forall a A(a) \rightarrow \exists a \neg A(a).$$

From Frege on, this rule had been regarded as an immediate consequence of the definitions of $\exists$ and $\forall$. For Hilbert, it is only immediate for specified finite domains as an instance of *tertium non datur*, the Law of Excluded Middle, and is otherwise a substantial manipulation on ideal propositions as an infinitary form of the Law.

Hilbert (1922, 157) had already expressed the need to formulate the Axiom of Choice so that it is as evident as 2 + 2 = 4. However, to say that $A(a) \rightarrow A(\epsilon(A))$ "contains the core" of the Axiom of Choice is misleading from the modern perspective, for it is after all just a variant of existential generalization.[30] However, it is indeed as a "choice function" that Hilbert had a particular *use* for his innovation in mind as part of a specific strategy for establishing consistency that he advanced along with his development of proof theory itself.

That strategy was first broached by Hilbert in his (1923), where before the ϵ-operator he had introduced his τ-operator through what he called the Transfinite Axiom:

$$A(\tau(A)) \rightarrow A(a).$$

The τ-operator encapsulated the universal quantifier as his later ϵ-operator would the existential quantifier. From this one axiom he derived all the quantifier rules, which he considered the source of non-finitary or "transfinite" reasoning.[31] Focusing on number-theoretic functions f, i.e. functions from the natural numbers into the natural numbers, he then defined an operator $\tau(f) = \tau_a(f(a) = 0)$, specifying the free variable a in the formula $f(a) = 0$, so that from the Transfinite Axiom we have

$$f(\tau(f)) = 0 \rightarrow f(a) = 0.$$

He interpreted $\tau(f)$ as a "function-of-functions", a *functional* we would now say, that had already appeared at the end of his (1922): This functional κ took number-theoretic functions f as arguments, with $\kappa(f) = 0$ if $f(a) = 0$ for every natural number a, and otherwise $\kappa(f)$ is the least a such that $f(a) \neq 0$. Evidently, the admissibility of κ rests on an infinitary form of *tertium non datur* and embodies Hilbert's use of non-constructive existence proofs. From the very beginning of his work on metamathematics Hilbert emphasized number-theoretic functions and substantial functionals operating on them, and this emphasis would soon extend to his attempt to establish the Continuum Hypothesis.

In terms of τ, Hilbert (1923, 159ff.) gave for a very weak subsystem of analysis an example of his strategy for establishing consistency: Starting with a putative proof of $0 \neq 0$, successive substitutions of numerals were made for the τ-terms appearing in the proof so that only a deductive sequence of true numerical formulas was left, and hence $0 \neq 0$ could not have appeared at the end after all. Hilbert had thus shown how to exploit the finiteness of proofs in a specific way, eliminating the "transfinite" τ-terms in favor of finitely many numerical instances. Ackermann (1925) undertook to carry out Hilbert's plan to apply this substitution strategy to the full system with quantification over number-theoretic functions; this would establish the consistency of analysis, with the number-theoretic functions construed as the real numbers. Hilbert had by then switched from τ-terms to ϵ-terms, which in the new rendition of his strategy were indeed interpreted as finite "choice functions". At the beginning of his career Hilbert had established a fundamental finiteness property with his basis theorem; he would now effect a new reduction to a "finite basis" to establish the consistency of mathematics.

Hilbert's strategy of eliminating ϵ-terms encountered a basic difficulty in the general setting: the possible nestings of ϵ-terms corresponding to quantifier dependence. In carrying out the substitution procedure, a numerical choice made for an ϵ-term t might typically conflict with a later choice made for an ϵ-term within which t occurs, necessitating a new substitution for t. This process can cycle in complicated ways, with the possibility that successive substitutions may not terminate. Ackermann's (1925) argument fell far short, failing to handle number-theoretic functions and even full induction for the natural numbers. John von Neumann (1927) then carried out a complex argument, based on Hilbert's (1905) approach to consistency as developed by Julius König (1914),[32] to establish the consistency of quantifier-free induction for the natural numbers. Thereupon Ackermann established the same result with his original approach. In (1928) Hilbert sketched this new argument of Ackermann's, and in succeeding comments Bernays (1928) elaborated on it.

Hilbert and his school (mainly Ackermann, Bernays, and von Neumann) believed at this time that Ackermann's new argument in fact established the consistency of full number theory (first-order Peano Arithmetic).[33] At the end of (1928) Hilbert wrote (as translated in van Heijenoort 1967, 479):

> For the foundations of ordinary analysis [Ackermann's] approach has been developed so far that only the task of carrying out a purely mathematical proof of finiteness [of the number of necessary substitutions of numerals for ϵ-terms] remains.

Thus Hilbert was also confident that his second (1900) problem, "the consistency of the arithmetical axioms" for the real numbers, would be solved. In his lecture at the 1928 International Congress of Mathematicians at Bologna, Hilbert (1929) assumed that the finiteness condition for the elimination of ϵ-terms had been established for number theory and made his first problem that of establishing the analogous finiteness condition for analysis. In a lecture given in December 1930, Hilbert (1931, 490) still thought that the consistency of number theory had been established.

However, in a lecture given in September 1930, Gödel (1930a) had announced his First Incompleteness Theorem, the existence of formally undecidable propositions of number theory. Von Neumann who was in the audience saw not only its broad significance but its particular relevance to the work of the Hilbert school. Some weeks after his lecture Gödel established his Second Incompleteness Theorem, the unprovability of consistency, and soon afterwards in November heard from von Neumann that he too had established this result.[34] The Second Incompleteness Theorem of Gödel (1931) implies in particular that for any theory subsuming the addition and multiplication of the natural numbers and for any putative proof of $0 \neq 0$ in that theory, no "proof of finiteness" (as in the quotation

above) is formalizable in that theory. Thus, there had to be something wrong with the assumption of the Hilbert school that Ackermann's new argument established the consistency of full number theory, and von Neumann soon produced an example for which the argument failed.[35] Beyond the common impression that Gödel's Second Incompleteness Theorem largely precluded Hilbert's consistency program, this close interplay between Gödel and von Neumann brings out the specific *mathematical* impact that Gödel's result had on a concerted effort then being made by the Hilbert school.

Gerhard Gentzen (1936; 1938; 1943) would show that there *is* a "purely mathematical proof" of the consistency of number theory. However, his method necessarily relied on a mathematical principle presumably non-finitary by Hilbert's standards, the principle of transfinite induction up to the ordinal ϵ_0.[36] Later Ackermann (1940) showed that for number theory Hilbert's original substitution method also provides a "purely mathematical proof of finiteness" and thereby establishes the consistency of number theory, but again by invoking transfinite induction up to ϵ_0. For number theory, Hilbert's goal of establishing consistency has been accomplished and through his substitution method – only the mathematical means were not finitary.[37]

7. CONTINUUM HYPOTHESIS

In (1923, 151) Hilbert had indicated that not only could his proof theory establish the consistency of analysis and set theory, but that it could also provide the means to solve "the great classical problems of set theory such as the Continuum Problem", the first of his (1900) problems. In (1926) Hilbert claimed to have established the Continuum Hypothesis with his "continuum theorem" and proceeded to sketch a proof. It is a failure,[35] but a notable one both for exhibiting the extent to which Hilbert thought he could extract mathematical content from formal proofs and for stimulating Gödel's work with L.

The Continuum Hypothesis would be established if the number-theoretic functions, functions from the natural numbers into the natural numbers, can be put into one-to-one correspondence with the countable ordinals. Hilbert apparently thought[39] that if he could show that from any given formalized putative disproof of the Continuum Hypothesis, he could prove the Continuum Hypothesis, then the Continuum Hypothesis would have been established. (At best, Hilbert's argument could only establish the *consistency* of the Continuum Hypothesis, but for him consistency is (mathematical) truth.[40])

According to Hilbert, the only way that the Continuum Hypothesis could be false is if there are non-constructively defined number-theoretic functions, i.e. functions defined using *tertium non datur* over existential quantifiers. A favorite example of Hilbert of such a function is $\varphi(a) = 0$ or 1 according to whether $a^{\sqrt{a}}$ is rational or not.[41] Hence, any proof of a proposition contradicting the Continuum Hypothesis would have to make use of such definitions of functions. Hilbert then asserted that the solvability of every well-posed mathematical problem is a "general lemma" of his metamathematics,[42] and that a "part of the lemma" is the following (as translated in van Heijenoort 1967, 385):

Lemma I. If a proof of a proposition contradicting the continuum theorem is given in a formalized version with the aid of functions defined by means of the transfinite symbol ϵ (axiom group III), then in this proof these functions can always be replaced by functions defined, without the use of the symbol ϵ, by means merely of ordinary and transfinite recursion, so that the transfinite appears only in the guise of the universal quantifier.

For establishing the consistency of arithmetic, Hilbert had started with a putative proof of $0 \neq 0$ and outlined a substitution procedure for replacing in effect its ϵ-terms by finite choice functions and showing that $0 \neq 0$ could not have appeared at the end after all. With Lemma I he would now start with a "proof of a proposition contradicting the continuum theorem", and presumably carry out a similar but more complex substitution procedure, this time replacing number-theoretic functions defined using ϵ symbols by a collection of functions defined by various forms of recursion. (Hilbert, we assume, was not making the stronger claim that for each given non-constructively definable function one can find an equivalent recursively definable function.) Hence, for Hilbert it remained to examine and to handle the functions so defined because (as translated in van Heijenoort 1967, 387):

in order to prove the continuum theorem, it is essential to correlate those definitions of number-theoretic functions that are free from the symbol ϵ one-to-one with Cantor's numbers of the second number class [the denumerable ordinals].

Hilbert was the first to consider number-theoretic functions defined through recursions more general than primitive recursion. He not only allowed definitions incorporating transfinite recursions through countable ordinals, but also higher type functionals. These are themselves defined recursively, a functional being a function whose arguments and values are previously defined functionals, and were classified by Hilbert into a hierarchy. Hilbert's logical beginnings in Russell's ramified theory of types is arguably discernible both in the preoccupation with definability, here reduced to recursions by Lemma I, and the introduction of a type hierarchy, though one extended into the transfinite.

In his hierarchy Hilbert classified functionals according to their *variable-type* by recursively considering their complexity of definition. He then recursively defined the *height* of a variable-type as the supremum of the heights plus 1 of the variable-types of the arguments and values. He argued that all definitions of functionals can be reduced to *substitution*, i.e. composition of functionals, and to *recursion*, i.e. primitive recursion allowing functionals. Hilbert next described how heights of certain variable-types, the *Z-types*, can be correlated with countable ordinals. The Z-types are those variable-types generated by the two processes of substitution and enumeration of a countable sequence of Z-types. Hilbert pointed out that in his correlation of heights with countable ordinals he had "presupposed" the theory of the latter. But he argued that only a formalization of the process of generating countable ordinals is necessary for his overall argument, and for that only those countable ordinals corresponding to Z-types matter. Hilbert then went on to describe how new variable-types, and therefore new ordinals, are generated by recursive enumeration of the variable-types up to a certain height and an application of "Cantor's diagonal procedure".

Hilbert next pointed out how his correlation of heights with countable ordinals was based on two apparent restrictions. First, he had only considered "ordinary recursion", not transfinite recursion directly through infinite ordinals, and second, he had only considered Z-types, those variable-types generated by enumeration of *countably* many variable-types. But he then claimed in his remarkable Lemma II that *all* number-theoretic functions defined by recursion can "also be defined by means of ordinary recursions and the exclusive use of Z-types".[43] But then, Hilbert has done what he said had to be done "in order to prove the continuum theorem". To recapitulate, from a formalized disproof of the Continuum Hypothesis Hilbert has "given" a proof of the Continuum Hypothesis!

The basic underlying difficulty with Hilbert's argument lies in his use of his Lemma I. Hilbert apparently thought that he can restrict his attention to only those number-theoretic functions that appear in purported disproofs of the Continuum Hypothesis. Whether such functions can be put in one-to-one correspondence with the countable ordinals gets us no closer to establishing even the consistency of the Continuum Hypothesis. However, Hilbert seems to have believed that there can be no number-theoretic functions unless definable in some formal proof. This is borne out by his later remark in (1928) (see van Heijenoort 1967, 476) that Lemma I is "useful in fixing the train of thought, but it is dispensable for the proof itself". He noted that the introduction of ϵ-terms does not affect the denumerability of the possible recursions in higher type functionals up to any particular height. Moreover, the ϵ-terms can be systematically "normalized", e.g. for those acting on number-theoretic functions, the functional κ (defined in Section 6) from his earliest paper (1922) in metamathematics can be used. The difficulty with Hilbert's attempted proof of the Continuum Hypothesis can arguably be reduced to his attempt to capture the force of functionals like κ in some constructive way by a collection of recursively defined functionals, whereas ironically κ, as mentioned earlier, embodies Hilbert's use of nonconstructive existence proofs.

There is a sense in which Hilbert's Lemma II is correct. Let us suppose, as his discussion would indicate, that the possible transfinite recursions that he speaks about are those given by recursive well-orderings. Then the number-theoretic functions that he was considering coincides with what today are called the general recursive functions. This is so because the class of general recursive functions is closed under recursions along any recursive well-ordering and is also closed under recursions in higher type functionals generated by primitive recursion using previously defined higher type functionals. But then, the conclusion of Lemma II was established independently by Myhill (1953) and Routledge (1953), who proved that every general recursive function is generated by recursion along primitive recursive well-orderings of ordertype ω.[44]

Hilbert broke fertile ground for the later, broad investigation of recursions. Ackermann (1928) showed that a scheme given in Hilbert (1926) does indeed define a non-primitive recursive function, now well-known as the Ackermann function. The association of ordinals with recursive definitions has become common place, with Gentzen's (1936; 1938; 1943) analysis of the consistency of number theory paradigmatic. And recursion in higher type functionals up to height ω in Hilbert's scheme was used by Gödel in his *Dialectica* interpretation (1958), already worked out in his (1941), to give a consistency proof of intuitionistic number theory and hence because of his (1933a) a consistency proof of (classical) number theory.[45]

8. GÖDEL

Kurt Gödel virtually completed the mathematization of logic by submerging metamathematical methods into mathematics.[46] The main vehicle was of course the direct coding, "the arithmetization of syntax", in his celebrated Incompleteness Theorem (1931), which transformed Hilbert's consistency program and led to the undecidability of the Decision Problem from Hilbert-Ackermann (1928) and the development of recursion theory. But starting an undercurrent, the earlier Completeness Theorem (1930) from his thesis answered affirmatively the Hilbert-Ackermann (1928) question

about semantic completeness, clarified the distinction between the formal syntax and model theory (semantics) of first-order logic, and secured its key instrumental property with the Compactness Theorem. This work would establish first-order logic as the canonical language for formalization because of its mathematical tractability, and higher order logics would become downgraded, now viewed as the workings of the power set operation in disguise. Skolem's earlier suggestion in (1923) that Zermelo's axiomatic set theory be based on first-order logic would be generally adopted, thus vindicating Hilbert's emphasis on first-order logic.

To pursue our earlier discussion of categoricity in connection with Hilbert's Completeness Axiom in geometry, say that a theory is *deductively complete* if each sentence of its language or its negation is provable from the axioms. In the Königsberg lecture (1930a) where Gödel discussed his Completeness Theorem and announced his First Incompleteness Theorem, he observed that the former implies that *for first-order theories categoricity implies deductive completeness*. The argument is simple: if there were a sentence such that neither it nor its negation can be proved from the axioms, then there would be two (non-isomorphic) models of the theory.[47] Now Hilbert's axioms for geometry inclusive of the Completeness Axiom and the Dedekind-Peano axioms for the natural numbers *are* categorical, but as second-order theories. However, Gödel's First Incompleteness Theorem established that no (decidable) set of axioms for first-order or higher order theories, which subsumes the arithmetic of the natural numbers and only proves true sentences of that arithmetic, can be deductively complete. Thus, the Incompleteness Theorem makes a distinction between first-order and higher order theories in terms of categoricity and deductive completeness. Although Gödel in his Incompleteness paper (1931) did not mention this distinction, he had made it the *motivation* for the Incompleteness Theorem in his Königsberg lecture (1930a, 29).

Footnote 48a of Gödel's (1931) was as follows:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (cf. D. Hilbert, "*Über das Unendliche*", Math. Ann. 95, p. 184), while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P [the simple theory of types superposed on the natural numbers as individuals satisfying the Peano axioms]). An analogous situation prevails for the axiom system of set theory.

This prescient note would be an early indication of a steady intellectual progress on Gödel's part that would take him from the Incompleteness Theorem through pivotal relative consistency results for set theory to speculations about its further possibilities. The reference to Hilbert (1926) and

Russell's theory of types foreshadows the strong influence that they would have on this progress.

In a subsequent lecture (1933), Gödel expanded on the theme of footnote 48a. He regarded the axiomatic set theory of Zermelo, Fraenkel, and von Neumann as "a natural generalization of the [simple] theory of types, or rather, what becomes of the theory of types if certain superfluous restrictions are removed".[48] First, instead of having separate types with sets of type $n+1$ consisting purely of sets of type n, sets can be *cumulative* in the sense that sets of type n can consist of sets of *all* lower types. If S_n is the collection of sets of type n, then: S_0 is the type of the individuals, and inductively, $S_{n+1} = S_n \bigcup \{X | X \subseteq S_n\}$. Second, the process can be continued into the transfinite, starting with the cumulation $S_\omega = \bigcup_n S_n$, proceeding through successor stages as before, and taking unions at limit stages. Gödel (1933, 46) credited Hilbert for pointing out the possibility of continuing the formation of types beyond the finite types. As for how far this cumulative hierarchy of sets is to continue, the "first two or three types already suffice to define very large ordinals" (Gödel 1933, 47) which can then serve to index the process, and so on. Gödel observed that although this process has no end, this "turns out to be a strong argument in favor of the theory of types" (Gödel 1933, 48). Implicitly referring to his incompleteness result Gödel noted that for a formal system S based on the theory of types a number-theoretic proposition can be constructed which is unprovable in S but becomes provable if to S is adjoined "the next higher type and the axioms concerning it" (Gödel 1933, 48).

In 1938 modern set theory was launched by Gödel's formulation of the model L of "constructible" sets, a model of set theory that established the consistency of the Axiom of Choice and the (Generalized) Continuum Hypothesis. In his first announcement Gödel (1938, 556) described L as a hierarchy "which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders". Indeed, with L Gödel had refined the cumulative hierarchy of sets described in his (1933) to a cumulative hierarchy of definable sets which is analogous to the orders of Russell's *ramified* theory. This hierarchy of definable sets was in the spirit of Hilbert (1926) as was the extension of the hierarchy into the transfinite. However, Gödel's further innovation was to continue the indexing of the hierarchy through *all* the ordinals to get a model of set theory.[49] The extent of the ordinals was highlighted in his monograph (1940), based on lectures in 1938, in which he formally generated L set by set using a sort of Gödel numbering in terms of ordinals. As with his proof of the Incompleteness Theorem, Gödel's careful coding of metamathematical features may have precluded any misinterpretations; however, it also served to purge the

intuitive underpinnings and historical motivations. In his (1939a), Gödel presented the hierarchy whose cumulation is L essentially as it is today:

$$M_0 = \{\emptyset\};\ M_\beta = \bigcup_{\alpha<\beta} M_\alpha \text{ for limit ordinals } \beta\text{; and } M_{\alpha+1} = M'_\alpha,$$

where M' is "the set of subsets of M defined by propositional functions $\phi(x)$ over M", these propositional functions having been precisely defined. Significantly, footnote 12 of (1939a) revealed that Gödel viewed his axiom A, that every set is constructible (now written $V = L$ following Gödel 1940), as deriving its sense from the cumulative hierarchy of sets regarded as an extension of the simple theory of types: "In order to give A an intuitive meaning, one has to understand by 'sets' all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders)."

The recent publication of hitherto unpublished lectures of Gödel on the Continuum Hypothesis has dramatically substantiated the strong influence of both Russell and Hilbert on him. Both figures loom large in Gödel's lecture (1939b) given at Hilbert's Göttingen. Gödel recalled at length Hilbert's work on the Continuum Hypothesis and cast his own as an analogical development, one leading however to the constructible sets as a model for set theory. Gödel (1939b, 131) pointed out that "*the model . . . is by no means finitary*; in other words, the transfinite and impredicative procedures of set theory enter into its definition in an essential way, and that is the reason why one obtains only a relative consistency proof [of the Continuum Hypothesis]".

To motivate the model Gödel referred to Russell's ramified theory of types. Gödel first described what amounts to the orders of that theory for the simple situation when the members of a countable collection of real numbers are taken as the "individuals" and new real numbers are successively defined via quantification over previously defined real numbers, and emphasized that the process can be continued into the transfinite. He then observed that this procedure can be applied to sets of real numbers, and the like, as "individuals", and moreover, that one can "intermix" the procedure for the real numbers with the procedure for sets of real numbers "by using in the definition of a real number quantifiers that refer to sets of real numbers, and similarly in still more complicated ways" (Gödel 1939b, 135). Gödel called a *constructible* set "the most general [object] that can at all be obtained in this way, where the quantifiers may refer not only to sets of real numbers, but also to sets of sets of real numbers and so on, *ad transfinitum*, and where the indices of iteration . . . can also be arbitrary transfinite ordinal numbers". Gödel considered that although this definition of constructible set might seem at first to be "unbearably complicated", "the *greatest generality yields*, as it so often does, at the same time the *greatest simplicity*" (Gödel 1939b, 137). Gödel was picturing Russell's ramified theory of types by first disassociating the types from the orders, with the orders here given through definability and the types represented by real numbers, sets of real numbers, and so forth. Gödel's intermixing then amounted to a recapturing of the complexity of Russell's ramification, the extension of the hierarchy into the transfinite allowing for a new simplicity.

Gödel went on to describe the universe of set theory, "the objects of which set theory speaks", as falling into "a transfinite sequence of Russellian [simple] types" (Gödel 1939b, 137), the cumulative hierarchy of sets that he had described in (1933). He then formulated the constructible sets as an analogous hierarchy, the hierarchy of (1939a), in effect introducing Russellian orders through definability. In a comment bringing out the intermixing of types and orders, Gödel pointed out that "there are sets *of lower type* that *can* only *be defined* with the help of *quantifiers for sets of higher type*" (Gödel 1939b, 141). This lecture of Gödel's is a remarkably clear presentation of both the mathematical and historical development of L.

Gödel's argument for the Continuum Hypothesis in the model L rests on (1939) "a generalization of Skolem's method for constructing enumerable models". It is arguably the next significant application of the Löwenheim-Skolem Theorem after Hilbert's anticipatory one with his countable interpretation for Euclidean geometry (*sans* the Completeness Axiom) and Skolem's own (1923) to get his "paradox" for set theory. Gödel showed that every subset of M_ω in L belongs to M_α for some $\alpha < \omega_1$. (Thus, every real number in L belongs to M_α for some $\alpha < \omega_1$.) In (1939b, 143) he asserted that "this fundamental theorem constitutes the corrected core of the so-called Russellian axiom of reducibility". Thus, Gödel established another connection between L and Russell's ramified theory of types. But while Russell had to *postulate* his Axiom of Reducibility for his finite orders, Gödel was able to *derive* an analogous form for his transfinite hierarchy. In his first announcement Gödel (1938, 556) had written: "The extension to transfinite orders has the consequence that the model satisfies the impredicative axioms of set theory, because an axiom of reducibility can be proved for sufficiently high orders". The beginnings of this was already hinted at in Gödel's Incompleteness paper (1931, 178), where he wrote of its Axiom IV: "This axiom plays the role of the axiom of reducibility (the comprehension axiom of set theory)". For Gödel, Russell's Axiom of Reducibility with its capability of replacing notationally specified objects of any order by equivalent objects of the lowest order of the same type was

the direct antecedent to "the comprehension axiom of set theory". As he said (1939b, 145),

> This character of the fundamental theorem as an axiom of reducibility is also the reason why *the axioms of classical* mathematics hold for the model of the constructible sets. For after all, as Russell showed, the axioms of reducibility, infinity and choice are the only axioms of classical mathematics that do not have a tautological character. To be sure, one must observe that the axiom of reducibility appears in different mathematical systems under different names and in different forms, for example, in Zermelo's system of set theory as the axiom of separation, in Hilbert's systems in the form of recursion axioms, and so on.

Hilbert and Russell also figure prominently in a later lecture (1940a) at Brown University on the Continuum Hypothesis. Gödel began by announcing that he had "succeeded in giving the [consistency] proof a new shape which makes it somewhat similar" to Hilbert's (1926) attempt, and proceeded to sketch the new proof, considering it "perhaps the most perspicuous". First, Gödel reviewed his construction of the model L. Once again he emphasized that his argument showing that the Continuum Hypothesis holds in L proves an axiom of reducibility.[50] Then Gödel turned to his new approach to the consistency proof, and introduced the concept of a relation being "recursive of order α" for ordinals α. This concept is a generalization of the notion of definability, a generalization obtained by interweaving the operation M', given five paragraphs above, with a recursion scheme akin to Hilbert's for his (1926) hierarchy of functionals. As Gödel (1940a, 180) said: "The difference between this notion of recursiveness and the one that Hilbert seems to have had in mind is chiefly that I allow quantifiers to occur in the definiens. This makes one [Lemma I] of Hilbert's lemmas superfluous and the other [Lemma II] demonstrable in a certain modified sense". Using this new concept of recursiveness – better, new concept of definability – Gödel gave a model of Russell's *Principia*, construed as his system P of his incompleteness paper (1931), in which the Continuum Hypothesis holds. (The types of this model were essentially coded versions of $M_{\omega_{n+1}} - M_{\omega_n}$.)

In his monograph (1940) Gödel had provided a formal presentation of L using an axiomatization of set theory with an antecedent in von Neumann (1925). Gödel's formalization not only recalled von Neumann's (1925, II) analysis of "subsystems", but also shed light on von Neumann's main concern: the categoricity of his axiomatization. Fraenkel (1922) had expressed the desirability of closing off the Zermelian generative axioms through an axiom of "restriction"; this required that there should be no further sets than those generated by the axioms, a notable move antithetical to the role played by Hilbert's Completeness Axiom in geometry. It was to pursue this that von Neumann had investigated subsystems for his axiomatization, but he concluded that there was probably no way to *formally* achieve

Fraenkel's idea of a minimizing, and hence categorical, axiomatization. Gödel's axiom A, that every set is constructible, can be viewed as formally achieving this sense of categoricity, since, as he essentially showed in (1940), in axiomatic set theory L is a definable class that together with the membership relation restricted to it is a model of set theory, and L is a submodel of every other such class.[51] In his first description of L Gödel wrote (1938, 557): "The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way."

However, Gödel came to regard L as primarily a contrivance for establishing relative consistency results. In his (1947) he suggested that the Continuum Hypothesis is false and in footnote 22 that a new axiom "in some sense directly opposite" to A might entail this. In a revision (1964, 266) of (1947), he expanded the footnote: "I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A states a minimum property. Note that only a maximum property would seem to harmonize with the concept of [arbitrary set]". This is related to Gödel's speculations with large cardinal hypotheses;[52] whereas his axiom A had enforced a kind of categoricity through minimization, large cardinals as maximum properties might establish the negation of the Continuum Hypothesis. Although the historical connection is now admittedly faint, just as the addition of the Completeness Axiom in geometry precludes Hilbert's countable interpretation, so maximum properties in set theory may preclude versions of Gödel's Skolem function argument for the consistency of the Continuum Hypothesis.

In an earlier letter to Ulam (see Ulam 1958, 13) Gödel had written of von Neumann's axiom (1925) that a class is proper exactly when it can be put into one-to-one correspondence with the entire universe:

> The great interest which this axiom has lies in the fact that it is a maximum principle, somewhat similar to Hilbert's axiom of completeness in geometry. For, roughly speaking, it says that any set which does not, in a certain well-defined way, imply an inconsistency exists. Its being a maximum principle also explains the fact that this axiom implies the axiom of choice. I believe that the basic problems of abstract set theory, such as Cantor's continuum problem, will be solved satisfactorily only with the help of stronger axioms of *this* kind, which in a sense are opposite or complementary to the constructivistic interpretation of mathematics.

Hilbert's Completeness Axiom thus fueled speculations about maximization for set theory, speculations resonating with his "consistency implies existence" view, speculations still being investigated to this day.

9. APPENDIX[53]

Recent developments have not only led to a precise logical analysis of Hilbert's basis theorem but to results that can be regarded as affirmatory for Hilbert's consistency program. In this appendix some of these developments are briefly described to recast Hilbert's results and initiatives in a new light.

Harvey Friedman (1975) observed that when a theorem of "ordinary" mathematics is proved from a very economical comprehension (or "set existence") axiom, then it should be possible to "reverse" the process by proving the axiom from the theorem over a weak ambient theory. Together with initial and continuing results by Friedman, Stephen Simpson and his collaborators since the late 1970's proceeded to carry out a program analyzing theorems in this spirit, the program of *reverse mathematics*. We first set the stage:

Primitive Recursive Arithmetic is the system in the language with the logical connectives (but no quantifiers), the constant 0, a unary function symbol for the successor function, and a function symbol for each (definition of a) primitive recursive function, where the axioms are the recursive defining equations for the functions symbols. First presented in Skolem (1923a) and extensively investigated in Hilbert-Bernays (1934), Primitive Recursive Arithmetic has been widely regarded as a characterization of Hilbert's "finitary" methods.

The *language of second-order arithmetic*[54] is a two-sorted language with *number variables* $i, j, m, n, \ldots$ and set variables $X, Y, Z, \ldots$. The number variables are intended to range over the natural numbers, and the set variables to range over sets of natural numbers. Numerical terms are generated as usual from the number variables, the constants 0 and 1, and the binary operations + and ×. The atomic formulas are $t = u$, $t < u$, and $t \in X$, where t, u are numerical terms. Finally, formulas are generated from the atomic formulas via logical connectives, number quantifiers $\forall n$ and $\exists n$, and the set quantifiers $\forall X$ and $\exists X$.

All the formal systems to be considered include the familiar axioms about $+, \times, 0, 1, <$ as well as the *induction axiom*:[55]

$$(0 \in X \land \forall n(n \in X \to n + 1 \in X)) \to \forall n(n \in X).$$

Full *second-order arithmetic*, or *analysis*, consists of these axioms together with the *full comprehension scheme*: For all formulas φ,

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)).$$

As shown in Hilbert-Bernays (1939), a great deal of classical mathematics can be faithfully recast in second-order arithmetic with codes for the real numbers. In what follows, certain subsystems are considered that exactly capture the strength of several basic mathematical results. An analysis of the complexity of formulas sets the stage:

A formula is Δ^0_0 if it has no set quantifiers and all of its number quantifiers are *bounded*, i.e. can be rendered in form $\forall m(m < t \rightarrow \cdots)$ or $\exists m(m < t \wedge \cdots)$. A formula is $\sum^0_1$ if it is of form $\exists m\varphi$ where φ is Δ^0_0, and $\prod^0_1$ if it is of form $\forall m\varphi$ where φ is Δ^0_0. For each natural number n, a formula is $\sum^0_{n+1}$ if it is of the form $\exists m\varphi$ where φ is $\prod^0_n$, and a formula is $\prod^0_{n+1}$ if it is of the form $\forall m\varphi$ where φ is $\sum^0_n$. A formula is *arithmetical* if it contains no set quantifiers, i.e. its prenex form is for some n a $\sum^0_n$ or $\prod^0_n$ formula. Finally, a formula is $\prod^1_1$ if it is of the form $\forall X\varphi$ where φ is arithmetical.

RCA_0 (Recursive Comprehension Axiom)[56] is the subsystem of second-order arithmetic consisting of the axioms of the $\sum^0_1$-induction scheme, i.e. for each $\sum^0_1$ formula φ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\varphi(n),$$

and axioms of the Δ^0_1-comprehension scheme, i.e. for $\sum^0_1$ formulas φ and $\prod^0_1$ formulas ψ,

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X\forall n(n \in X \leftrightarrow \varphi(n)).$$

RCA_0 just suffices to establish the existence of the (general) recursive sets and also to develop some basic theory of real-valued continuous functions and of countable algebraic structures. However, with its parsimonious form of induction it can only establish the totality of number-theoretic functions in a restricted class. It is essentially a result of Charles Parsons (1970) that the provably total general recursive functions of RCA_0 are exactly the primitive recursive functions.[57] RCA_0 proves that the ordinal ω^n is well-ordered for each particular natural number n, but not that ω^ω is.[58] For RCA_0 proves that ω^ω is well-ordered implies the totality of Ackermann's function, the paradigmatic non-primitive recursive function.

WKL_0 (Weak König's Lemma) is the subsystem consisting of the axioms of RCA_0 together with: Every infinite tree of finite sequences of 0's and 1's ordered by extension has an infinite path. WKL_0 provides a better theory of continuous functions and suffices for the development of ideal theory for countable commutative rings.

ACA_0 (Arithmetical Comprehension Axiom) is the subsystem consisting of the axioms of the arithmetical comprehension scheme, i.e. for each arithmetical formula φ,

$$\exists X\forall n(n \in X \leftrightarrow \varphi(n)).$$

(In what follows, other comprehension schemes based on formula complexity have analogous formulations.) ACA_0 subsumes WKL_0. Since RCA_0 can encode functions as sets of ordered pairs, it follows that over this base theory ACA_0 is equivalent to the $\sum^0_1$-comprehension scheme. In terms of well-orderings, ACA_0 proves that every ordinal less than ϵ_0 is well-ordered, but not ϵ_0 itself.[59] ACA_0 has the same consequences for analysis as the system explored by Weyl (1918).

THEOREM 1 (Friedman, Simpson). The following are equivalent over RCA_0:

(a) WKL_0.
(b) The Heine-Borel Theorem: Every covering of the unit interval of reals by a countable sequence of open sets has a finite subcover.
(c) Every continuous real function on the unit interval has a supremum.
(d) Every countable commutative ring has a prime ideal.
(e) The Gödel Completeness Theorem.
(f) The Hahn-Banach Theorem for separable Banach spaces.

THEOREM 2 (Friedman, Simpson). The following are equivalent over RCA_0:

(a) ACA_0.
(b) The Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.
(c) Every bounded sequence of real numbers has a least upper bound.
(d) Every countable commutative ring has a maximal ideal.
(e) König's Lemma: Every infinite, finitely branching tree consisting of finite sequences of natural numbers ordered by extension has an infinite path.

Theorem 2(e) highlights the new strength beyond WKL_0, which draws the same conclusion for finite sequences of 0's and 1's.

Simpson (1988a) provided the following analysis of Hilbert's basis theorem:

THEOREM 3 (Simpson). The following are equivalent over RCA_0:

(a) Hilbert's basis theorem in the following sense: For countable fields K and $x_1, \ldots, x_n$, the (commutative) ring of polynomials $K[x_1, \ldots, x_n]$ is finitely generated.
(b) The ordinal ω^ω is well-ordered.

The proof incidentally is similar to Gordan's (1899) proof of the basis theorem. By our previous remarks about RCA_0, (a) thus just transcends RCA_0 and implies the totality of Ackermann's function.

Friedman (unpublished) has in fact established an equivalence between a variant of Hilbert's basis theorem and the totality of Ackermann's function. Friedman showed: *For any natural number k there is a natural number n such that for every sequence of n polynomials in k variables over any field, where the ith term of the sequence has degree at most i, some polynomial is in the ideal generated by the previous polynomials. With h(k) denoting the least such n, the function h is essentially Ackermann's function.* Note that h does not depend on the field. The assertion cast for polynomials over the two-element field is formalizable as a Π^0_2 sentence, as is the assertion of the totality of Ackermann's function. This is a remarkable historical confluence of Hilbert's mathematics and metamathematics, in that a variant of his first major result is seen to be equivalent to the totality of the first recursive function that he (1926) had considered for transcending primitive recursion, and hence just transcends Primitive Recursive Arithmetic, the common characterization of Hilbert's "finitary" methods.

Hilbert's Nullstellensatz has also been analyzed, though in a different setting. Following a major reduction of the theorem to an effective form by W. Dale Brownawell (1987), Michael Shub and Stephen Smale in their (1995) observed that that effective form is equivalent to an algebraic version for the real numbers of the well-known NP $\neq$ P assertion in theoretical computer science.

Perhaps the main triumphs of reverse mathematics are the following two conservation results:

THEOREM 4

(a) (Friedman: Kirby and Paris 1976) WKL_0 is a conservative extension of Primitive Recursive Arithmetic with respect to Π^0_2 sentences, i.e. every Π^0_2 sentence provable in WKL_0 is already provable in Primitive Recursive Arithmetic.

(b) (Harrington) For every model of RCA_0 there is a model of WKL_0 with the same "natural numbers". In particular, (Friedman 1975, 238) WKL_0 is a conservative extension of RCA_0 with respect to Π^1_1 sentences, i.e. every Π^1_1 sentence provable in WKL_0 is already provable in RCA_0.

As emphasized by Simpson (1985, 469), (a) provides a significant advance towards the realization of Hilbert's consistency program in the sense that strong ideal propositions can be eliminated from the proofs of substantial

assertions of Primitive Recursive Arithmetic. One can apply the powerful methods of Riemann integration, the ideal theory for countable commutative rings, and Gödel's Completeness Theorem available in WKL_0 to establish results of a rich logical complexity as in (a) and (b). In the simplest case, one cannot derive $0 \neq 0$ in WKL_0 if one cannot already derive it in Primitive Recursive Arithmetic.

Theorem 4 was established by model-theoretic means; Sieg (1991) provided systematic proof-theoretic proofs based on Herbrand and Gentzen. Feferman (1988) gives a detailed account of constructive consistency proofs for various powerful subsystems of analysis.

NOTES

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[1] Henri Poincaré, Hilbert's only rival for preeminence, died in 1912.

[2] See Kanamori (1996) for the development of set theory from Cantor to Cohen.

[3] That central problem emanated from the work of Arthur Cayley. It had been known that for a polynomial $ax^2 + 2bxy + cy^2$ in x and y, if $x = \alpha x' + \beta y'$ and $y = \gamma x' + \delta y'$ are substituted to get $a'x'^2 + 2b'x'y' + c'y'^2$, then

$$b'^2 - a'c' = (b^2 - ac)(\alpha\delta - \beta\gamma)^2,$$

i.e. the new *discriminant* $b'^2 - a'c'$ equals the old discriminant $b^2 - ac$ times a constant factor (in fact the square of the *determinant* of the transformation).

Generalizing, a *form* (Cayley's *quantic*), is a polynomial in $x_1, \ldots, x_n$ which is homogeneous (i.e. there is a fixed k such that the sum of the powers of the variables in each summand is k). A *linear transformation* of $x_1, \ldots, x_n$ to $x'_1, \ldots, x'_n$ is given by a system of equations, each x_i being equated to a linear form in $x'_1, \ldots, x'_n$. For a form P, a polynomial Q in the coefficients and variables of P is an *invariant* if and only if for every linear transformation, if the corresponding substitutions are made to get a corresponding form P' in $x'_1, \ldots, x'_n$ and a Q' corresponding to Q, then Q and Q' differ only by a constant factor. A *complete system of invariants* for P is a collection C of such invariants such that every invariant is a linear combination of members of C. Finally, the central problem of invariant theory solved by Hilbert (1890) was for any form P to find a *finite*, complete system of invariants.

Of course, it is straightforward to generalize the foregoing in modern terms to polynomials over a field and groups of linear transformations, and then to vector spaces on which groups act linearly, and this is how invariant theory was eventually reactivated.

[4] In Hilbert's original (1893) form the Nullstellensatz states that if $f, f_1, \ldots, f_r$ are in $C[x_1, \ldots, x_n]$, the ring of polynomials in $x_1, \ldots, x_n$ over the complex field, and f vanishes at all the common roots of $f_1, \ldots, f_r$, then some power f^k is a linear combination $f^k = h_1 f_1 + \cdots + h_r f_r$. The assertion is equivalent to the special case when $f = 1$

and the f_i's having no common roots. In modern terms, this special case amounts to the assertion that if F is a field, I is the ideal of $F[x_1, \ldots, x_n]$ generated by $\{f_1, \ldots, f_r\}$ (and all ideals of that polynomial ring are generated by some such finite collection by Hilbert's basis theorem), and the f_i's have no common roots in the algebraic closure of F, then the ideal is the unit ideal, i.e. the whole ring.

[5] Hilbert's basis theorem would stimulate the search for algebraic generalizations, with an optimistic one suggested by the 14th of Hilbert's (1900) problems, and much progress would be made. Section 3 discusses the first two of Hilbert's (1900) problems; see Mumford (1976) for the 14th problem.

As for invariant theory itself, Hilbert's comprehensive result there was to leave the field fallow for most of Hilbert's lifetime, only revived by his brilliant student Hermann Weyl (1939) for the classical Lie groups as part of their representation theory. The subject was then fully reactivated by David Mumford (1965) with his incisive investigation of groups of automorphisms on algebraic varieties (see also Mumford-Fogarty Kirwan 1994). Notably it was the approach of (1893) rather than the initial (1890) that was to inspire Mumford (1965), which can be considered as perpetuating in geometric terms the 19th Century view of invariant theory as a constructive theory.

[6] See Ellison (1971) for a history of Waring's Problem. The conjecture is that $g(k) = [(\frac{3}{2})^k] + 2^k - 2$, and according to recent research literature this has been verified for $k \leq 471{,}600{,}000$ and for sufficiently large k.

[7] According to Blumenthal (1935, 403) Hilbert already in 1891 uttered his aphorism portending his axiomatic and formalist leanings: "One must always be able to say for *points, line, plane*: *table, chair, beer-mug*."

[8] The Archimedean Axiom asserts that for any two line segments s and t a finite number of contiguous copies of s along the ray of t will subsume t.

[9] The embeddability of an axiomatically presented geometry into Euclidean space was Hilbert's first "meta" result in mathematics. In his (1895), appearing as Appendix I from the second edition (1903) on of the *Grundlagen*, what amounts in modern terms to a homeomorphism of a "general geometry" with a finite convex part of Euclidean space played a crucial role. (1895) dealt with the problem of "the straight line as the shortest distance between two points", and a general version of this became the fourth of Hilbert's (1900) problems. See Busemann (1976) for the fourth problem.

[10] After *defining* the real numbers as the cuts, Russell (1919, 73) continued: "The above definition of real numbers is an example of 'construction' as against 'postulation', of which we had another example in the definition of cardinal numbers. The great advantage of this method is that it requires no new assumptions, but enables us to proceed deductively from the original apparatus of logic."

[11] The arithmetical version of the Archimedean Axiom for ordered fields states that for any $a > 0$ and $b > 0$, a can be added to itself a (finite) number of times so that: $a + a + \cdots + a > b$. Ordered fields having this property are now called *Archimedean*. The arithmetical version of the Completeness Axiom in (1900b) states that the reals cannot be properly extended if the Archimedean ordered field properties are to be maintained.

[12] Hilbert in (1900b) actually asserted that his axioms characterize the real numbers since its version of the Completeness Axiom implies the existence of limit points; this was the first statement along these lines. In connection with the discussion above at the end of Section 2 but shifting from geometry to the real numbers, the later polemic of Hilbert (1905, 185) (as translated in van Heijenoort 1967, 138) is notable: "[The Completeness Axiom] expresses the fact that the totality [Inbegriff] of real numbers contains, in the sense of one-to-one correspondence between elements, any other set whose elements satisfy also the axioms that precede; thus considered, the completeness axiom, too, becomes a stipulation expressible by formulas constructed like those constructed above, and the axioms for the totality of real numbers do not differ qualitatively in any respect from, say, the axioms necessary for the definition of the integers. In the recognition of this fact lies, I believe, the real refutation of the conception of the foundations of arithmetic associated with L. Kronecker and characterized at the beginning of my lecture as dogmatic."

[13] Hilbert (1900b,184) wrote: "In order to prove the consistency of the given axioms all that is needed is a suitable modification of known methods of inference". When (1900b) appeared as Appendix VI in later editions of the *Grundlagen*, this sentence is missing.

[14] Browder (1976) is a compendium on the mathematical developments arising from Hilbert's problems.

[15] This contention did not become clear until the development of the theory of real-closed fields by Emil Artin and Otto Schreier in their (1926; 1927). This development was resonating: Real-closed fields have a maximal property analogous to Hilbert's Completeness Axiom; the theory was crucial for Artin's (1927) non-constructive solution of Hilbert's 17th Problem; and constructive solutions were later given. See Pfister (1976) for more on Hilbert's 17th Problem.

[16] See Ehrlich (1995).

[17] Wang (1987) describes Gödel's admiration for Hahn.

[18] See Moore (1982, 89ff.) for more about Zermelo.

[19] Hilbert wrote to Frege (see Frege 1980, 39ff.):

You write: "I call axioms propositions that are true but are not proved because our knowledge of them flows from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict each other." I found it very interesting to read this sentence in your letter, for as long as I have been thinking, writing and lecturing on these things, I have been saying the exact opposite: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by the axioms exist. For me this is the criterion of truth and existence.

Cantor (1883, Section 8) had written:

Mathematics is completely free in its development and only bound by the self-evident consideration that its concepts must on the one hand be consistent in themselves and on the other stand in orderly relation, fixed through definitions, to the previous formed concepts already present and tested.

[20] Hallett (1997, Section 3) corroborates, through notes to Hilbert's lectures during this period, his more favorable attitude toward the genetic method of building up mathematical objects.

[21] See Goldfarb (1979) and the note of Dreben and van Heijenoort in Gödel (1986, 44–59) for a discussion of logic in the 1910's and 1920's. And Hylton (1990) for a discussion of the metaphysics underlying Russell's logic.

[22] Russell wrote (Whitehead-Russell 1910, 41):

By a 'propositional function' we mean something which contains a variable x, and expresses a *proposition* as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. It agrees with the ordinary functions of mathematics in the fact of containing an unassigned variable; where it differs is in the fact that the values of the function are propositions. . . . The question as to the nature of a [propositional] function is by no means an easy one.

It would seem, however, that the essential characteristic of a [propositional] function is *ambiguity*.

A few pages on Russell declared (Whitehead-Russell 1910, 50): "A [propositional] function, in fact, is not a definite object ...; it is a mere ambiguity awaiting determination".

In a later book Russell (1919, 157) wrote: "We do not need to ask, or attempt to answer, the question: 'What *is* a propositional function?' A propositional function standing all alone may be taken to be a mere schema, a mere shell, an empty receptacle for meaning, not something already significant".

[23] Russell wrote (continuing in Whitehead-Russell 1910, 41–42 after the quotation from there in the previous note):

a [propositional] function is not ... well-defined unless all its values are already well-defined. It follows from this that no [propositional] function can have among its values anything which presupposes the function .... This is a particular case, perhaps the most fundamental case, of the vicious-circle principle. A [propositional] function is what ambiguously denotes some one of a certain totality, namely the values of the [propositional] function; hence this totality cannot contain any members which involve the [propositional] function, since, if it did, it would contain members involving the totality, which, by the vicious-circle principle, no totality can do.

[24] See for example Church (1976).

[25] It is from the simple theory that the terms "first-order logic", "second-order logic", and so forth evolved, with "order" retained instead of "type". For example, with the zeroth order comprised of individuals and the first order consisting of the (predicative) objects consisting of individuals, first-order logic treats quantification over individuals of a domain. Similarly, second-order logic treats in addition quantification over objects consisting of individuals.

[26] The first such presentation of *Principia* in print was Gödel's system P in his Incompleteness paper (1931).

[27] Ramsey is not mentioned in the text of Hilbert-Ackermann (1928), but his paper (1926) in which he suggested that the ramified theory be replaced by the simple theory and Axiom of Reducibility be dropped is cited in their bibliography.

[28] Significantly, Weyl (1910, 112ff.) had begun his foundational investigations by trying to provide a satisfactory formulation for Zermelo's definite property for the Separation Axiom and had suggested building up the concept from $\in$ and $=$ by a finite number of generating principles. It was in the course of developing these principles that Weyl (1918, 36) found that he could not avoid presupposing the natural numbers – a primordial vicious circle. Weyl (1918, 35) acknowledged that his hierarchy "corresponds" to Russell's, but rejected the Axiom of Reducibility.

[29] As the Athenians were wont to say of Aristides, if there is an honest man, then it is he.

[30] The Second ϵ-Theorem of Hilbert-Bernays (1939) would establish that in first-order logic with ϵ-terms, if neither the premises nor the conclusion of a deduction contains such terms, then there is a deduction not using such terms. In order to derive the Axiom of Choice using ϵ-terms, the crucial set-theoretic feature of the Axiom, the existence of a *set* of choices, or concomitantly a choice function, must be incorporated. One approach is to allow ϵ-terms in the Replacement Axiom, an essential feature of modern set theory. Hilbert (1923, 164) himself used an informal variant of this approach to argue for the Axiom of Choice for sets of reals. (Wang 1955 discusses the interplay of ϵ-terms and the Axiom of Choice in axiomatic set theory.) Interestingly, Zermelo (1930, 31) in his final axiomatizations of set theory also regarded the Axiom of Choice as a logical principle and did not list it explicitly among his axioms.

In later years a fully Tarskian semantics was developed by Günter Asser (1957) and Hans Hermes (1965) for the ϵ-operator with its interpretations being global choice functions for the structure at hand. More in the spirit of Hilbert's intention was Rudolf Carnap's (1961) indeterminate use of the ϵ-operator as an interpretation of his T-, or theoretical, terms.

[31] Hilbert (1923, 161) specifically asserted that transfinite reasoning was necessary for his solution of the central invariant theory problem discussed in Section 1, and that although Gordan thought that he had removed this "theological" aspect of the argument with his own version of the proof, it remained embedded in his "symbolic" approach. Hilbert's view of the complexity of his proof was substantiated; see Theorem 3 in the appendix.

[32] von Neumann (1927, 22) acknowledges König (1914).

[33] This was corroborated in oral communication from Bernays to Dreben in 1965, and in a letter from Bernays to the editor of a projected Spanish translation of van Heijenoort (1967), dated 15 June 1974.

[34] See Gödel (1986, 137).

[35] The example is given in Hilbert-Bernays (1939, 123ff.).

[36] ϵ_0 is the supremum of the ordinals $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$. There is a primitive recursive ordering $\prec$ of the natural numbers which is isomorphic to ϵ_0. The *principle of transfinite induction up to* ϵ_0 asserts that for any formula $\varphi(v)$,

$$\forall n(\forall m(m \prec n \to \varphi(m)) \to \varphi(n)) \to \forall n\varphi(n).$$

This assertion is formalizable as a schema in any first-order theory of number theory that subsumes primitive recursion, of which the minimal is Primitive Recursive Arithmetic described in the appendix. Gentzen showed that a single instance of the schema for a certain quantifier-free φ implies the consistency of number theory.

[37] A more intuitive, constructive model-theoretic version of Hilbert's substitution method was provided by Jacques Herbrand (1930) with his Fundamental Theorem; in particular, Herbrand gave a much simpler proof of the result of von Neumann (1927). Expanding on Dreben and John Denton's analysis in their (1966, 1970) of Herbrand's Theorem, Thomas Scanlon (1973) provided a Herbrand-style proof for the full number theory result of Ackermann (1940).

[38] In the reprintings of (1926) and the related (1928) in the seventh edition (1930) of the *Grundlagen*, Hilbert excised all reference to his purported proof of the Continuum Hypothesis.

[39] Paul Lévy (1964, 89) remarked, as pointed out by van Heijenoort (1967, 368): "Zermelo told me in 1928 that even in Germany nobody understood what Hilbert meant".

[40] See note 19 for Hilbert's attitude about consistency and truth. With the metamathematical viewpoint slow to filter into mathematical practice only Nikolai Luzin (1933) among the early commentators saw that Hilbert's argument was really aimed at the consistency of the Continuum Hypothesis. To Gödel (1939b, 129) this was clear: "the first to outline a *program* for a consistency proof of the continuum hypothesis was *Hilbert*".

[41] This example occurred in Hilbert's lectures and in his (1923). For natural numbers a with $\sqrt{a}$ irrational, it was unknown then whether $a^{\sqrt{a}}$ is rational or not. The seventh of Hilbert's (1900) problems was to establish that if α is an algebraic number and β an algebraic irrational, then α^β is transcendental, or at least irrational. This problem was to stimulate the development of transcendental number theory. Aleksander Gel'fond (1934) and Theodor Schneider (1934) independently solved the problem by showing that under the hypotheses (and excluding the trivial cases $\alpha \neq 0, 1$) α^β is in fact transcendental. See Tijdeman (1976) for more on Hilbert's seventh problem.

[42] However, Hilbert never claimed that there is an algorithm, a general method, for solving

every mathematical problem. Indeed, he asserted in (1926) (as translated in van Heijenoort (1967, 384) that there is no "general method for solving every mathematical problem; that does not exist". Presumably neither Hilbert nor any of his school thought that a positive solution to the decision problem for first-order logic would yield such an algorithm.
[43] The lemma states in full (as translated in van Heijenoort 1967, 391):

Lemma II. In the formation of functions of a number-theoretic variable transfinite recursions are dispensable; in particular, not only does ordinary recursion (that is, the one that proceeds on a number-theoretic variable) suffice for the actual formation process of the functions, but also the substitutions call merely for those variable-types whose definition requires only ordinary recursion. Or, to express ourselves with greater precision and more in the spirit of our finitist attitude, if by adducing a higher recursion or a corresponding variable-type we have formed a function that has only an ordinary number-theoretic variable as argument, then this function can always be defined also by means of ordinary recursions and the exclusive use of Z-types.

[44] Both Myhill (1953) and Routledge (1953) pointed out that the natural hierarchy generating the recursive functions already terminates in ω stages. Kleene (1958) formulated a hierarchy of recursive functions which may be closer to Hilbert's intentions. Hilbert had argued that his scheme leads to new functions by applying "Cantor's diagonal procedure" on a recursive enumeration of the functions previously constructed. Kleene's hierarchy is based on enumeration and diagonalization, the former according to a fixed system of primitive recursive codes for well-orderings ("Kleene's $\mathcal{O}$"). Feferman (1962) showed that Kleene's hierarchy encompasses all the recursive functions. He showed moreover that such hierarchies terminate rather quickly so that they do not provide an informative hierarchical analysis of the general recursive functions. In his later years Gödel considered providing such an analysis to be a major problem of mathematical logic.
[45] Clifford Spector (1962) extended the *Dialectica* interpretation to full analysis, bringing in certain basic ideas of Brouwer.
[46] Alfred Tarski shares the honor.
[47] Actually, the assertion that for first-order theories categoricity implies deductive completeness is largely vacuous, since a now well-known consequence of the Compactness Theorem is that *any first-order theory with infinite models is not categorical* However, call a first-order theory $\aleph_0$*-categorical iff* it has a unique countably infinite model up to isomorphism. Then by the argument given in the text as sharpened by the Löwenheim-Skolem Theorem, *for first-order theories* $\aleph_0$*-categoricity implies deductive completeness*. This assertion is not vacuous, and also applicable to the distinction to be made in the text between first-order and higher order logics.
[48] For this view Gödel (1933, 46) mainly acknowledged von Neumann (1929), although Zermelo (1930) would have been a better source.
[49] Years later in 1968 Gödel wrote to Hao Wang (1974, 8ff.): "there was a special obstacle which *really* made it *practically impossible* for constructivists to discover my consistency proof. It is the fact that the ramified hierarchy, which had been invented *expressly for constructive purposes*, had to be used in an *entirely nonconstructive way*." Gödel (1947, 518) mentioned in a footnote that the transfinite iteration of the procedure for constructing sets in Weyl (1918) results exactly in the real numbers of L.
[50] He further said (1940a, 178): "So since an axiom of reducibility holds for constructible sets it is not surprising that the axioms of set theory hold for the constructible sets, because the axiom of reducibility or its equivalents, e.g., Zermelo's Aussonderungsaxiom, is really the only essential axiom of set theory".

[51] In (1940a, 176), Gödel wrote: "One may at first doubt that this assertion [A] has a meaning at all, because A is apparently a metamathematical statement since it involves the manifestly metamathematical term 'definable' or 'constructible'. But now it has been shown in the last few years how metamathematical statements can be translated into mathematics, and this applies also to the notion of constructibility and the proposition A, so that its consistency with the axioms of mathematics is a meaningful assertion."
[52] See Kanamori (1994) for the recent work in set theory on large cardinal hypotheses.
[53] This appendix is mostly drawn from Simpson (1985), to which we refer for more details and references. See also Simpson (1988).
[54] "Arithmetic" here refers to number theory, the structure of addition and multiplication of the natural numbers. As mentioned in Section 3, Hilbert used "arithmetic" to refer to analysis, which in the present setting corresponds to "second-order arithmetic" if sets of natural numbers are construed as real numbers.
[55] The full induction scheme, which is *not* assumed, is: For all formulas φ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n).$$

The subscript 0 in the acronyms for the subsystems distinguished below is an evolutionary artifact, indicating that only the induction axiom is being assumed and not the full scheme.
[56] See the previous note for the use of the subscript 0.
[57] A recursive function $f : \omega \rightarrow \omega$ has the Kleene normal form $f(i) = U(\mu m(T(i,m) = 0))$ where U and T are primitive recursive functions and μ is the least number operator, specifying the least m such that $T(i,m) = 0$. That f is *total* is the assertion $\forall i \exists m T(i,m) = 0$, and f is *provably total* in a system of arithmetic if that system proves this assertion.
[58] Let $\prec$ be a primitive recursive ordering of the natural numbers which is isomorphic to the ordinal ϵ_0 (cf. note 36). For an ordinal $\alpha \leq \epsilon_0$, "α is well-ordered" is the Π^1_1 assertion that every set consisting of natural numbers corresponding via $\prec$ to ordinals less than α has a $\prec$-least element.
[59] See notes 36 and 58 for the terminology.

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