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Does GCH Imply AC Locally?

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We explore how the Generalized Continuum Hypothesis might imply the Axiom of Choice *locally*. The focal question is whether for any set X, if there is no strictly intermediate cardinality between $|X|$ and $2^{|X|}$, then X is well-orderable. This question remains unresolved, and we describe related results and relative consistencies, taking the opportunity to survey themes from classical to modern set theory.

That the Generalized Continuum Hypothesis (GCH) implies the Axiom of Choice (AC) is a classical result of set theory. A sharp, local version of this result remains open and serves here as the focus for a largely expository survey, one which is almost completely self-contained and weaves together basic concepts and results of classical set theory as refined by modern techniques and sensibilities. The emphasis here is on simple, transparent arguments, and most of this paper can be absorbed with surprisingly little sophistication in set theory. With his emphasis on the concrete and the combinatorial in mathematics, Paul Erdős would presumably have enjoyed this very arithmetical approach to the study of infinite cardinality.

We proceed in Zermelo-Fraenkel set theory (ZF), quickly recapitulating the theory of cardinality in the setting without AC: $|X|$ denotes the *cardinal* of the set X, and the fraktur letters $\mathfrak{m}, \mathfrak{n}, \ldots$ are used to denote cardinals. Informally, $|X|$ is the collection of all sets Y that have a bijection with X, but then $|X|$ is a proper class when X is not empty. Formally, one can avoid quantification over these classes altogether by recasting the theory in terms of sets and bijections. Officially, we can adopt Scott's trick, letting $|X|$ be the collection of those sets Y of *minimal rank* bijective with X; such collections are sets, and so we can quantify over them. A cardinal $\mathfrak{m}$ is *finite iff* it is the cardinal of a natural number, else it is *infinite;* finite cardinals are identified with their natural numbers.

The arithmetic of cardinals is essentially as Cantor defined it: With $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$, $\mathfrak{m} + \mathfrak{n}$ is the cardinal of the union of disjoint copies of X and Y; $\mathfrak{m} \cdot \mathfrak{n}$ is the cardinal of the Cartesian product of X and Y; and $\mathfrak{m}^{\mathfrak{n}}$ is the cardinal of the set of functions from Y into X. In particular, the power set $\mathcal{P}(X)$ of the set X has cardinality $|\mathcal{P}(X)| = 2^{|X|}$. Continuing, $\mathfrak{m} \leq \mathfrak{n}$ *iff* there is an injection of X into Y, and $\mathfrak{m} < \mathfrak{n}$ *iff* $\mathfrak{m} \leq \mathfrak{n}$ yet $\mathfrak{m} \neq \mathfrak{n}$. Beyond the most straightforward results, the two main results of classical set theory are Cantor's Theorem that $\mathfrak{m} < 2^{\mathfrak{m}}$, and the Schröder-Bernstein Theorem that $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{n} \leq \mathfrak{m}$ implies $\mathfrak{m} = \mathfrak{n}$.

Generalizing Cantor's Continuum Hypothesis, we take CH ($\mathfrak{m}$) to be the proposition that there is no intermediate cardinal between $\mathfrak{m}$ and $2^{\mathfrak{m}}$ and GCH to be the proposition that this holds for all infinite $\mathfrak{m}$:

$$\mathrm{CH}(\mathfrak{m}): \ \neg\exists\mathfrak{n}(\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}).$$

$$\mathrm{GCH}: \ \forall\mathfrak{m}(\mathfrak{m} \text{ is infinite} \longrightarrow \mathrm{CH}(\mathfrak{m})).$$

To connect with the Axiom of Choice, we take WO ($\mathfrak{m}$) to be the proposition that $\mathfrak{m}$ "is an aleph":

$$\mathrm{WO}(\mathfrak{m}): \ \mathfrak{m} = |X| \text{ for some infinite, well-orderable set } X.$$

The Greek letters $\kappa, \lambda, \ldots$ are used to denote such cardinals; they themselves fit into a well-ordered sequence $\aleph_0, \aleph_1, \ldots$; and $\forall\mathfrak{m}(\mathfrak{m}$ is infinite $\longrightarrow$ WO $(\mathfrak{m}))$ is a reformulation of AC.

Turning finally to the connection between GCH and AC, Lindenbaum–Tarski [6] announced the at first surprising result that GCH implies AC. Only two decades later did Sierpiński [8] provide the first published proof. Sierpinski actually established that CH $(\mathfrak{m}) \wedge$ CH $(2^{\mathfrak{m}}) \wedge$ CH $(2^{2^{\mathfrak{m}}})$ implies WO $(\mathfrak{m})$. Lindenbaum and Tarski had announced similar local implications. Then Specker [10] sharpened these various results to establish:

$$\mathrm{CH}(\mathfrak{m}) \ \wedge \ \mathrm{CH}(2^{\mathfrak{m}}) \quad \text{implies} \quad \mathrm{WO}(2^{\mathfrak{m}}).$$

The conclusion *a fortiori* implies WO $(\mathfrak{m})$, and hence we have a local implication of two CH hypotheses about $|X|$ implying the well-orderability of X. This leaves open the following question, the focus of this paper:

Question 0.1. Does CH $(\mathfrak{m})$ imply WO $(\mathfrak{m})$?

§1 reviews the main idea behind Zermelo's Well-Ordering Theorem, partly to make points of independent interest bearing on the beginnings of set theory and partly to establish corollaries that will be applied in later sections. §2 is devoted to a simple, perspicuous proof of Specker's result. §3 discusses the strength of a proposition related to 0.1, CH $(\mathfrak{m})$ and $\neg$ WO $(2^{\mathfrak{m}})$. In particular, we establish an equi-consistency result with the existence of an inaccessible cardinal. Finally, §4 is devoted to a sketch of a consequence of this last proposition, a consequence that imposes further constraints on a negative answer to 0.1.

§1. On Zermelo's Well-Ordering Theorem

Zermelo's first proof of his Well-Ordering Theorem, appearing in his [13], was epochal both in articulating the Axiom of Choice and applying it in a specific way to generate well-orderings. The proof provoked considerable controversy, mainly having to do with the exacerbation of a growing conflict among mathematicians about the use of arbitrary functions (see [7], Chapter 2). Out of this to and fro would emerge Zermelo's initial axiomatization [14] and *abstract* set theory. For more on the import of Zermelo's Well-Ordering Theorem see [4], from which much of this section is drawn.

The following result isolates that part of Zermelo's proof which does not depend on AC. Several significant corollaries will be drawn that emphasize the importance of Zermelo's argument, both for classical results as well as for arguments of this paper. Well-orderings here are strict, i.e. irreflexive.

Theorem 1.1. *Suppose that $F : \mathcal{P}(X) \to X$. Then there is a unique $\langle W, \prec\rangle$ such that $W \subseteq X$, $\prec$ is a well-ordering of W, and:*

(a) *For any $x \in W$, $F(\{y \in W \mid y \prec x\}) = x$.*

(b) *$F(W) \in W$.*

Proof. Call $Y \subseteq X$ an *F-set iff* there is a well-ordering R of Y such that for each $x \in Y$, $F(\{y \in Y \mid y\,R\,x\}) = x$. The following are thus F-sets (some of which may be the same):

$$\{F(\emptyset)\}; \quad \{F(\emptyset), F(\{F(\emptyset)\})\};$$

$$\{F(\emptyset), F(\{F(\emptyset)\}), F(\{F(\emptyset), F(\{F(\emptyset)\})\})\}.$$

We shall establish:

(*) If Y is an F-set with a witnessing well-ordering R and Z is an F-set with a witnessing well-ordering S, then $\langle Y, R\rangle$ is an initial segment of $\langle Z, S\rangle$, or the converse.

(Taking $Y = Z$ it will follow that any F-set has a unique witnessing well-ordering.)

For establishing (*), we continue to follow Zermelo: By the comparability of well-orderings, we can assume without loss of generality that there is an order-preserving injection $e : Y \to Z$ with range an S-initial segment of Z. It then suffices to show that e is in fact the identity map: If not, let t be the R-least member of Y such that $e(t) \neq t$. It follows that $\{y \in Y \mid y\,R\,t\} = \{z \in Z \mid z\,S\,e(t)\}$. But then,

$$e(t) = F\big(\{z \in Z \mid z\,S\,e(t)\}\big) = F\big(\{y \in Y \mid y\,R\,t\}\big) = t,$$

a contradiction.

To conclude the proof, let W be the union of all the F-sets. Then W is itself an F-set by (*) and so, with $\prec$ its witnessing well-ordering, satisfies (a). For (b), note that if $F(W) \notin W$, then $W \cup \{F(W)\}$ would be an F-set, contradicting the definition of W. Finally, that (a) and (b) uniquely specify $\langle W, \prec\rangle$ also follows from (*). ■

Zermelo of course focused on choice functions as given by the Axiom of Choice to well-order the entire set:

Corollary 1.2 (Zermelo [13]) (The Well-Ordering Theorem). *If $\mathcal{P}(X)$ has a choice function, then X is well-orderable.*

Proof. Suppose that $G : \mathcal{P}(X) \to X$ is a choice function, and define a function $F : \mathcal{P}(X) \to X$ to "choose from complements" by: $F(Y) = G(X - Y) \in X - Y$ for $Y \neq X$, and $F(X)$ some specified member of X. Then the resulting W of the theorem must be X itself. ■

It is noteworthy that 1.1 leads to a new proof and a positive form of Cantor's Theorem $\mathfrak{m} < 2^{\mathfrak{m}}$:

Corollary 1.3. *For any $F : \mathcal{P}(X) \to X$, there are two distinct sets W and Y both definable from F such that $F(W) = F(Y)$. Hence, of course, $|X| < |\mathcal{P}(X)|$.*

Proof. Let $\langle W, \prec\rangle$ be as in 1.1, and let $Y = \{x \in W \mid x \prec F(W)\}$. Then by 1.1(a) $F(Y) = F(W)$, yet $F(W) \in W - Y$. ■

This corollary provides a *definable* counterexample $\langle W, Y\rangle$ to injectivity. In the $F : \mathcal{P}(X) \to X$ version of Cantor's diagonal argument, one would consider the definable set

$$A = \{x \in X \mid \exists Z(x = F(Z) \ \wedge\ F(Z) \notin Z)\}.$$

By querying whether or not $F(A) \in A$, one deduces that there must be some $Y \neq A$ such that $F(Y) = F(A)$. However, no such Y is provided with a *definition.*

Another notable consequence of the argument for 1.1 is that since the F there need only operate on the *well-orderable* subsets of X or on the well-orderings of subsets of X, the $\mathcal{P}(X)$ in 1.1 can be replaced by the following sets:

$$\mathcal{W}(X) = \{Z \subseteq X \mid Z \text{ is well-orderable}\}$$

$$\mathcal{O}(X) = \{R \subseteq X \times X \mid R \text{ is a well-ordering (of a subset of } X)\}$$

Consequently, by the argument for 1.3 we have:

Corollary 1.4. *For any set X,*
(a) $|X| < |\mathcal{W}(X)|$.
(b) $|X| < |\mathcal{O}(X)|$.

1.4(a) was noted, with a less direct proof, by Tarski [11]; 1.4(b) will be applied in §4.

Question 1.5. *Beyond $|\mathcal{W}(X)| \leq |\mathcal{P}(X)|$, are the order relationships among $|\mathcal{P}(X)|$, $|\mathcal{W}(X)|$, and $|\mathcal{O}(X)|$ independent of* ZF*?*

The argument for 1.1 has yet further consequences for cardinals. Remarkably, the following generalization of Cantor's Theorem was only first observed in the 1950's by Specker, using a more complicated argument.

Corollary 1.6 (Specker [10]). *For $\mathfrak{m} > 1$, $\mathfrak{m} + 1 < 2^{\mathfrak{m}}$.*

Proof. Assume that for some set X, $F : \mathcal{P}(X) \to X \cup \{a\}$ is an injection, where $a \notin X$. By interchanging values if necessary, we can assume that $F(X) = a$. Applying the argument for 1.1 to this F, we get a corresponding W. If $W \neq X$, then the argument for 1.3 leads to a counterexample for injectivity. Otherwise, $W = X$ so that X is well-orderable, and we again have a contradiction, since infinite, well-orderable X always satisfy $|X| + 1 = |X|$. ■

It is easily seen that this argument has simple augmentations that lead to stronger conclusions. Specker actually showed that $2^{\mathfrak{m}} \not\leq \mathfrak{m}^2$ for $\mathfrak{m} \geq 5$ using a diagonalization argument and deduced that for infinite $\mathfrak{m}$ and finite n, $n \cdot \mathfrak{m} < 2^{\mathfrak{m}}$. Halbeisen–Shelah [2] recently provided various improvements of Specker's non-injectibility result. Although we only need Specker's result in the sequel, we establish one of those improvements, since no extra effort is needed, using the argument for 1.1 together with diagonalization. $\mathrm{Seq}(X)$ denotes the set of finite sequences of members of X, identifiable with the set of functions from some natural number into X. For infinite, well-orderable Y we have $|Y| = |\mathrm{Seq}(Y)|$; in fact, to every infinite well-ordering of a set Y we can canonically associate a bijection between Y and $\mathrm{Seq}(Y)$.

Proposition 1.7 (Halbeisen–Shelah [2]). *If* $\aleph_0 \leq |X|$, *then* $|\mathcal{P}(X)| \not\leq |\mathrm{Seq}(X)|$.

Proof. Assume to the contrary that there is an injection $G : \mathcal{P}(X) \to \mathrm{Seq}(X)$. Using G we shall define a function F on well-orderings of infinite subsets of X such that:

(∗) If R is a well-ordering of an infinite subset Y of X, then $F(R) \in X - Y$.

With such an F we can start with an infinite well-ordering R_0 as given by $\aleph_0 \leq |X|$ and apply the argument for 1.1 to extend R_0 to a well-ordering W of all of X (cf. 1.2, 1.4). But $F(W) \notin X$ by (∗), which is a contradiction.

To establish (∗), let R be a well-ordering of an infinite subset Y of X. As noted above, we can canonically associate to R a bijection $H : Y \to \mathrm{Seq}(Y)$. Now let

$$D = \left\{ x \in Y \mid G^{-1}(H(x)) \quad \text{is defined} \quad \wedge \ x \notin G^{-1}(H(x)) \right\}.$$

If $G(D) \in \mathrm{Seq}(Y)$, then $H(x_0) = G(D)$ for some $x_0 \in Y$, and we would have the paradigmatic diagonal contradiction:

$$x_0 \in D \quad \text{iff} \quad x_0 \notin D.$$

Hence, $G(D) \notin \mathrm{Seq}(Y)$. But then, we can define $F(Y)$ to be the least member of the sequence $G(D)$ not in Y. ■

Diagonalization first enters our exposition only here, since by 1.3 it is not needed to establish Cantor's Theorem. Both 1.6 and 1.7 will be applied in the next section.

§2. GCH implies AC

In this section we present a simple, perspicuous proof of Specker's theorem, the currently best known local implication from CH hypotheses to well-orderability. Such implications all turn on how CH hypotheses entail the existence of bijections and on the well-known concept of the *Hartogs' Aleph* of $\mathfrak{m}$:

$$\aleph(\mathfrak{m}) \text{ is the least cardinal } \kappa \text{ such that } \kappa \not\leq \mathfrak{m}.$$

With $|X| = \mathfrak{m}$, $\aleph(\mathfrak{m})$ corresponds to the supremum of the well-orderings of subsets of X, i.e. the supremum of the members of $\mathcal{O}(X)$, defined before 1.4. Hartogs [3] himself showed how this supremum can be constituted as a set in Zermelo's [14] axiomatization. We have

$$2^{\aleph(\mathfrak{m})} \leq 2^{2^{(\mathfrak{m}^2)}},$$

since with $|X| = \mathfrak{m}$, $|Y| = \aleph(\mathfrak{m})$, and W a well-ordering of Y, any *subset* Z of Y can be injectively associated with the *set* of all well-orderings $\subseteq X \times X$ isomorphic to some initial segment determined by some member of Z.

Theorem 2.1 (Specker [10]). $\mathrm{CH}(\mathfrak{m})$ *and* $\mathrm{CH}(2^{\mathfrak{m}})$ *implies* $2^{\mathfrak{m}} = \aleph(\mathfrak{m})$, *and so* $\mathrm{WO}(2^{\mathfrak{m}})$.

The theorem will follow from a calculation based on two lemmas of wider applicability:

Lemma 2.2. *If* $\mathrm{CH}(\mathfrak{m})$, *then* $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m}$.

Proof. By 1.6 and $\mathrm{CH}(\mathfrak{m})$, we have $\mathfrak{m}+1 = \mathfrak{m}$. Hence, if $|X| = \mathfrak{m}$, there is a injection f of X into a proper subset of itself (X is "Dedekind-infinite") and so X has a countable subset, namely $\{f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots\}$ for an $x_0 \in X$ not in the range of f. The consequent $\aleph_0 \leq \mathfrak{m}$ allows us to apply 1.7 in what follows.

Next,

$$\mathfrak{m} \leq \mathfrak{m} + \mathfrak{m} \leq 2^{\mathfrak{m}} + 2^{\mathfrak{m}} = 2^{\mathfrak{m}+1} = 2^{\mathfrak{m}}$$

using $\mathfrak{m} + 1 = \mathfrak{m}$ at the end. But $\mathfrak{m} + \mathfrak{m} = 2^{\mathfrak{m}}$ easily contradicts 1.7, and so by $\mathrm{CH}(\mathfrak{m})$ we can conclude that $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}$.

Finally,

$$\mathfrak{m} \leq \mathfrak{m} \cdot \mathfrak{m} \leq 2^{\mathfrak{m}} \cdot 2^{\mathfrak{m}} \leq 2^{\mathfrak{m}+\mathfrak{m}} = 2^{\mathfrak{m}},$$

using $\mathfrak{m}+\mathfrak{m}=\mathfrak{m}$ at the end. But $\mathfrak{m}\cdot\mathfrak{m}=2^{\mathfrak{m}}$ easily contradicts 1.7, and so by CH$(\mathfrak{m})$ we can conclude that $\mathfrak{m}\cdot\mathfrak{m}=\mathfrak{m}$. ■

Lemma 2.3. *If* $\mathfrak{m}+\mathfrak{m}=\mathfrak{m}$ *and* $\mathfrak{m}+\mathfrak{n}=2^{\mathfrak{m}}$, *then* $\mathfrak{n}=2^{\mathfrak{m}}$.

Proof. $2^{\mathfrak{m}}\cdot 2^{\mathfrak{m}}=2^{\mathfrak{m}+\mathfrak{m}}=2^{\mathfrak{m}}=\mathfrak{m}+\mathfrak{n}$. With $|X|=\mathfrak{m}$ and $|Y|=\mathfrak{n}$ and $X\cap Y=\emptyset$ there is a corresponding bijection

$$f:\mathcal{P}(X)\times\mathcal{P}(X)\to X\cup Y.$$

By Cantor's Theorem there must be an $X_0\in\mathcal{P}(X)$ that does not serve as the first coordinate of any member of the preimage $f^{-1}(X)$. But then, the restriction of f to pairs with first coordinate X_0 induces an injection of $\mathcal{P}(X)$ into Y, so that $2^{\mathfrak{m}}\le\mathfrak{n}$. The conclusion now follows by the Schröder–Bernstein Theorem. ■

Proof of 2.1. Using 2.2 to get $2^{\aleph(\mathfrak{m})}\le 2^{2^{(\mathfrak{m}^2)}}=2^{2^{\mathfrak{m}}}$, we have

$$2^{\mathfrak{m}}\le 2^{\mathfrak{m}}+\aleph(\mathfrak{m})<2^{(2^{\mathfrak{m}}+\aleph(\mathfrak{m}))}=2^{2^{\mathfrak{m}}}\cdot 2^{\aleph(\mathfrak{m})}=2^{2^{\mathfrak{m}}}\cdot 2^{2^{\mathfrak{m}}}=2^{2^{\mathfrak{m}}+2^{\mathfrak{m}}}=2^{2^{\mathfrak{m}}},$$

using $2^{\mathfrak{m}}+2^{\mathfrak{m}}=2^{\mathfrak{m}+1}=2^{\mathfrak{m}}$ at the end. Hence, $2^{\mathfrak{m}}+\aleph(\mathfrak{m})=2^{\mathfrak{m}}$ by CH$(2^{\mathfrak{m}})$, so that $\aleph(\mathfrak{m})\le 2^{\mathfrak{m}}$. But then, $\mathfrak{m}<\mathfrak{m}+\aleph(\mathfrak{m})\le 2^{\mathfrak{m}}+2^{\mathfrak{m}}=2^{\mathfrak{m}}$ implies that $\mathfrak{m}+\aleph(\mathfrak{m})=2^{\mathfrak{m}}$ by CH$(\mathfrak{m})$. Hence, $\aleph(\mathfrak{m})=2^{\mathfrak{m}}$ by 2.3. ■

§3. On CH$(\mathfrak{m})$ and $\neg$WO$(2^{\mathfrak{m}})$

Absent a direct argument for an affirmative answer to the main question 0.1, what remains is to try to establish the consistency of CH$(\mathfrak{m})$ and $\neg$WO$(\mathfrak{m})$ by forcing. In this section and the next we consider what constraints are already imposed on this project by considering the consequences of a weaker hypothesis which *is* known to be consistent, namely CH$(\mathfrak{m})$ and $\neg$WO$(2^{\mathfrak{m}})$. Indeed, it is well-known that the Axiom of Determinacy implies CH$(\aleph_0)$ and $\neg$WO$(2^{\aleph_0})$. In this section we observe that this proposition is consistent relative to just ZF (see 3.3(c)). Moreover, we observe that with a local choice principle adjoined, we can get an equi-consistency result with the existence of an inaccessible cardinal (3.2 and 3.3(a)(b)).

The following simple observation imposes a limitation on forcing possibilities.

Proposition 3.1. *Suppose* CH$(\mathfrak{m})$, CH$(\mathfrak{n})$, *and* $\mathfrak{m}<\mathfrak{n}$. *Then* $2^{\mathfrak{m}}\le\mathfrak{n}$. *In particular, there is at most one* κ, *a cardinal of a well-orderable set, such that* CH(κ) *and* $\neg$WO(2^{κ}).

Proof. With $\mathfrak{m}<\mathfrak{n}$ we have

$$\mathfrak{n}\le 2^{\mathfrak{m}}+\mathfrak{n}\le 2^{\mathfrak{n}}+2^{\mathfrak{n}}=2^{\mathfrak{n}+1}=2^{\mathfrak{n}},$$

since $\mathfrak{n}+1=\mathfrak{n}$ by 2.2. It follows that either $2^{\mathfrak{m}}+\mathfrak{n}=\mathfrak{n}$ or $2^{\mathfrak{m}}+\mathfrak{n}=2^{\mathfrak{n}}$. But the latter implies $2^{\mathfrak{m}}=2^{\mathfrak{n}}$ by 2.2 and 2.3, so that $\mathfrak{m}<\mathfrak{n}<2^{\mathfrak{m}}$ contradicting CH$(\mathfrak{m})$. Hence, we must have $2^{\mathfrak{m}}+\mathfrak{n}=\mathfrak{n}$, and so $2^{\mathfrak{m}}\le\mathfrak{n}$.

The latter statement of 3.1 follows by assuming that $\mathfrak{m}$ and $\mathfrak{n}$ are two cardinals of well-orderable sets; $2^{\mathfrak{m}}\le\mathfrak{n}$ would then contradict $\neg$WO$(2^{\mathfrak{m}})$. ■

The next, more substantial results have to do with the metamathematical interplay with inner models of set theory. M is an *inner model iff* it is a transitive class containing all the ordinals such that for any axiom σ of ZF, ZF establishes σ^M, the relativization of σ resulting from restricting its quantifiers to M. M is an *inner model of* ZFC (ZF together with AC) *iff* it is an inner model such that ZF also establishes AC^M. Gödel's constructible universe L is the archetypal inner model of ZFC.

The following local versions of the Axiom of Choice will be germane:

AC$(\mathfrak{m})$: Every family of non-empty sets indexed by a set of cardinal $\mathfrak{m}$ has a choice function.

AC$(<\mathfrak{m})$: $\forall\mathfrak{n}\big(\mathfrak{n}<\mathfrak{m}\longrightarrow \mathrm{AC}(\mathfrak{n})\big)$.

Also, a cardinal $\mathfrak{m}$ is *regular iff* no X with $|X|=\mathfrak{m}$ is a union of form $\bigcup_{i\in Y}X_i$ where $|Y|<|X|$ and each $|X_i|<|X|$. $\mathfrak{m}$ is a *strong limit iff* whenever $\mathfrak{n}<\mathfrak{m}$, $2^{\mathfrak{n}}<\mathfrak{m}$. Finally, $\mathfrak{m}$ is *inaccessible iff* $\mathfrak{m}$ is both regular and a strong limit.

Although we did not need to do this until now, whenever WO$(\mathfrak{m})$ we henceforth *identify* $\mathfrak{m}$, as usually done, with its least (von Neumann) ordinal and regard our $<$ as extended to ordinals. In an inner model of ZFC every cardinal is thus an ordinal, but of course that ordinal may not be a cardinal in the full universe as there may be more bijections.

Theorem 3.2. *Suppose* CH$(\mathfrak{m})$, $\neg$WO$(2^{\mathfrak{m}})$, *and* M *is an inner model of* ZFC. *Then* $\aleph(\mathfrak{m})$ *is a strong limit cardinal in the sense of* M. *If in* *addition* AC $(<\aleph(\mathrm{m}))$, *then* $\aleph(\mathrm{m})$ *is regular, and hence in the sense of* M, *inaccessible.*

Proof. Suppose that $|X| = \mathrm{m}$ and α is an ordinal less than $\aleph(\mathrm{m})$. There is thus a $Y \subseteq X$ well-orderable in ordertype α, and so a $Z \subseteq \mathcal{P}(Y)$ well-orderable in ordertype type $\beta = (2^\alpha)^M$, the cardinal of the power set of α in the sense of M. Clearly, $|\beta| \leq 2^{\mathrm{m}}$ by embeddability. We thus have $\mathrm{m} \leq |\beta| + \mathrm{m} \leq 2^{\mathrm{m}}$. But $|\beta| + \mathrm{m} = 2^{\mathrm{m}}$ would imply by 2.2 and 2.3 that $|\beta| = 2^{\mathrm{m}}$, contradicting $\neg\,\mathrm{WO}\,(2^{\mathrm{m}})$. Hence, $\mathrm{m} = |\beta| + \mathrm{m}$, and so $|\beta| \leq \mathrm{m}$, i.e. $(2^\alpha)^M < \aleph(\mathrm{m})$ by definition of $\aleph(\mathrm{m})$.

Suppose next that AC $(<\aleph(\mathrm{m}))$ holds. Assume to the contrary that, with the identification with ordinals, $\aleph(\mathrm{m}) = \bigcup_{\alpha<\lambda}\kappa_\alpha$ where λ and each κ_α is less than $\aleph(\mathrm{m})$. Let $\langle y_\alpha \,|\, \alpha < \lambda\rangle$ be a sequence of distinct elements of X. Also, using AC (λ) let $\langle R_\alpha \,|\, \alpha < \lambda\rangle$ be a sequence with each R_α a well-ordering in ordertype κ_α of a corresponding $Y_\alpha \subseteq \{y_\alpha\} \times X$. Then $\bigcup_{\alpha<\lambda} Y_\alpha \subseteq X \times X$ is well-orderable in ordertype $\aleph(\mathrm{m})$ by taking the lexicographic "sum" of the well-orderings. However, this contradicts $\mathrm{m}^2 = \mathrm{m}$ from 2.2. ■

The next theorem provides converses for the necessary implications about inner models given by 3.2; we in particular get an equi-consistency result with the existence of an inaccessible cardinal. The theorem requires familiarity with forcing, in particular the Levy collapse and Solovay's model for "every set of reals is Lebesgue measurable". Solovay [9] started with the universe satisfying ZFC + "there is an inaccessible cardinal" and Levy collapsed such a cardinal κ to $\aleph_1$, i.e. applied Levy's idea of forcing every cardinal less than κ to be countable. Solovay then took the inner model HOD $({}^\omega\,\mathrm{ON})$ of the sets hereditarily definable from some countable sequence of ordinals and established that every set of reals is Lebesgue measurable there. Solovay's procedure can also be applied to the Levy collapse of κ to $\aleph_2$ and the taking of the inner model HOD $({}^{\omega_1}\mathrm{ON})$ of the sets hereditarily definable from some ω_1 sequence of ordinals. Finally, the Levy collapse was first applied to collapse $\aleph_\omega$ to $\aleph_1$ in a "symmetric" fashion (not adding the entire sequence of collapsing functions but only adding the functions individually) by Fefferman–Levy [1] to get a model of ZF + "the set of reals is a countable union of countable sets."

Theorem 3.3.

(a) *Suppose that an inaccessible cardinal* κ *is Levy collapsed to* $\aleph_1$. *Then*

$$\mathrm{HOD}\,({}^\omega\,\mathrm{ON}) \models \mathrm{CH}\,(\aleph_0) \;\wedge\; \neg\,\mathrm{WO}\,(2^{\aleph_0}) \;\wedge\; \aleph(\aleph_0) = \aleph_1 \;\wedge\; \mathrm{AC}\,(\aleph_0).$$

(b) *Suppose that an inaccessible cardinal* κ *is Levy collapsed to* $\aleph_2$. *Then*

$$\mathrm{HOD}\,({}^{\omega_1}\mathrm{ON}) \models \mathrm{CH}\,(\aleph_1) \;\wedge\; \neg\,\mathrm{WO}\,(2^{\aleph_1}) \;\wedge\; \aleph(\aleph_1) = \aleph_2 \;\wedge\; \mathrm{AC}\,(\aleph_1).$$

(c) *Suppose that* $\aleph_\omega$ *is a strong limit. Then in the symmetric Levy collapse* M *of* $\aleph_\omega$ *to* $\aleph_1$,

$$M \models \mathrm{CH}\,(\aleph_0) \;\wedge\; \neg\,\mathrm{WO}\,(2^{\aleph_0}) \;\wedge\; \aleph(\aleph_0) = \aleph_1 \;\wedge\; \neg\,\mathrm{AC}\,(\aleph_0).$$

Proof (Sketch).

(a) is a procedural observation about Solovay's model. The model in fact satisfies the Perfect Set Property, which is stronger than CH $(\aleph_0)$, and the Principle of Dependent Choices, which is stronger than AC $(\aleph_0)$.

(b) is established by adapting Solovay's argument. For CH $(\aleph_1)$, one establishes the direct analogue of the Perfect Set Property for subsets of $\aleph_1$. (In terms of Kanamori [4], one first establishes the analogue for 11.12 and then proceeds with the argument on p.141 for building a perfect tree.) The rest follows straightforwardly from properties of HOD $({}^{\omega_1}\mathrm{ON})$.

(c) is a procedural observation about the symmetric Levy collapse. That $\aleph_\omega$ is a strong limit is needed to establish CH $(\aleph_0)$. Moreover, AC $(\aleph_0)$ fails in the strong sense that the set of reals is a countable union of countable sets. ■

3.3(b) shows that the specificity in 3.3(a) of $\kappa = \aleph_0$ for CH $(\kappa) \wedge \neg\,\mathrm{WO}\,(2^\kappa)$ is not peculiar, and 3.3(c) also has a similar analogue.

§4. A Long Sequence of Cardinals

In this final section, we provide a sketch of a technical result that establishes a further constraint imposed by CH (m) and $\neg\,\mathrm{WO}\,(2^{\mathrm{m}})$. The result is a strong negation of Specker's result 2.1, which in a contraposition asserts that CH (m) and $\neg\,\mathrm{WO}\,(2^{\mathrm{m}})$ implies that there is at least one cardinal intermediate between 2^{m} and $2^{2^{\mathrm{m}}}$. Recall that for an ordinal α, the ***cofinality*** of α, denoted cf (α), is the least ordinal β such that there is function: $\beta \to \alpha$ whose range is cofinal in α.

Theorem 4.1. *Suppose that* CH(m) *and* $\neg$WO(2^{m}). *Then there is an increasing sequence of cardinals of length* cf$(\aleph(\mathrm{m}))$ *between* 2^{m} *and* $2^{2^{\mathrm{m}}}$.

Proof (Sketch). For each ordinal ξ, let $\mathcal{O}^{\xi}$ denote ξ iterations of the operation $\mathcal{O}$ defined before 1.4, with unions taken at limits. For cardinals m, set $\mathcal{O}^{\xi}(\mathrm{m}) = |\mathcal{O}^{\xi}(X)|$ for some set X with $|X| = \mathrm{m}$. By 2.2 we have: $\mathcal{O}(\mathrm{m}) \leq 2^{(\mathrm{m}^2)} = 2^{\mathrm{m}}$. By applications of 1.4 we then have

$$\mathrm{m} < 2^{\mathrm{m}} = \mathcal{O}(\mathrm{m}) < \mathcal{O}^2(\mathrm{m}) \ldots.$$

We proceed to establish a series of properties about these $\mathcal{O}^{\xi}(\mathrm{m})$'s. First, a simple transfinite induction shows:

(i) For any ordinal ξ, $|\xi| \leq \mathcal{O}^{\xi}(\mathrm{m})$.

Second, the last part of the proof of 2.1 shows that with CH(m) one must have $\aleph(\mathrm{m}) \not\leq 2^{\mathrm{m}}$. Hence, the definition of the Hartogs' Aleph implies that $\aleph(\mathrm{m}) = \aleph(2^{\mathrm{m}})$. Since $2^{\mathrm{m}} = \mathcal{O}(\mathrm{m})$ from above, we thus have

(ii) $\aleph(\mathrm{m}) = \aleph(\mathcal{O}(\mathrm{m}))$.

Third, a simple argument about reconstituting well-orderings of well-orderings shows that in general, $\aleph(\mathrm{n}) \leq \aleph(\mathrm{n}')$ implies $\aleph(\mathcal{O}(\mathrm{n})) \leq \aleph(\mathcal{O}(\mathrm{n}'))$. Consequently,

(iii) $\aleph(\mathrm{n}) = \aleph(\mathrm{n}')$ implies $\aleph(\mathcal{O}(\mathrm{n})) = \aleph(\mathcal{O}(\mathrm{n}'))$.

The next property has a detailed, though straightforward, proof by transfinite induction.

(iv) For any infinite cardinal n and ordinal ξ, $\mathcal{O}^{\xi}(\mathrm{n}) \leq 2^{\mathrm{n}\cdot\aleph(\mathcal{O}^{\xi}(\mathrm{n}))}$.

(With $|A| = \mathrm{n}$ say, every member of $\mathcal{O}(A)$ can be canonically associated with a subset of $A \times \aleph(\mathrm{n})$. Then every member of $\mathcal{O}^2(A)$ can be canonically associated with a subset of $\aleph(\mathcal{O}(\mathrm{n})) \times A \times \aleph(\mathrm{n})$. But for (well-ordered) cardinals $\kappa \leq \lambda$ we have $\kappa \times \lambda = \lambda$, and so $|\aleph(\mathcal{O}(\mathrm{n})) \times A \times \aleph(\mathrm{n})| = |A \times \aleph(\mathcal{O}(\mathrm{n}))|$. The general argument can be based on a recursively defined representation of members of $\mathcal{O}^{\xi}(A)$. That we do not bother to give the cumbersome details is the reason why this proof is flagged as a sketch.)

Turning to the main line of argument, by (ii) and (iii) we have

$$\aleph(\mathrm{m}) = \aleph(\mathcal{O}(\mathrm{m})) = \aleph(\mathcal{O}^2(\mathrm{m})) = \ldots,$$

yet by (i) the $\mathcal{O}^{\xi}(\mathrm{m})$'s are arbitrarily large. Hence, there must be a least γ satisfying $\aleph(\mathrm{m}) < \aleph(\mathcal{O}^{\gamma}(\mathrm{m}))$, and this γ must be an infinite limit ordinal. Using (iv), we then have for $1 < \xi < \gamma$:

$$2^{\mathrm{m}} < \mathcal{O}^{\xi}(\mathrm{m}) \leq 2^{\mathrm{m}\cdot\aleph(\mathcal{O}^{\xi}(\mathrm{m}))} = 2^{\mathrm{m}\cdot\aleph(\mathrm{m})} = \left(2^{\aleph(\mathrm{m})}\right)^{\mathrm{m}} \leq \left(2^{2^{\mathrm{m}}}\right)^{\mathrm{m}} = 2^{2^{\mathrm{m}}},$$

using $2^{\mathrm{m}}\cdot\mathrm{m} \leq 2^{\mathrm{m}+1} = 2^{\mathrm{m}}$ at the end. Thus, $\langle \mathcal{O}^{\xi}(\mathrm{m}) \mid \xi < \gamma \rangle$ is an increasing sequence of cardinals between 2^{m} and $2^{2^{\mathrm{m}}}$. The following establishes the theorem:

(v) $\gamma \geq \mathrm{cf}(\aleph(\mathrm{m}))$.

To prove this, let $|X| = \mathrm{m}$. Noting that γ a limit ordinal, there is an injection $f : \aleph(\mathrm{m}) \to \bigcup_{\xi<\gamma} \mathcal{O}^{\xi}(X)$. For each $\xi < \gamma$, let η_{ξ} be that ordinal order isomorphic to the pre-image $f^{-1}(\mathcal{O}^{\xi}(X) - \bigcup_{\zeta<\xi} \mathcal{O}^{\zeta}(X))$. Let η be the ordinal sum $\Sigma_{\xi<\gamma}\eta_{\xi}$. Then $|\eta| = \aleph(\mathrm{m})$. Moreover, η is the supremum of partial sums each less than $\aleph(\mathrm{m})$, by the definition of γ. Hence $\eta = \aleph(\mathrm{m})$, and so $\gamma \geq \mathrm{cf}(\aleph(\mathrm{m}))$. ■

The question 0.1 remains. None of the forcing models in which AC fails seem applicable, and the results of this and the previous section restrict various possibilities.

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