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ULTRAFILTERS OVER A MEASURABLE CARDINAL

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0. Introduction

The extensive theory that exists on $\beta\omega$, the set of ultrafilters over the integers, suggests an analogous study of the family of κ-complete ultrafilters over a measurable cardinal $\kappa > \omega$. This paper is devoted to such a study, with emphasis on those aspects which make the uncountable case interesting and distinctive.

Section 1 is a preliminary section, recapitulating some known concepts and results in the theory of ultrafilters, while Section 2 introduces the convenient framework of Puritz for discussing elementary embeddings of totally ordered structures. Section 3 then begins the study in earnest, and introduces a function τ on ultrafilters which is a measure of complexity. Section 4 is devoted to p-points; partition properties akin to the familiar Ramsey property of normal ultrafilters are shown to yield non-trivial p-points, and examples are constructed. In Section 5 sum and limit constructions are considered; a new proof of a theorem of Solovay and a generalization are given, and it is shown that the Rudin–Frolik tree cannot have much height. Finally, Section 6 discusses filter related formulations of the well-known Jonsson and Rowbottom properties of cardinals.

The notation used in this paper is much as in the most recent set theoretical literature, but the following are specified: The letters α, β, γ, δ ... denote ordinals whereas κ, λ, μ, ν ... are reserved for cardinals. If x and y are sets, ${}^xy$ denotes the set of functions from x to y, so that κ^λ is the cardinality of ${}^\lambda\kappa$. If x is a set, $\mathcal{P}(x)$ denotes its power set. id

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will denote the identity function with the domain appropriate to the particular context, and if f and g are functions, fg denotes the application of g and then f.

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1. Ultrafilters

This initial section quickly reviews the relevant concepts, definitions and results in the theory of ultrafilters which form the basic preliminary material for the paper.

Definition 1.1. If I is an infinite set, $\mathcal{FR}_I = \{X \subseteq I \mid |I - X| < |I|\}$ is the *Frechet filter for I.* A filter $\mathcal{F}$ over I is *uniform iff* $X \in \mathcal{F} \to |X| = |I|$. Let

$$\beta I = \{\mathcal{U} \mid \mathcal{U} \text{ is an ultrafilter over } I\}$$
$$\beta_u I = \{\mathcal{U} \mid \mathcal{U} \text{ is a uniform ultrafilter over } I\}.$$

Topologically, βI with the topology generated by the sets $\{\mathcal{U} \mid X \in \mathcal{U}\}$ for $X \subseteq I$ is the Cech compactification of I (with the discrete topology), and $\beta_u I$ is a closed subspace which is identifiable with the Stone space of the Boolean algebra $\mathcal{P}(I)/\mathcal{FR}_I$.

The initial work on the structure theory of ultrafilters was done in a topological context on $\beta\omega$, the ultrafilters over the integers. However, with the advent of the ultraproduct construction in model theory, some of the attention has since focused on a more general set theoretical approach and, recently, on considerations of other index sets I. The following are some of the references in this trend: W. Rudin [30], M.E. Rudin [29], Booth [5], Kunen [21], Blass [1], and Ketonen [17]. For definiteness I state the following:

Definition 1.2. If $\mathcal{U} \in \beta I$, $i_u : V \to V^I/\mathcal{U}$ is the usual elementary embedding of the set theoretic universe into its ultrapower by $\mathcal{U}$, E_u denotes the induced membership relation for $V^I/\mathcal{U}$, and if $f : I \to V$, $[f]_u$ denotes the equivalence class of f in $V^I/\mathcal{U}$. The subscripts will often be dropped if it is clear from the context what $\mathcal{U}$ is being discussed.

The following partial order was defined independently by M.E. Rudin and Keisler, and is helpful in evaluating the complexity of ultrafilters.

Definition 1.3. The *Rudin–Keisler ordering* (RK) on ultrafilters is defined as follows:

If $\mathcal{U} \in \beta I$ and $\mathcal{V} \in \beta J$, $\mathcal{V} \leqslant \mathcal{U}$ *iff* there is a function $f : I \to J$ so that $\mathcal{V} = f_*(\mathcal{U})$, where

$$f_*(\mathcal{U}) = \{X \subseteq J \mid f^{-1}(X) \in \mathcal{U}\}.$$

Let $\mathcal{U} \cong \mathcal{V}$ *iff* both $\mathcal{U} \leqslant \mathcal{V}$ and $\mathcal{V} \leqslant \mathcal{U}$; in this case, $\mathcal{U}$ is said to be *isomorphic* to $\mathcal{V}$. Finally, let $\mathcal{U} < \mathcal{V}$ *iff* $\mathcal{U} \leqslant \mathcal{V}$ and $\mathcal{U} \not\cong \mathcal{V}$.

As $\leqslant$ is transitive, $\cong$ is an equivalence relation; the use of the term isomorphism is justified by:

Proposition 1.4. *If $\mathcal{U} \in \beta I$ and $\mathcal{V} \in \beta J$, $\mathcal{U} \cong \mathcal{V}$ iff there are $f : I \to J$, $X \in \mathcal{U}$ and $Y \in \mathcal{V}$ so that $f_*(\mathcal{U}) = \mathcal{V}$ and f restricted to X is* 1–1 *onto Y, i.e. f is* 1–1 (mod $\mathcal{U}$).

For a proof of this result and more details on $\leqslant$, see M.E. Rudin [29]. Notice that is is reasonable to consider only uniform ultrafilters; since if $\mathcal{U} \in \beta I$ and $J \in \mathcal{U}$ is of least cardinality, then by the proposition $\mathcal{U}$ is isomorphic to $\mathcal{U} \cap \mathcal{P}(J) \in \beta_u J$.

It is also interesting to note that if $\mathcal{U} \in \beta I$, $\mathcal{V} \in \beta J$ and $\mathcal{V} \leqslant \mathcal{U}$, to every f so that $f_*(\mathcal{U}) = \mathcal{V}$ there corresponds an elementary embedding $\phi : V^J/\mathcal{V} \to V^I/\mathcal{U}$; define ϕ by $\phi([g]_{\mathrm{v}}) = [gf]_{\mathrm{u}}$. Note also that the composition of i_{v} and then ϕ equals i_{u}.

The following concepts were first used in the study of $\beta_u\,\omega$; see W. Rudin [30] and Choquet [6,7].

Definitions 1.5. Let λ be regular and $\mathcal{U} \in \beta_u \lambda$.

(i) $f \in {}^\lambda\lambda$ is *unbounded* (mod $\mathcal{U}$) *iff* for every $\alpha < \lambda$ $\{\xi < \lambda \mid \alpha < f(\xi)\} \in \mathcal{U}$

(ii) $f \in {}^\lambda\lambda$ is *almost 1–1 iff* for every $\alpha < \lambda$, $|f^{-1}(\{\alpha\})| < \lambda$. $f \in {}^\lambda\lambda$ is *almost 1–1* (mod $\mathcal{U}$) *iff* there is an $X \in \mathcal{U}$ so that $f \mid X$ is almost 1–1, i.e. for every $\alpha < \lambda$

$$|f^{-1}(\{\alpha\}) \cap X| < \lambda.$$

(iii) $\mathcal{U}$ is a *p-point iff* every function $\in {}^\lambda\lambda$ unbounded (mod $\mathcal{U}$) is almost 1–1 (mod $\mathcal{U}$).

(iv) $\mathcal{U}$ is a *q-point iff* every almost 1–1 function $\in {}^\lambda\lambda$ is 1–1 (mod $\mathcal{U}$).

(v) $\mathcal{U}$ is $\beta_u\lambda$-*minimal iff* every function $\in {}^\lambda\lambda$ unbounded (mod $\mathcal{U}$) is 1–1 (mod $\mathcal{U}$).

It is obvious that $\mathcal{U}$ is $\beta_u\lambda$-minimal *iff* $\mathcal{U}$ is a p-point and a q-point. Actually, (v) is stated so that this connection is apparent, though the term $\beta_u\lambda$-minimal refers to the equivalent condition of $\mathcal{U}$ being minimal in the Rudin–Keisler ordering restricted to $\beta_u\lambda$, i e. $\mathcal{V} \leq \mathcal{U} \to \mathcal{V} \cong \mathcal{U}$ or $\mathcal{V} \notin \beta_u\lambda$. $\beta_u\lambda$-minimal ultrafilters may seem special, but it is known that if $2^\lambda = \lambda^+$, there are 2^{2^λ} of them. The following are easy generalizations of known characterizations.

Theorem 1.6. *Let λ be regular and $\mathcal{U} \in \beta_u\lambda$.*

(i) *$\mathcal{U}$ is a p-point iff whenever $\{X_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{U}$ are such that $\alpha < \beta < \lambda \to X_\beta \subseteq X_\alpha$, there is a $Y \in \mathcal{U}$ so that $|Y - X_\alpha| < \lambda$ for every $\alpha < \lambda$.*

(ii) (Kunen, see [5]) *$\mathcal{U}$ is $\beta_u\lambda$-minimal iff whenever $\{X_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{U}$ are such that $\alpha < \beta < \lambda \to X_\beta \subseteq X_\alpha$, there is a $Y \in \mathcal{U}$ so that $\alpha, \beta \in Y$ and $\alpha < \beta \to \beta \in X_\alpha$.*

Consider the following definition:

Definition 1.7. For any cardinal κ,

$$\beta_m\kappa = \{\mathcal{U} \in \beta_u\kappa \mid \mathcal{U} \text{ is } \kappa\text{-complete}\}\,.$$

κ-complete means, of course, that if $\mu < \kappa$ and $\{X_\alpha \mid \alpha < \mu\} \subseteq \mathcal{U}$, $\bigcap_{\alpha<\mu} X_\alpha \in \mathcal{U}$. Note that $\beta_m\omega = \beta_u\omega$, and that for $\kappa > \omega$, $\beta_m\kappa$ is not empty *iff* κ is a measurable cardinal. This latter case is the main subject of this paper. When considering $\mathcal{U} \in \beta_m\kappa$, several simplications are possible, for example, the sets $\{X_\alpha \mid \alpha < \kappa\}$ in 1.6 need no longer be descending, and the term $\beta_u\kappa$-minimal can be properly replaced by *minimal*, since $\mathcal{V} \leq \mathcal{U}$ and $\mathcal{V} \notin \beta_m\kappa \to \mathcal{V}$ is principal. Interesting considerations involving partitions also arise.

Proposition 1.8. *If $\mathcal{U} \in \beta_m\kappa$, $\mathcal{U}$ is minimal iff $\mathcal{U}$ is Ramsey, i.e. for any function $F : [\kappa]^2 \to 2$ there is an $X \in \mathcal{U}$ so that $|F''[X]^2| = 1$. In addition, if $\kappa > \omega$, $\mathcal{U}$ is isomorphic to a normal ultrafilter $\in \beta_m\kappa$.*

The main part follows from 1.6(ii). Thus, in $\beta_m\kappa$ for $\kappa > \omega$, below any element there is a minimal one in the RK ordering. However, Mathias [23] has shown with CH that there exist elements of $\beta_u\omega$ with not even p-points below them.

Topologically, $\beta_m \kappa$ is quite special because of its basic separation property: given distinct ultrafilters $\{\mathcal{U}_\alpha \mid \alpha < \mu\} \subseteq \beta_m \kappa$ where $\mu < \kappa$, there exists a partition $\langle X_\alpha \mid \alpha < \mu \rangle$ of κ so that $X_\alpha \in \mathcal{U}_\alpha$. Also, p-points in the context of $\beta_m \kappa$ have a topological definition equivalent to the one given: the intersection of any κ open sets containing $\mathcal{U}$ also contains a neighborhood of $\mathcal{U}$. Indeed, p-points were originally considered from this viewpoint in the theory of $\beta_u \omega$. Their topological invariance was used to prove that $\beta_u \omega$ is not homogeneous (W. Rudin [30]; CH is assumed here to get the existence of p-points).

With the exception of some elegant constructions by Kunen [21] involving independent sets, most of the interesting results in the theory of $\beta_u \omega$ depend on CH and can often be generalized to follow from Martin's Axiom (MA) as well (see for example Blass [3]). Roughly, these hypotheses allow inductive constructions which adequately take care of 2^ω conditions compatibly. Usually, if there is no direct proof of some assertion, a counterexample can thus be constructed. However, without CH or MA it is not even known whether p-points can exist.

Note that in constructing ultrafilters by gradually extending filters, the finite intersection property persists through limits as one takes unions of filters. But, κ-completeness is not preserved in general, so that such inductive methods are not available in the theory of $\beta_m \kappa$ for $\kappa > \omega$. However, there is a new advantage that offsets this somewhat: the well-foundedness of ultrapowers. This is the new factor which makes $\beta_m \kappa$ for $\kappa > \omega$ interesting and distinctive, and will be used repeatedly in this paper.

There are simple processes for constructing new ultrafilters from given ones.

Definition 1.9. Let $\mathcal{D} \in \beta I$ and $\mathcal{E}_i \in \beta J$ for $i \in I$.

(i) *The $\mathcal{D}$-limit of* $\langle \mathcal{E}_i \mid i \in I \rangle$ is the ultrafilter $\mathcal{D}\text{-}\lim \mathcal{E}_i$ over J defined by

$$X \in \mathcal{D}\text{-}\lim \mathcal{E}_i \ \text{ iff } \ \{i \mid X \in \mathcal{E}_i\} \in \mathcal{D} \ .$$

(ii) *The $\mathcal{D}$-sum of* $\langle \mathcal{E}_i \mid i \in I \rangle$ is the ultrafilter $\mathcal{D}\,\Sigma_i\, \mathcal{E}_i$ over $I \times J$ defined by

$$X \in \mathcal{D} \sum_i \mathcal{E}_i \ \text{ iff } \ \{i \mid \{j \mid \langle i, j \rangle \in X\} \in \mathcal{E}_i\} \in \mathcal{D} \ .$$

The indexing variable under the summation sign Σ will be suppressed unless it is not clear from the context what the variable is. When considering cartesian products like $I \times J$, π_1 will denote the projection onto the first coordinate, and π_2, the second coordinate.

(iii) When each $\mathcal{E}_i = \mathcal{E}$ in (ii) we get the *product of* $\mathcal{D}$ *and* $\mathcal{E}$, denoted by $\mathcal{D} \times \mathcal{E}$. For $n \in \omega$, $\mathcal{U}^n$ is defined by induction: $\mathcal{U}^n = \mathcal{U}^{n-1} \times \mathcal{U}$.

Note that sums can always be written as limits. Taking an ultrapower of a structure by $\mathcal{D}\ \Sigma\ \mathcal{E}_i$ corresponds to first taking ultrapowers by each $\mathcal{E}_i$ and then taking their ultraproduct by $\mathcal{D}$. Note that when λ is regular and each ultrafilter involved is in $\beta_u\lambda$, $\mathcal{D}\ \Sigma\ \mathcal{E}_i$ is not a p-point since the projection π_1 is not almost 1–1 mod($\mathcal{D}\ \Sigma\ \mathcal{E}_i$). Also, the product $\mathcal{D} \times \mathcal{E}$ is not a q-point either, since

$$\{\langle\alpha, \beta\rangle \mid \alpha < \beta < \lambda\} \in \mathcal{D} \times \mathcal{E}$$

and π_2 is almost 1–1 on this set, but it cannot be 1–1 (mod $\mathcal{D} \times \mathcal{E}$).

Sum considerations lead in a natural way to another partial ordering on ultrafilters, first defined by Frolik and M.E. Rudin.

Definition 1.10.

(i) If $\mathcal{E}_i \in \beta J$ for $i \in I$, $\{\mathcal{E}_i \mid i \in I\}$ is a *discrete* family of ultrafilters *iff* there is a partition $\langle X_i \mid i \in I\rangle$ of J so that $X_i \in \mathcal{E}_i$ for each $i \in I$.

(ii) The *Rudin–Frolik* ordering (RF) on ultrafilters is defined as follows: if $\mathcal{D} \in \beta I$, $\mathcal{D} \leqslant_{\mathrm{RF}} \mathcal{U}$ *iff* for some J and discrete family $\{\mathcal{E}_i \mid i \in I\} \subseteq \beta J$, $\mathcal{U} = \mathcal{D}$-$\lim \mathcal{E}_i$. $\mathcal{D} <_{\mathrm{RF}} \mathcal{U}$ *iff* $\mathcal{D} \leqslant_{\mathrm{RF}} \mathcal{U}$ and $\mathcal{D} \not\cong \mathcal{U}$.

Whenever the conditions in (ii) are satisfied for $\mathcal{D}$ and the $\mathcal{E}_i$'s, $\mathcal{D}$-$\lim \mathcal{E}_i \cong \mathcal{D}\ \Sigma\ \mathcal{E}_i$ by a simple argument using discreteness. $\leqslant_{\mathrm{RF}}$ is well defined for Rudin–Keisler equivalence classes of ultrafilters. It is known that $\leqslant_{\mathrm{RF}}$ is a sub-ordering of the Rudin–Keisler ordering, but the most interesting fact about it is the following.

Theorem 1.11. (M.E. Rudin) *The Rudin–Frolik ordering restricted to elements of $\beta_m\kappa$ for some κ is a tree, i.e. the predecessors of any element are linearly ordered.*

For a proof, see Booth [5] or Blass [1]; a more detailed formulation will appear later (5.5). Frolik [11] used the topological nature of $\leqslant_{\mathrm{RF}}$ to show without CH that $\beta_u\omega$ is not homogeneous. Booth [5] later showed that there are elements of $\beta_u\omega$ with an infinite number of $\leqslant_{\mathrm{RF}}$ predecessors, and even a $\leqslant_{\mathrm{RF}}$ chain isomorphic to the reals. No such results exist for $\kappa > \omega$.

The following proposition is stated here for future reference.

Proposition 1.12. *A family of κ distinct p-points $\in \beta_m\kappa$ is a discrete family.*

Proof. It suffices to establish the following fact: If $\mathcal{U} \in \beta_m \kappa$ is a *p-point* and for $\alpha < \kappa$ $\mathcal{U}_\alpha \neq \mathcal{U}$ and $\mathcal{U}_\alpha \in \beta_u \kappa$, then there is an $X \in (\mathcal{U} - \bigcup\{\mathcal{U}_\alpha \mid \alpha < \kappa\})$. An easy inductive argument using κ-completeness can then be used to construct the partition of κ that demonstrates discreteness.

So, let $X_\alpha \in \mathcal{U} - \mathcal{U}_\alpha$. By 1.6(i) and κ-completeness of $\mathcal{U}$, there is an $X \in \mathcal{U}$ so that $|X - X_\alpha| < \kappa$ for each $\alpha < \kappa$. But then $X \notin \mathcal{U}_\alpha$ for $\alpha < \kappa$, since $\mathcal{U}_\alpha$ is uniform, and we are done.

Note that we actually proved that no p-point in $\beta_m \kappa$ is a limit of κ or fewer other elements in $\beta_u \kappa$, a consequence of the topological definition of p-points. This immediately implies that p-points in $\beta_m \kappa$ are minimal in the Rudin–Frolik ordering. Kunen [19] has proved with CH that the converse is not true for $\kappa = \omega$.

Turning briefly to filters, there will be occasion to use the following.

Definition 1.13. Let λ be regular.

(i) A filter $\mathcal{F}$ over λ is a *q-point filter iff* whenever $f \in {}^\lambda\lambda$ is almost $1-1$, f is $1-1$ on a set in $\mathcal{F}$.

(ii) If $\lambda > \omega$, $\mathcal{C}_\lambda$ denotes the λ-complete filter generated by the closed unbounded subsets of λ.

Stationary subsets of λ are just those with positive $\mathcal{C}_\lambda$ measure, and a well known result of Fodor states that any function regressive (i.e. strictly less than the identity function) on a stationary set is constant on a stationary subset.

The following are some large cardinal definitions, special cases of which will be used.

Definition 1.14.

(i) κ is λ-*supercompact iff* there is an elementary embedding $j : V \to M$ of the universe into a transitive subclass so that:

(a) $j(\alpha) = \alpha$ for $\alpha < \kappa$, but $\kappa < j(\kappa)$.

(b) M is closed under λ sequences, i.e. if $\langle x_\alpha \mid \alpha < \lambda\rangle \subseteq M$, $\langle x_\alpha \mid \alpha < \lambda\rangle \in M$.

(ii) κ is λ-*compact iff* every κ-complete filter over λ can be extended to a κ-complete ultrafilter over λ.

For details, see Keisler–Tarski [14] and Solovay–Reinhardt–Kanamori [28]. A different, but equally natural, definition of λ-compact is often seen in the literature.

2. Skies and constellations

This section introduces some concepts essentially due to Puritz for discussing elementary embeddings of totally ordered structures. I present the situation in some generality to suggest potential uses in model theory, but with enough speciality so that cumbersome notation can be avoided and direct applications are possible in succeeding sections. Thus for example, only regular cardinals as domains will be considered. The minimal structure on such a cardinal κ adequate for the discussion seems to be $\langle \kappa, \in, X \dots \rangle$ where the $X \dots$ are names for every unary and binary relation on κ. In particular, every subset of κ and function $\in {}^\kappa\kappa$ has a name. So, let

$$\mathrm{i} : \langle \kappa, \in, X \dots \rangle \to \langle \mathrm{i}(\kappa), E, \mathrm{i}(X) \dots \rangle$$

be any elementary embedding (where the second structure need not be E-well founded). To suggest ordinality $<$ will be used both for $\in$ between elements of κ and E between E-members of $\mathrm{i}(\kappa)$; similarly, $\leqslant, =, >, \geqslant$ will have their derived meanings. The following definitions and propositions (2.1 through 2.9) are due to Puritz [26,27] for the case $\kappa = \omega$ but generalize with trivial modifications.

Definition 2.1. Set

$$\mathrm{i}^{\sim}(\kappa) = \{x \mathrel{E} \mathrm{i}(\kappa) \mid \alpha < \kappa \to \mathrm{i}(\alpha) < x\} .$$

Then for $x, y \in \mathrm{i}^{\sim}(\kappa)$ define:

$$x \sim y \quad \textit{iff} \text{ for some } f, g \in {}^\kappa\kappa,\ \mathrm{i}(f)(x) \geqslant y \text{ and } \mathrm{i}(g)(y) \geqslant x .$$

$$x \lessdot y \quad \textit{iff } f \in {}^\kappa\kappa \text{ implies } \mathrm{i}(f)(x) < y .$$

$$x \leftrightarrow y \quad \textit{iff} \text{ for some } f, g \in {}^\kappa\kappa,\ \mathrm{i}(f)(x) = y \text{ and } \mathrm{i}(g)(y) = x .$$

The *sky*, *constellation* and *exact range* of $x \in \mathrm{i}^{\sim}(\kappa)$ are then defined respectively as follows:

$$\mathrm{sk}(x) = \{y \in \mathrm{i}^{\sim}(\kappa) \mid y \sim x\} ,$$

$$\mathrm{con}(x) = \{y \in \mathrm{i}^{\sim}(\kappa) \mid y \leftrightarrow x\} ,$$

$$\mathrm{er}(x) = \{\mathrm{i}(f)(x) \mid f \in {}^\kappa\kappa \text{ and } \mathrm{i}(f)(x) \in \mathrm{i}^{\sim}(\kappa)\} .$$

Thus, two elements are in the same sky if they are close enough to each other to be mutually accessible by "standard" functions. It is evident that though the definition of $x \sim y$ is symmetric in x and y, at least one of f, g can in fact be taken to be id, the identity function. The following lemma is simple.

Lemma 2.2.
(i) $\sim$ *and* $\leftrightarrow$ *are equivalence relations.*
(ii) *If* $x, y \in \mathrm{i}``(\kappa)$, $x \lessdot y$ *iff* $a \in \mathrm{sk}(x)$ *and* $b \in \mathrm{sk}(y) \to a < b$.

Proof. Of (i) for $\sim$. To show that $\sim$ is transitive, suppose $x \sim y$ and $y \sim z$. Let $y \lessdot \mathrm{i}(f)(x)$ and $z \lessdot \mathrm{i}(g)(y)$, where by regularity of κ, one can assume g is an increasing function. Hence, $z \lessdot \mathrm{i}(g)(y) \lessdot \mathrm{i}(g)\, \mathrm{i}(f)(x) = \mathrm{i}(gf)(x)$. Similarly, there is an $h \in {}^{\kappa}\kappa$ so that $x \lessdot \mathrm{i}(h)(z)$.

By (ii) of the lemma, skies can be naturally ordered by $\mathrm{sk}(x) < \mathrm{sk}(y)$ *iff* $x \lessdot y$. Thus, $\mathrm{i}``(\kappa)$ can be decomposed into ordered sub-intervals called skies, which in turn are made up of (whole) constellations. However, note that if $f \in {}^{\kappa}\kappa$ is $1-1$ and increasing, $\mathrm{i}(f)(x) \in \mathrm{con}(x)$, and for any $y \in \mathrm{sk}(x)$ there is such an f so that $y \lessdot \mathrm{i}(f)(x)$. Hence, unless a sky consists of just one constellation, its constellations will not themselves be subintervals, but will be spread out cofinally within the sky. The following propositions provide more information.

Proposition 2.3. *Let* $x \in \mathrm{i}``(\kappa)$.
(i) *If* $f \in {}^{\kappa}\kappa$ *is almost* $1-1$, *then* $x \sim \mathrm{i}(f)(x)$.
(ii) $\{\mathrm{i}(f)(x) \mid f \in {}^{\kappa}\kappa$ *is almost* $1-1$ *and non-decreasing*$\}$ *is both coinitial and cofinal in* $\mathrm{sk}(x)$.
(iii) *If* $S \subseteq \mathrm{sk}(x)$ *but* $|S| \lessdot \kappa$, *then* S *is not cofinal in* $\mathrm{sk}(x)$. *If in addition* S *contains no least element, then* S *is not coinitial in* $\mathrm{sk}(x)$.

Proof. (i) Set $g(\alpha) = \sup\{\beta \mid f(\beta) \lessdot \alpha\}$. Then $g \in {}^{\kappa}\kappa$ and $x \lessdot \mathrm{i}(g)\, \mathrm{i}(f)(x) = \mathrm{i}(gf)(x)$.

(ii) Suppose $y \in \mathrm{sk}(x)$. Then there is an f $1-1$ and increasing so that $y \lessdot \mathrm{i}(f)(x)$, and $\mathrm{i}(f)(x) \sim x$ by (i). Also, there is a g so that $x \lessdot \mathrm{i}(g)(y)$. If $g^0 \in {}^{\kappa}\kappa$ is defined by:

$$g^0(\alpha) = \text{least } \beta(g(\beta) \geqslant \alpha)\,,$$

$\mathrm{i}(g^0)(x) \lessdot y$ and $\mathrm{i}(g^0)(x) \sim x$ also by (i).

(iii) Suppose $S = \{a_\xi \mid \xi < \kappa\}$. For each $\xi < \kappa$ there is an f_ξ so that

$i(f_\xi)(a_\xi) > x$. If we set $f(\alpha) = \sup\{f_\xi(\beta) \mid \xi,\beta < \alpha\}$, an easy elementarity argument shows that $i(f)(a_\xi) > i(f_\xi)(a_\xi) > x$. Now associate to f a function f^0 as in the proof of (ii). Then $i(f^0)(x) < a_\xi$ for every $\xi < \kappa$, and $i(f^0)(x) \sim x$. Hence, if S contains no least element, then S is not coinitial in $\mathrm{sk}(x)$. A similar but shorter proof shows that S is not confinal.

Proposition 2.4. *For $x \in i``(\kappa)$,*

$$\mathrm{con}(x) = \{i(\pi)(x) \mid \pi \text{ is a permutation of } \kappa\}\ .$$

Proof. Let $y \in \mathrm{con}(x)$. It suffices to find a permutation π so that $i(\pi)(x) = y$. Assume $i(f)(x) = y$ and $i(g)(y) = x$, and set $S = \{\alpha < \kappa \mid gf(\alpha) = \alpha\}$. f is 1–1 on S, and $x\,E\,i(S)$. As S is infinite, let $S_0 \cup S_1 = S$, $S_0 \cap S_1 = \emptyset$, and $|S_1| = |S_0| = S$. Say for example that $x\,E\,i(S_0)$. Let $h : (\kappa - S_0) \leftrightarrow (\kappa - f''S_0)$ be bijective, and set

$$\pi(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in S_0\ , \\ h(\alpha) & \text{otherwise.} \end{cases}$$

Proposition 2.5. *If $x,y \in i``(\kappa)$, $x \sim y$ iff there is an f almost 1–1 and non-decreasing so that $i(f)(x) = i(f)(y)$.*

Proof. One direction follows from 2.3(i). For the other, assume $x \sim y$. One can suppose $x < y$ and $y < i(g)(x)$ for some g strictly increasing. Define a function $h \in {}^\kappa\kappa$ by induction as follows: $h(0) = g(0)$, $h(\alpha+1) = gh(\alpha)$, and $h(\gamma) = \sup\{h(\alpha) \mid \alpha < \gamma\}$ at limits γ. The range of $i(h)$ is then closed and cofinal in $i(\kappa)$, so let $a < i(\kappa)$ be such that $i(h)(a)$ is largest $< x$. Then

$$i(h)(a) < x < y < i(g)(x) < i(g)i(h)(a\,i(+)\,i(1))$$
$$= i(h)(a\,i(+)\,i(2))\ .$$

So, for example, if a is an even "ordinal", set $f(\xi) = \alpha$ *iff* $h(\alpha) < \xi < h(\alpha+2)$, for α even $< \kappa$. Clearly $i(f)(x) = i(f)(y)$.

In the situation we have been considering, notice that any $x \in i``(\kappa)$ can be considered a "generic" element which generates a uniform ultrafilter $\mathcal{U}$ over κ, defined by

$$X \in \mathcal{U} \quad \textit{iff} \quad x\,E\,i(X)\ .$$

When this idea is pursued further in the case where i itself arises from the ultrapower construction with respect to a uniform ultrafilter over κ, we will get another formulation of the Rudin–Keisler ordering (and indeed, this was Keisler's original method).

Let $\mathcal{U} \in \beta_u\kappa$ where again, κ is a regular cardinal. The previous notions and results will now be applied with i and E specialized respectively to i_{u} and E_{u} restricted to the appropriate domains; the other notation will be retained. The following are evident:

$$x < \mathrm{i}_{\mathrm{u}}(\kappa) \quad \textit{iff} \quad x = [f]_{\mathrm{u}} \text{ for some } f \in {}^\kappa\kappa\,,$$

$$[f]_{\mathrm{u}} \in \mathrm{i}_{\mathrm{u}}^{\sim}(\kappa) \quad \textit{iff} \quad f \in {}^\kappa\kappa \text{ and } f \text{ is unbounded } (\mathrm{mod}\, \mathcal{U}\,),$$

$$\mathrm{i}_{\mathrm{u}}(g)([f]_{\mathrm{u}}) = [gf]_{\mathrm{u}} \quad \text{for} \quad g \in {}^\kappa\kappa\,.$$

So, for example, $[h]_{\mathrm{u}} \in \mathrm{er}([f]_{\mathrm{u}})$ *iff* $[h]_{\mathrm{u}} \in \mathrm{i}_{\mathrm{u}}^{\sim}(\kappa)$ and f is a refinement of h when both are considered as partitions of κ. Also, observe that there is a highest sky, the sky of the identity function, and in fact

$$\mathrm{sk}([\mathrm{id}]_{\mathrm{u}}) = \{[f]_{\mathrm{u}} \mid f \text{ is almost } 1\!-\!1\}\,,$$

for suppose $f \sim \mathrm{id}$; then there is a g so that $[gf]_{\mathrm{u}} \geq [\mathrm{id}]_{\mathrm{u}}$, i.e. $\{\xi < \kappa \mid gf(\xi) \geq \xi\} \in \mathcal{U}$ and on this set f is almost $1-1$. From 2.4 it is also clear that

$$\mathrm{con}([\mathrm{id}]_{\mathrm{u}}) = \{[f]_{\mathrm{u}} \mid f \text{ is } 1\!-\!1\}\,.$$

These remarks immediately lead to the following characterization of p-points, q-points and $\beta_u\kappa$-minimal ultrafilters.

Proposition 2.6.
(i) *$\mathcal{U}$ is a p-point iff* $\mathrm{i}_{\mathrm{u}}^{\sim}(\kappa)$ *is one sky.*
(ii) *$\mathcal{U}$ is a q-point iff the highest sky is one constellation.*
(iii) *$\mathcal{U}$ is $\beta_u\kappa$-minimal iff* $\mathrm{i}_{\mathrm{u}}^{\sim}(\kappa)$ *is one constellation.*

Proof. Obvious from the definitions.

So, the sky structure of an ultrafilter can be considered a measure of its complexity: the more skies there are and the more constellations there are in each sky, the more complex the ultrafilter. As noted before, any "large" element of $\mathrm{i}_{\mathrm{u}}(\kappa)$ generates an element of $\beta_u\kappa$. It is now evident that

$$\mathcal{V} \leq \mathcal{U} \text{ and } \mathcal{V} \in \beta_u\kappa \textit{ iff } \text{there is an } f \in {}^\kappa\kappa \text{ unbounded } (\mathrm{mod}\, \mathcal{U}\,) \text{ such that } f_*(\mathcal{U}) = \mathcal{V},$$

and that

$$X \in \mathcal{V} \text{ iff } f^{-1}(X) \in \mathcal{U} \text{ iff } [f]_u \in i_u(X).$$

Also, if $[f]_u, [g]_u \in i_u^{\sim}(\kappa)$,

$$\mathrm{con}([f]_u) = \mathrm{con}([g]_u) \to f_*(\mathcal{U}) \cong g_*(\mathcal{U}).$$

When $f_*(\mathcal{U}) = \mathcal{V}$, by the remark just before 1.5, the map $\phi : V^\kappa/\mathcal{V} \to V^\kappa/\mathcal{U}$ defined by $\phi([g]_v) = [gf]_u$ is an elementary embedding. When $\mathcal{V} \in \beta_u\kappa$, the following facts about the action of ϕ on $i_v(\kappa)$ are easy to ascertain:

(i) ϕ preserves $\sim$ and $\leftrightarrow$.

(ii) $\phi'' i_v^{\sim}(\kappa) = \mathrm{er}([f]_u)$.

(iii) ϕ sends constellations onto constellations.

(iv) $\phi''\ \mathrm{sk}([g]_v)$ is coinitial and cofinal in $\mathrm{sk}([gf]_u)$, and no two skies are sent into one.

Bearing these facts in mind, the following proposition corresponds to 2.6.

Proposition 2.7. *If $f_*(\mathcal{U}) = \mathcal{V}$ and $\mathcal{V} \in \beta_u\kappa$, then*

(i) *$\mathcal{V}$ is a p-point iff* $\mathrm{er}([f]_u) \subseteq \mathrm{sk}([f]_u)$.

(ii) *$\mathcal{V}$ is a q-point iff* $\mathrm{er}([f]_u) \cap \mathrm{sk}([f]_u) = \mathrm{con}([f]_u)$.

(iii) *$\mathcal{V}$ is $\beta_u\kappa$-minimal iff* $\mathrm{er}([f]_u) = \mathrm{con}([f]_u)$.

Corollary 2.8.

(i) *If $\mathcal{U}$ is a p-point, $\mathcal{V} < \mathcal{U}$ and $\mathcal{V} \in \beta_u\kappa$, then $\mathcal{V}$ is a p-point.*

(ii) *If $f_*(\mathcal{U})$ is a q-point and $[g]_u \in \mathrm{sk}([f]_u)$, then $[f]_u \in \mathrm{er}([g]_u)$. Hence, at most one constellation in each sky can consist of $[f]_u$ so that $f_*(\mathcal{U})$ is a q-point.*

Proof. For (ii), notice that by 2.5, there is an h so that $[hf] = [hg]$ and $[hf] \in \mathrm{con}([f])$. So, there is permutation π so that $[f] = [\pi hf] = [\pi hg]$. Hence, $[f] \in \mathrm{er}([g])$.

Note that by 2.8(ii), any two $\beta_u\kappa$-minimal ultrafilters below a p-point are isomorphic. The next result essentially generalizes 2.8(ii) to q-points not necessarily ultra.

Theorem 2.9. *If $\mathcal{F}$ is a q-point filter over κ, then any sky of $\mathcal{U}$ contains at most one element $[f]_u$ so that $f_*(\mathcal{U}) \supseteq \mathcal{F}$.*

Proof. Suppose $f_*(\mathcal{U}) \supseteq \mathcal{F}$ and $g_*(\mathcal{U}) \supseteq \mathcal{F}$, but $\mathrm{sk}([f]) = \mathrm{sk}([g])$. By 2.5 there is an almost 1–1 function h so that $i_u(h)([f]) = i_u(h)([g])$. Since

$\mathcal{F}$ is a q-point, there is an $X \in \mathcal{F}$ so that

$$\alpha, \beta \in X \quad \text{and} \quad h(\alpha) = h(\beta) \to \alpha = \beta .$$

Then, by elementarity,

$$x, y \; E_u \; i_u(X) \quad \text{and} \quad i_u(h)(x) = i_u(h)(y) \to x = y .$$

But $f_*(\mathcal{U}) \supseteq \mathcal{F}$ and $g_*(\mathcal{U}) \supseteq \mathcal{F}$, so that $[f]_u$, $[g]_u$ E_u $i_u(X)$, and hence $[f]_u = [g]_u$.

The following propositions mark the point of departure from the case $\kappa = \omega$ and hence from Puritz's results. The assumption from now is that κ is regular and uncountable.

Proposition 2.10. *$\mathcal{C}_\kappa$, the closed unbounded filter over κ, is a q-point.*

Proof. Suppose f is almost $1-1$ and let $g(\alpha) = \sup\{\beta \mid f(\beta) \leqslant \alpha\} < \kappa$ for $\alpha < \kappa$. Then $\mathcal{C} = \{\alpha < \kappa \mid f : \alpha \to \alpha \text{ and } g : \alpha \to \alpha\}$ is closed unbounded and $\alpha, \beta \in \mathcal{C}$ and $\alpha < \beta \to f(\alpha) < \beta \leqslant f(\beta)$, i.e. f is $1-1$ on $\mathcal{C}$.

The next proposition is really a special case of 2.9.

Proposition 2.11. *Suppose $f \in {}^\kappa\kappa$ is unbounded* (mod $\mathcal{U}$). *Then $[f]_u$ is the least element of* sk$([f]_u)$ *iff $f_*(\mathcal{U}) \supseteq \mathcal{C}_\kappa$.*

Hence, if a sky has no least element, there are no $[g]_u$ in the sky such that $g_*(\mathcal{U})$ extends the closed unbounded filter.

Proof. If there is an element below $[f]_u$ in its sky, by 2.3(ii) there is a g almost $1-1$ so that $[gf]_u < [f]_u$. g is regressive on a set X in $f_*(\mathcal{U})$, and if X were stationary, g would be constant on an unbounded subset of X, a contradiction since g is almost $1-1$. Hence $f_*(\mathcal{U}) \not\supseteq \mathcal{C}_\kappa$.

Conversely, if $f_*(\mathcal{U})$ contains some X which is the complement of a closed unbounded set, $g(\alpha) = \sup(\alpha \cap (\kappa - X))$ defines an almost $1-1$ function regressive on X. Thus, $[gf]_u < [f]_u$ and $[gf]_u \sim [f]_u$.

Corollary 2.12. *No distinct extensions of $\mathcal{C}_\kappa$ in $\beta_u\kappa$ can be isomorphic.*

To conclude this section I make two remarks relating skies to recent work in the theory of ultrafilters over ω:

(a) M.E. Rudin's ordering $\subseteq$ whose minimal points are precisely the

p-points (see [29]) can be succinctly characterized by

$$\mathcal{D} \subseteq \mathcal{E} \quad \textit{iff} \text{ there is an } f \in {}^{\omega}\omega \text{ so that } \mathcal{D} = f_*(\mathcal{E}),$$
$$\text{and } \mathrm{sk}([f]_{\mathcal{E}}) < \mathrm{sk}([\mathrm{id}]_{\mathcal{E}}).$$

(b) The ultrafilter version of the main theorem in Blass [2] is as follows: If a countable number of p-points have a common p-point upper bound, then they have a common lower bound. Using skies, a short proof of this result is possible.

Suppose $\mathcal{D}$ and $\mathcal{E}_n$ for $n \in \omega$ are p-points such that there exist f_n with $f_{n*}(\mathcal{D}) = \mathcal{E}_n$ for every n. By 2.3(iii) $S = \{[f_n] \mid n \in \omega\}$ is neither coinitial nor cofinal in *the* sky $\mathrm{i}_{\mathcal{D}}{}^{\sim}(\omega)$, so let $[g]_{\mathcal{D}} < [f_n]_{\mathcal{D}} < [h]_{\mathcal{D}}$ for each n. By 2.5 there is a t nondecreasing so that $[tg]_{\mathcal{D}} = [th]_{\mathcal{D}} =$ some $[F]_{\mathcal{D}}$, i.e. for each n $[tf_n]_{\mathcal{D}} = [F]_{\mathcal{D}}$.

Hence, $F_*(\mathcal{D})$ is a lower bound for each $\mathcal{E}_n$, and note that there is *one* function t which can be used to simultaneously project all the $\mathcal{E}_n$'s down to a common lower bound, a fact that also follows from 1.12.

3. κ-ultrafilters and the function τ

This section begins in earnest the study of $\beta_m\kappa$ where κ denotes a typical measurable cardinal $>\omega$. To simplify the presentation the following definition will be used throughout:

Definition 3.1. If X is a set such that $|X| = \kappa$, a *κ-ultrafilter over X* is a non-principal κ-complete ultrafilter over X, and a *κ-ultrafilter* is just a κ-ultrafilter over κ itself, i.e. a member of $\beta_m\kappa$.

As remarked in Section 1 many aspects of the theory of $\beta_u\omega$ will have analogues, but there is now an essentially new advantage, the well-foundedness of ultrapowers. I assume the reader's acquaintance with the model theoretic techniques involved, and as is common practice, I do not distinguish between a well-founded ultrapower and its transitive isomorph. In particular, if $\mathcal{U}$ is a κ-ultrafilter $\mathrm{i_u} : V \to M \cong V^\kappa/\mathcal{U}$ will now be a non-trivial elementary embedding of the universe into a transitive submodel which first moves κ to some ordinal $\mathrm{i_u}(\kappa) > \kappa$. Note that $\mathrm{i_u}{}^{\sim}(\kappa)$ is now just $\mathrm{i_u}(\kappa) - \kappa$. Also, I often do not distinguish between an equivalence class of functions and a typical member, e.g. f is called the least non-constant function (mod $\mathcal{U}$) when (the transitization of) $[f]_{\mathrm{u}}$ equals κ.

In considering the family of κ-ultrafilters a natural question to ask is how rich it can be, how much, for example, the structure of $\beta_u(\omega)$ with CH or MA can be copied. As noted in Section 1, without much in the way of inductive procedures which can be adequately controlled at each stage, it seems more difficult to construct various interesting kinds of κ-ultrafilters. An obstacle in this regard is the existence of a simple inner model of measurability, the model $L[\mathcal{U}]$ of sets constructible from a normal κ ultrafilter $\mathcal{U}$. Kunen [20] has shown that in $L[\mathcal{U}]$ there are only κ^+ κ-ultrafilters, and each one is equivalent to a finite product of $\mathcal{U} \cap L[\mathcal{U}]$. Thus in $L[\mathcal{U}]$ the Rudin–Keisler ordering on κ-ultrafilters has order type ω. In fact, it will soon be clear that every κ-ultrafilter there has a finite number of skies, there are no non-minimal p-points or q-points, and the only extension of the closed unbounded filter is $\mathcal{U} \cap L[\mathcal{U}]$ itself. Hence, though $L[\mathcal{U}]$ is extremely interesting in many respects (especially in exhibiting strong similarities to L; see Devlin [8], for example), it is a rather barren landscape to search for κ ultrafilters, and shows that mere measurability is not enough to prove the existence of any essentially non-trivial κ-ultrafilters.

A fully adequate hypothesis seems to be the assertion that κ is 2^κ-supercompact, and indeed, Kunen and Solovay have both constructed very interesting examples of κ-ultrafilters from this hypothesis. Another hypothesis rich in possibilities is the assertion that κ is κ-compact. The relative provability strengths of these hypotheses are not yet sufficiently clarified. Perhaps the main question in this connection is still whether it is consistent to have κ κ-compact and carry only one normal κ-ultrafilter. In this paper these hypotheses will be intermittently used to provide examples on which a rich and interesting structure theory for κ-ultrafilters can rest.

To begin the development, some initial remarks are in order. If $\mathcal{U}$ is a κ-ultrafilter and $f \in {}^\kappa\kappa$, $f_*(\mathcal{U})$ is a κ-ultrafilter *iff* $[f]_u > \kappa$ (and principal otherwise), so we are only interested in such f. Concerning the sky structure of $\mathcal{U}$, the following observations are evident:

(i) each sky sk($[f]$) has a least element, which by 2.3(ii) is of the form $[hf]$ where h is almost 1–1 and non-decreasing, and by 2.11 is such that $(hf)_*(\mathcal{U}) \geq \mathcal{C}_\kappa$; (ii) each constellation also has a least element, and though constellations within a sky are not convex subsets but cofinally spread out, constellations can now be naturally ordered as per their least elements. The next proposition is also easy.

Proposition 3.2. *Let $\mathcal{U}$ be a κ-ultrafilter.*

(i) $\mathcal{U}$ is a p-point iff its least non-constant function is almost $1-1$ (mod $\mathcal{U}$).

(ii) $\mathcal{U}$ is a q-point iff its least almost $1-1$ function is $1-1$ (mod $\mathcal{U}$).

(iii) $\mathcal{U}$ is minimal iff its least non-constant function is $1-1$ (mod $\mathcal{U}$).

Proof. (i) is evident, since the least non-constant function is in the highest, and hence only, sky. For (ii), if f is the least almost $1-1$ function (mod $\mathcal{U}$), for any almost $1-1$ function g there is an h so that $[hg] = [f]$. But then $[f^{-1}hg] = [\mathrm{id}]$ and g is $1-1$ (mod $\mathcal{U}$).

The following well-foundedness result is much deeper, and perhaps somewhat surprising. Due to Solovay, it is a basic tool in the theory of κ-ultrafilters. I state it in general form and include a short proof.

Theorem 3.3. (Solovay). *The Rudin–Keisler ordering on countably complete ultrafilters (over arbitrary sets) is well founded.*

Proof. For each $n \in \omega$ let $\mathcal{U}_n$ be countably complete over a set I_n, and for $n < m$ let $f_{nm} : I_n \to I_m$ be such that $f_{nm*}(\mathcal{U}_n) = \mathcal{U}_m$ and $f_{nm} f_{0n} = f_{0m}$. It suffices to find some n such that for every $m > n$ f_{nm} is $1-1$ (mod $\mathcal{U}_n$).

Consider an equivalence relation $\simeq$ defined on I_0 by $x \simeq y$ *iff* there is an n such that $f_{0n}(x) = f_{0n}(y)$. Fix one element in each equivalence class as a representative, and for $x \in I_0$ set $f(x)$ = the least n such that $f_{0n}(x) = f_{0n}(r)$ and $r \simeq x$ is the representative of the class of x. By countable completeness there is an n_0 such that $X = \{x \in I_0 \mid f(x) = n_0\} \in \mathcal{U}_0$. Then it is simple to see that for every $m > n_0$ $f_{n_0 m}$ is $1-1$ on $f_{0n_0}{}'' X \in \mathcal{U}_{n_0}$.

I now consider for each κ-ultrafilter an associated set of ordinals first defined by Ketonen, and show its direct relationship to skies.

Definition 3.4. (Ketonen [17]) For a κ-ultrafilter $\mathcal{U}$ set

$$\Gamma(\mathcal{U}) = \{[f]_{\mathrm{u}} \mid f_*(\mathcal{U}) \supseteq \mathcal{C}_\kappa\}$$

$$\tau(\mathcal{U}) = \text{order-type of } \Gamma(\mathcal{U}).$$

Note that if $\mathcal{V} \leqslant \mathcal{U}$, the canonical embedding sends $\Gamma(\mathcal{V})$ into $\Gamma(\mathcal{U})$ and so, for example, $\mathcal{V} \cong \mathcal{U}$ implies $\tau(\mathcal{V}) = \tau(\mathcal{V})$.

Proposition 3.5. *If $\mathcal{U}$ is a κ-ultrafilter, $\tau(\mathcal{U})$ is the order type of the skies in their natural ordering, and in fact the least element of each sky is the unique one in $\Gamma(\mathcal{U})$.*

Proof. See 2.11.

Corollary 3.6. *$\Gamma(\mathcal{U})$ is a closed set of ordinals with a highest element, and so $\tau(\mathcal{U})$ is always a successor ordinal.*

The following interesting theorem on $\tau(\mathcal{U})$ is due to Ketonen.

Theorem 3.7. (Ketonen [17]) *Let $\mathcal{U}$ be a κ-ultrafilter and μ a regular cardinal $< \kappa$.*

(i) *If $\tau(\mathcal{U}) > \mu$, there is a $\mathcal{V} \leq \mathcal{U}$ such that $\mathcal{V} \supseteq \mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$.*

(ii) *If $\mathcal{U}$ is a q-point, $\tau(\mathcal{U}) = \mu + 1$ iff $\mathcal{U}$ is isomorphic to an* RK*-minimal extension of $\mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$.*

Proof. For (i), let $\langle [f_\xi] \mid \xi < \mu \rangle$ be any increasing sequence of elements of $\Gamma(\mathcal{U})$, and define $f \in {}^\kappa\kappa$ by $f(\alpha) = \sup\{f_\xi(\alpha) \mid \xi < \mu\}$. Then $[f] \in \Gamma(\mathcal{U})$ by 3.6, and $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\} \in f_*(\mathcal{U})$.

For (ii), if $\mathcal{U}$ is a minimal extension of $\mathcal{C}_\kappa \cup \{\{\alpha \mid \mathrm{cf}(\alpha) = \mu\}\}$, then first of all $\tau(\mathcal{U}) \leq \mu + 1$: if not, by the argument of the preceding paragraph there is an f such that $[f] < [\mathrm{id}]$ and $[f]$ is the supremum of μ elements in $\Gamma(\mathcal{U})$; but then $f_*(\mathcal{U}) < \mathcal{U}$ yet $f_*(\mathcal{U}) \supseteq \mathcal{C}_\kappa \cup \{\{\alpha \mid \mathrm{cf}(\alpha) = \mu\}\}$, contradicting the minimality of $\mathcal{U}$. Secondly, $\tau(\mathcal{U}) \geq \mu + 1$: otherwise, since $\{\alpha \mid \mathrm{cf}(\alpha) = \mu\} \in \mathcal{U}$, let $\langle [f_\xi] \mid \xi < \mu \rangle$ be any sequence cofinal in $[\mathrm{id}]$. Since we are assuming that there are less than μ skies, some final segment of the sequence must be in a single sky $< \mathrm{sk}([\mathrm{id}])$, but this contradicts 2.3(iii). Thus, $\tau(\mathcal{U}) = \mu + 1$.

Conversely, if $\tau(\mathcal{U}) = \mu + 1$, by (i) there is a $\mathcal{V} \leq \mathcal{U}$ so that $\mathcal{V} \supseteq \mathcal{C}_\kappa \cup \{\{\alpha \mid \mathrm{cf}(\alpha) = \mu\}\}$ and $\mathcal{V}$ can be taken minimal in this respect. If $\mathcal{V} < \mathcal{U}$, $\tau(\mathcal{V}) < \tau(\mathcal{U})$ as $\mathcal{U}$ is a *q-point* and the highest sky of $\mathcal{U}$ is left out in the embedding of $\Gamma(\mathcal{V})$ into $\Gamma(\mathcal{U})$. But $\tau(\mathcal{V}) < \mu + 1$ contradicts the conclusion of the previous paragraph; hence $\mathcal{V} \cong \mathcal{U}$, and the result follows.

The assumption that $\mathcal{U}$ is a *q*-point is necessary in (ii) since, for example, if $\mathcal{V}$ and $\mathcal{U}$ are such that $\tau(\mathcal{V}) < \mu$ and $\tau(\mathcal{U}) = \mu + 1$, it will follow from forthcoming results that $\tau(\mathcal{V} \times \mathcal{U}) = \mu + 1$, yet $\mathcal{V} \times \mathcal{U}$ is not an RK-minimal extension of $\mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$.

Corollary 3.8. (Ketonen [17]) (i) *If μ is regular $< \kappa$ and no κ-ultrafilter extends $\mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$, then every κ-ultrafilter has less than μ normal κ-ultrafilters below it.*

(ii) *if $\nu < \mu < \kappa$ are regular and $\mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$ can be ex-*

tended to a κ-ultrafilter, then $\mathcal{C}_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \nu\}\}$ *can also be extended to a κ-ultrafilter.*

Thus, as μ gets larger it becomes harder to extend $\mathcal{C}_\kappa \cup \{\{\alpha \mid \mathrm{cf}(\alpha) = \mu\}\}$ to κ-ultrafilters. Essentially, more and more skies must be constructed.

3.9. When κ is κ-compact it is immediate from the above that for arbitrarily large $\mu < \kappa$ there exist κ-ultrafilters with μ skies. In fact, the following "brute force" argument shows that for arbitrarily large $\mu < (2^\kappa)^+$ there exist κ-ultrafilters $\mathcal{U}$ with $\tau(\mathcal{U}) \geq \mu$:

Let $\mathcal{F} \subseteq {}^\kappa\kappa$ be a family of 2^κ κ-independent functions (see Ketonen [15] for details); that is, if $\{f_\xi \mid \xi < 2^\kappa\}$ enumerates $\mathcal{F}$, for any $\gamma < \kappa$, any set of distinct ordinals $\{\xi_\alpha \mid \alpha < \gamma\} \subseteq 2^\kappa$ and any set of ordinals $\{\eta_\alpha \mid \alpha < \gamma\} \subseteq \kappa$,

$$\{\beta \mid f_{\xi_\alpha}(\beta) = \eta_\alpha \text{ for all } \alpha < \gamma\} \neq \emptyset .$$

Then given $2^\kappa \leq \mu < (2^\kappa)^+$, let $\{g_\xi \mid \xi < \mu\}$ enumerate $\mathcal{F}$ in type μ and let $\mathcal{U}$ be any κ-ultrafilter which includes the sets

$$\{\alpha < \kappa \mid hg_\xi(\alpha) < g_\eta(\alpha)\}$$

for $\xi < \eta < \mu$, and h ranging over all functions $\in {}^\kappa\kappa$ unbounded in κ. Then for $\xi < \eta < \mu$, $\mathrm{sk}([g_\xi]) < \mathrm{sk}([g_\eta])$ and so $\tau(\mathcal{U}) \geq \mu$. Of course, this kind of construction does not give much information, and in 5.3 there are better woven examples of κ-ultrafilters with a large number of skies.

Finally, concerning the extent of skies in absolute terms, consider the following proposition.

Proposition 3.10. *Let* $\mathcal{U}$ *be a κ-ultrafilter and* $\mathrm{i}_\mathrm{u} : \mathcal{V} \to M \cong \mathcal{V}^\kappa/\mathcal{U}$ *the associated embedding. Then if* $\kappa \leq \alpha, \beta \leq (2^\kappa)^M$, *there exists an* $f \in {}^\kappa\kappa$ *such that* $\mathrm{i}_\mathrm{u}(f)(\beta) = \alpha$. *(It is well known that* $2^\kappa \leq (2^\kappa)^M < \mathrm{i}_\mathrm{u}(\kappa) < (2^\kappa)^+$.*)*

Proof. Let $F : \mathcal{P}(\kappa) \to \delta$ be some well ordering of $\mathcal{P}(\kappa)$, δ being some ordinal, so that the following two conditions are satisfied for any cardinal $\mu < \kappa$:

(i) If $\eta \leq 2^\mu$, there is an $X \subseteq \mu$ such that $F(X) = \eta$.

(ii) $X \subseteq \mu$ implies that $F(X) < \kappa$.

For $\xi < \kappa$ let $G(\xi)$ = the least cardinal ν so that $\xi \leq 2^\nu$. Finally, for $X \subseteq \kappa$ define $f_X \in {}^\kappa\kappa$ by:

$$f_X(\xi) = F(X \cap G(\xi)) < \kappa .$$

Now if $\alpha < (2^\kappa)^M$, there is an $X \subseteq \kappa$ so that $i_u(F)(X) = \alpha$ by (i). Then for any β such that $\kappa < \beta < (2^\kappa)^M$,

$$\begin{aligned} i_u(f_X)(\beta) &= i_u(F)(i_u(X) \cap i_u(G)(\beta)) \\ &= i_u(F)(i_u(X) \cap \kappa) \\ &= i_u(F)(X) \\ &= \alpha . \end{aligned}$$

Corollary 3.1 ł. *The first sky (indeed, the first constellation) extends beyond* 2^κ.

Of course, the proposition can be generalized to show that each constellation includes long definable intervals of ordinals, but this method will not yield any characterizations, since skies and constellations are essentially 'non-standard' objects.

4. *P*-points

This section deals with p-points, i.e. the case $\tau(\mathcal{U}) = 1$. The main interest here is essentially in the possible complexities of structure within one sky.

As previously noted, to get any interesting (i.e. not minimal) p-points assumptions stronger than measurability will have to be used. But once in a sufficiently rich situation the next proposition is relevant. But first, a lemma due to Solovay and used by him in the initial proof of 3.3. It is of independent interest, as it shows that con([id]) is always the highest constellation for κ-ultrafilters.

Lemma 4.1. (Solovay) *If* λ *is regular and* $\mathcal{U} \in \beta_u\lambda$, *then for every* f *unbounded* (mod $\mathcal{U}$) *there is a* $[g] \in \mathrm{con}([f])$ *such that* $[g] \leq [\mathrm{id}]$.

Proof. Set $g(\alpha) = \text{least } \beta(f(\beta) = f(\alpha))$. Then $[g] \leq [\mathrm{id}]$ and $[fg] = [f]$. Also, if h is defined by $h(\alpha) = \text{least } \beta(f(\beta) = \alpha)$ then $[hf] = [g]$. Hence $[g] \in \mathrm{con}([f])$.

Suppose now that $\mathcal{U}$ is a non-minimal p-point. If $\mathcal{N}$ is normal $< \mathcal{U}$ and $\mathcal{V}$ is such that $\mathcal{N} < \mathcal{V} \leq \mathcal{U}$ and minimal in this respect, then $\mathcal{V}$

would be a two constellation p-point. The following proposition describes how to get one canonically.

Proposition 4.2. *If $\mathcal{U}$ is a non-minimal p-point and* $\mathrm{con}([f]_u)$ *is the second constellation, then $f_*(\mathcal{U}) = \mathcal{V}$ is a two constellation p-point.*

Proof. One can assume that $[f]_u$ is the least element of $\mathrm{con}([f]_u)$. Because of the properties of the embedding $V^\kappa/\mathcal{V} \to V^\kappa/\mathcal{U}$ defined from f, it suffices to show : whenever $[g]_\nu > \kappa$ and $[g]_\nu \notin \mathrm{con}([\mathrm{id}]_\nu)$, then $[gf]_u$ is in some fixed constellation of $i_u\tilde{}(\kappa)$. But for such $[g]_\nu$, by the Lemma there is an h such that $[h]_\nu \in \mathrm{con}([g]_\nu)$ and $[h]_\nu < [\mathrm{id}]_\nu$, i.e. $[hf]_u < [f]_u$. By the choice of f, $[hf]_u$ is in the first constellation of $i_u\tilde{}(\kappa)$ and hence $[gf]_u$ is as well.

I do not know in general how to get a $\mathcal{V} \leq \mathcal{U}$ with exactly three constellations (assuming $\mathcal{U}$ had at least three), and indeed, this may not always be possible. The following definitions will yield some nice p-points for which such questions can be answered.

Definitions 4.3. For a κ-ultrafilter $\mathcal{U}$ and a positive integer n, define $\mathcal{U}$ is *n-Ramsey iff* for any $F : [\kappa]^2 \to n+1$ there is an $X \in \mathcal{U}$ so that $|F''[X]^2| \leq n$, and n is minimal in this respect. Define by induction on n $\mathcal{U}$ is *strictly n-Ramsey iff* $\mathcal{U}$ is n-Ramsey and either $n = 1$, or there is an f so that $f_*(\mathcal{U})$ is a strictly $(n-1)$-Ramsey κ-ultrafilter.

These notions are clearly well defined for Rudin–Keisler equivalence classes of κ-ultrafilters, and it is easy to see that 2-Ramsey is the same as strictly 2-Ramsey. 2-Ramsey ultrafilters for the ω case were first defined (as 'weakly Ramsey' ultrafilters) by Blass [4]. He shows that 2-Ramsey ultrafilters are two constellation p-points, and that CH implies there are many 2-Ramsey ultrafilters. The notion is extended here through all positive integers, and examples in the measurable cardinal case are considered.

4.4. n-Ramsey ultrafilters are not necessarily p-points. For example, if $\mathcal{U}$ is normal, $\mathcal{U} \times \mathcal{U}$ is 6-Ramsey: Let $F : [\kappa \times \kappa]^2 \to 7$. There are 6 ways that four ordinals $\alpha_0, \beta_0, \alpha_1, \beta_1$ such that $\alpha_0 < \beta_0, \alpha_1 < \beta_1$ and $\langle \alpha_0, \beta_0 \rangle < \langle \alpha_1, \beta_1 \rangle$ (in the lexigraphic ordering) can be ordered. For $1 \leq n \leq 6$, a corresponding function $f_n : [\kappa]^4$ (or $[\kappa]^3) \to 7$ can be defined by $f_n(\{\alpha_0, \beta_0, \alpha_1, \beta_1\}) = F(\{\langle \alpha_0, \beta_0 \rangle, \langle \alpha_1, \beta_1 \rangle\})$. Hence, by normality there is an X_n homogeneous for f_n and if $Y = \bigcap_{1 \leq n \leq 6} X_n$, $|F''[Y \times Y]^2| \leq 6$. At the same time, $[\kappa \times \kappa]^2$ can be partitioned into 6 parts according to which ordering the four component ordinals assume, and any $X \in \mathcal{U}$ is such that $[X \times X]^2$ contains sets ordered in each of the 6 ways. Hence, 6 was minimal, and $\mathcal{U} \times \mathcal{U}$ is 6-Ramsey.

Thus, some n-Ramsey ultrafilters are not particularly special (though many interesting characterization and existence questions can be asked concerning n-Ramseyness in general), and hence the introduction of strictly n-Ramsey ultrafilters.

The proof of the next theorem has some new details beyond the $n = 2$ case that must be taken care of.

Theorem 4.5. (Blass [4] for $n = 2$) *If a κ-ultrafilter $\mathcal{U}$ is strictly n-Ramsay, it is a p-point with exactly n constellations such that* $\mathrm{con}([f]) \leqslant \mathrm{con}([g])$ *implies* $[f] \in \mathrm{er}([g])$.

Proof. By induction on n. Let $f_*(\mathcal{U}) < \mathcal{U}$ be strictly $(n-1)$-Ramsey and let $[f_1], \ldots, [f_{n-1}] = [f]$ be in $\mathrm{er}([f])$ and in different constellations such that: f_1 is the least function of $\mathrm{er}([f])$, and $0 < i < j < n \to [f_i] \in \mathrm{er}([f_j])$.

To prove the theorem it suffices to verify two facts: (i) f_1 is almost $1-1 \pmod{\mathcal{U}}$, and (ii) if g is not constant and not $1-1 \pmod{\mathcal{U}}$, there is an $i < n$ such that $[g] \in \mathrm{er}([f_i])$. Whence, to show that $\mathcal{U}$ is a p-point it suffices to show that f_1 is the least function, by 3.2(i). But if $\kappa \leqslant [g] < [f_1]$, $\mathrm{sk}([g]) < \mathrm{sk}([f_1])$, and since by (ii) $[hf_i] = [g]$ for some $i < n$ and a function h, $\mathrm{er}([f_i])$ would contain at least two skies. Hence, $f_{i*}(\mathcal{U})$ would not be a p-point, contrary to the inductive hypothesis. The rest of the theorem would follow from (ii).

I now turn to the proofs of (i) and (ii). To show (i), for $\alpha, \beta < \kappa$ set

$$S_{\alpha\beta} = \begin{cases} \text{least } i < n \quad \text{such that } f_i(\alpha) \neq f_i(\beta) & \text{if it exists,} \\ n \quad \text{otherwise.} \end{cases}$$

Define $F : [\kappa]^2 \to n + 1$ by

$$F(\{\alpha,\beta\}) = \begin{cases} 0 & \text{if } S_{\alpha\beta} = 1 \text{ and } \alpha < \beta \text{ and } f_1(\alpha) < f_1(\beta) \\ 1 & \text{if } S_{\alpha\beta} = 1 \text{ and } \alpha \geqslant \beta \text{ and } f_1(\alpha) < f_1(\beta) \\ S_{\alpha\beta} & \text{if } S_{\alpha\beta} \neq 1. \end{cases}$$

Since $\mathcal{U}$ is n-Ramsey, there is an $X \in \mathcal{U}$ such that $F''[X]^2$ omits one value. If that value were n, an easy argument shows that f_{n-1} would be

$1-1$ on X, a contradiction. If the value were not n but still greater than 1, then for some j such that $1 < j < n$,

$$\alpha, \beta \in X \quad \text{and} \quad f_{j-1}(\alpha) = f_{j-1}(\beta) \to f_j(\alpha) = f_j(\beta)\,.$$

Hence, there is a function h such that $[hf_{j-1}] = [f_j]$, contradicting the assumption that $[f_{j-1}]$ and $[f_j]$ are in different constellations.

Suppose now that the omitted value were 0. Let $\alpha \in X$ be such that $f_1(\alpha)$ is least. Then for any $\beta \in X$, $f_1(\alpha) < f_1(\beta)$ implies $\alpha > \beta$, once again a contradiction, since f_1 must be unbounded on every set in $\mathcal{U}$.

So we conclude that the omitted value must be 1. It is then easy to see that for $\alpha, \beta \in X$, $\alpha < \beta \to f_1(\alpha) \leqslant f_1(\beta)$. Thus, f_1 is almost $1-1$ on X, and (i) is proved.

To show (ii), let $[g] \geqslant \kappa$. Let $S_{\alpha\beta}$ be as before, and for $0 < i \leqslant n$ define $F_i : [\kappa]^2 \to n+1$ by

$$F_i(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } S_{\alpha\beta} = i \text{ and } g(\alpha) = g(\beta) \\ 1 & \text{if } S_{\alpha\beta} = i \text{ and } g(\alpha) \neq g(\beta) \\ S_{\alpha\beta} + 1 & \text{if } S_{\alpha\beta} < i \\ S_{\alpha\beta} & \text{if } S_{\alpha\beta} > i\,. \end{cases}$$

For $0 < i \leqslant n$ there is an $Y_i \in \mathcal{U}$ such that $F_i''[Y_i]^2$ omits some value. As before, we can deduce that the omitted value must either be 0 or 1. Let $Y = \bigcap_{0<i\leqslant n} Y_i \in \mathcal{U}$. If for $0 < i \leqslant n$ $F''[Y]^2$ always omits the value 1, then g is constant on Y; so we can assume that for some i_0, $0 < i_0 \leqslant n$, the omitted value is 0, and i_0 is maximal in this respect.

I claim that for $0 < j < i_0$, the value omitted for F_j is again 0: Since $Y \in \mathcal{U}$, there are $\alpha, \beta, \gamma \in Y$ so that $S_{\alpha\beta} = i_0$ and $S_{\alpha\gamma} = j$. Note that also $S_{\beta\gamma} = j$. Since $g(\alpha) \neq g(\beta)$, either $g(\gamma) \neq g(\beta)$ or $g(\gamma) \neq g(\beta)$. Thus, $F_j''[Y]^2$ does not omit 1, so it must omit 0, which was the claim.

Finally, by the claim if $\alpha, \beta \in Y$,

$$g(\alpha) \neq g(\beta) \quad \textit{iff} \quad S_{\alpha\beta} \leqslant i_0\,.$$

Hence, if $i_0 = n$, g is $1-1$ on Y. But if $i_0 < n$, $f_{i_0}(\alpha) = f_{i_0}(\beta) \to g(\alpha) = g(\beta)$ on Y, and so $[g] \in \mathrm{er}([f_{i_0}])$.

The proof of the theorem is now complete.

The following example of a non-minimal p-point is due to Ketonen.

Example 4.6. (Ketonen [17]). Assume κ is a measurable cardinal and a limit of measurable cardinals. Let $\mathcal{N}$ be a normal κ-ultrafilter and for μ

measurable $< \kappa$ let $\mathcal{U}_\mu$ be a normal μ-ultrafilter. For $\alpha < \kappa$ set $m(\alpha) =$ the least measurable cardinal $> \alpha$. Finally, define $\mathcal{D}$ over κ by

$$X \in \mathcal{D} \quad \textit{iff} \quad \{\alpha < \kappa \mid X \cap m(\alpha) \in \mathcal{U}_{m(\alpha)}\} \in \mathcal{U} .$$

If $A \subseteq \kappa$ is the closure (in the order topology) of the set of measurable cardinals $< \kappa$,

$$\bigcup\{(\alpha, m(\alpha)) \mid \alpha \in A\} \in \mathcal{D} .$$

On this set, define a function ϕ by

$$\phi(\beta) = \alpha \quad \textit{iff} \quad \beta \in (\alpha, m(\alpha)) \text{ and } \alpha \in A .$$

Then $\phi_*(\mathcal{D}) = \mathcal{U}$ and ϕ is the least non-constant function (mod $\mathcal{D}$): Assume $[f]_\mathcal{D} < [\phi]_\mathcal{D}$. For α in a set in $\mathcal{U}$ $\{\beta < m(\alpha) \mid f(\beta) < \alpha\} \in \mathcal{U}_{m(\alpha)}$, so that by $m(\alpha)$-completeness there is an $h(\alpha) < \alpha$ so that $\{\beta < m(\alpha) \mid f(\beta) = h(\alpha)\} \in \mathcal{U}_{m(\alpha)}$. But since $\mathcal{U}$ is normal, h is constant (mod $\mathcal{U}$), and so f is constant (mod $\mathcal{D}$).

Since ϕ is almost 1–1 but not 1–1 (mod $\mathcal{D}$), $\mathcal{D}$ is a non-minimal p-point. Ketonen goes on to show that $\mathrm{con}([\phi]_\mathcal{D})$ and $\mathrm{con}([\mathrm{id}]_\mathcal{D})$ are the only constellations. This will now be a consequence of the following theorem.

Theorem 4.7. *The κ-ultrafilter $\mathcal{D}$ in 4.6 is a 2-Ramsey κ-ultrafilter.*

Proof. Let $F : [\kappa]^2_* \to 3$.

Step 1: There is an $X_0 \in \mathcal{D}$ such that for some fixed $i < 3$: $\alpha, \beta \in X_0$ and $\phi(\alpha) \neq \phi(\beta) \to F(\{\alpha, \beta\}) = i$.

To show this, for $\beta < \kappa$ let $S_\beta \in \mathcal{D}$ be such that for a fixed $i_\beta < 3$:

$$\delta \in S_\beta \to \beta < \delta \quad \text{and} \quad F(\{\beta, \delta\}) = i_\beta .$$

There is an $i < 3$ so that $Y = \{\beta \mid i_\beta = i\} \in \mathcal{D}$. Now let

$$T_\alpha = \bigcap \{S_\beta \mid \phi(\beta) = \alpha\} \in \mathcal{D} .$$

Since ϕ is the least non-constant function (mod $\mathcal{D}$), $Z = \{\beta \mid \alpha < \phi(\beta) \to \beta \in T_\alpha\} \in \mathcal{D}$.

Then $X_0 = Y \cap Z \in \mathcal{D}$, and if $\alpha, \beta \in X_0$, $\phi(\alpha) < \phi(\beta) \to \beta \in T_{\phi(\alpha)} \subseteq S_\alpha$, that is

$$F(\{\alpha, \beta\}) = i_\beta = i .$$

Step 2: There is an $X_1 \in \mathcal{D}$ so that for some fixed $j < 3$, if $\alpha, \beta \in X$, $\phi(\alpha) = \phi(\beta) \to F(\{\alpha, \beta\}) = j$.

To show this, for $\alpha \in A$ since $\mathcal{U}_{m(\alpha)}$ is normal, let $Y_\alpha \in \mathcal{U}_{m(\alpha)}$ be such that $Y_\alpha \subseteq (\alpha, m(\alpha))$ and for a fixed $j_\alpha < 3$, $F''[Y_\alpha]^2 = \{j_\alpha\}$. There is a $j < 3$ so that $K = \{\alpha \in A \mid j_\alpha = j\} \in \mathcal{U}$. Now let

$$X_1 = \bigcup\{Y_\alpha \mid \alpha \in K\} \in \mathcal{D} .$$

If $\alpha, \beta \in X_1$, $\phi(\alpha) = \phi(\beta) \to F(\{\alpha, \beta\}) = j_{\phi(\alpha)} = j$.

The proof is now complete since

$$F''[X_0 \cap X_1]^2 = \{i, j\} .$$

A weak converse to this theorem exists. Suppose $\mathcal{E}$ is a 2-Ramsey κ-ultrafilter and ψ its least function, which we can take to be non-decreasing. Suppose, in addition, that there are filters $\mathcal{F}_\alpha$ over the sets $\psi^{-1}(\{\alpha\})$ so that

$$X \in \mathcal{E} \quad \text{iff} \quad \{\alpha < \kappa \mid X \cap \psi^{-1}(\{\alpha\}) \in \mathcal{F}_\alpha\} \in \psi_*(\mathcal{E}) .$$

Then, for α in a set in $\psi_*(\mathcal{E})$, $\mathcal{F}_\alpha$ is a Ramsey ultrafilter and so $|\psi^{-1}(\{\alpha\})|$ is a measurable cardinal.

To show this, it suffices to get a contradiction from the assertion that for α in a set in $\psi_*(\mathcal{E})$, $\mathcal{F}_\alpha$ are not Ramsey. So for these α let F_α : $[\psi^{-1}(\{\alpha\})]^2 \to 2$ be without homogeneous sets in $\mathcal{F}_\alpha$. Set

$$G(\{\delta, \eta\}) = \begin{cases} F_\alpha(\{\delta, \eta\}) & \text{if} \quad \psi(\delta) = \psi(\eta) = \alpha \quad \text{for some } \alpha, \\ 2 & \text{otherwise.} \end{cases}$$

Let $X \in \mathcal{E}$ be such that $G''[X]^2 \neq 3$. The omitted value cannot be 2, as ψ is not constant (mod $\mathcal{E}$). Say, for example, that it is 0. Then for α in a set in $\psi_*(\mathcal{E})$,

$$\delta, \eta \in X \cap \psi^{-1}(\{\alpha\}) \to F_\alpha(\{\delta, \eta\}) = 1,$$

and $X \cap \psi^{-1}(\{\alpha\}) \in \mathcal{F}_\alpha$, a contradiction.

Question 4.8. Can 2-Ramsey κ-ultrafilters always be written as a discrete limit of ultrafilters over smaller cardinals, as above?

This is closely related to the following more general problem:

Question 4.9. If there is a non-minimal p-point over a measurable cardinal κ, does Solovay's $0^\dagger$ exist. [1] If κ is κ-compact, is there a non-minimal p-point over κ?

[1] For more on $0^\dagger$, see 5.13. This is the proper question to ask, since T.K. Menas has recently shown that if it is consistent that a measurable cardinal which is a limit of measurable cardinals exists, then it is consistent that the least measurable cardinal carries a non-minimal p-point.

Concerning generalizations, it follows by induction on n that if in 4.6 the $\mathcal{N}_\mu$'s were replaced by strictly $(n-1)$-Ramsey μ-ultrafilters $\mathcal{D}_\mu$, the resulting $\mathcal{D}$ will be a strictly n-Ramsey κ-ultrafilter:

The proof of 4.7 goes through with the appropriate modification in step 2, and to show that there is an f so that $f_*(\mathcal{D})$ is strictly $(n-1)$-Ramsey, for $\alpha \in A$ let $f_\alpha : (\alpha, m(\alpha)) \to (\alpha, m(\alpha))$ be such that $f_{\alpha*}(\mathcal{D}_{m(\alpha)})$ is strictly $(n-2)$-Ramsey. Then if

$$f = \bigcup_{\alpha \in A} f_\alpha ,$$

we have

$$\begin{aligned} X \in f_*(\mathcal{D}) \quad &\textit{iff} \quad \{\alpha \mid f^{-1}(X) \cap m(\alpha) \in \mathcal{D}_{m(\alpha)}\} \in \mathcal{N} \\ &\textit{iff} \quad \{\alpha \mid X \cap m(\alpha) \in f_{\alpha*}(\mathcal{D}_{m(\alpha)})\} \in \mathcal{N} . \end{aligned}$$

Hence, by induction $f_*(\mathcal{D})$ is strictly $(n-1)$-Ramsey.

I have proved:

Theorem 4.10. *If κ is a measurable cardinal and a limit of measurable cardinals which carry strictly $(n-1)$-Ramsey ultrafilters, $n > 1$, then for every normal ultrafilter $\mathcal{N}$ over κ, there is a $\mathcal{D} > \mathcal{N}$ which is strictly n-Ramsey.*

By the constructions so far, it does not seem possible to get p-points with an infinite number of constellations. I now present Kunen's example of p-points which have this property and many more. It is relevant to our context because it shows the richness of structure under the assumption of 2^κ-supercompactness.

Theorem 4.11. (Kunen, unpublished) *If κ is 2^κ-supercompact, there is an ascending Rudin–Keisler chain $\langle \mathcal{X}_\alpha \mid \alpha < (2^\kappa)^+ \rangle$ of p-points of length $(2^\kappa)^+$ such that for any $\beta < (2^\kappa)^+$,*

$$\mathcal{U} < \mathcal{X}_\beta \quad \textit{iff} \quad \mathcal{U} \cong \mathcal{X}_\alpha \quad \textit{for some } \alpha < \beta .$$

Note that $(2^\kappa)^+$ is the maximal length possible. This example shows that it is possible to have κ-ultrafilters with exactly μ constellations for every cardinal $\mu \leq 2^\kappa$. Indeed, for any ordinal $\rho < (2^\kappa)^+$ there can be p-points with the constellations in their one sky ordered in type ρ.

Proof. By definition of 2^κ-supercompactness let $j : V \to M$ be an elementary embedding which first moves κ, where M is transitive and closed under 2^κ sequences. By standard arguments, $(2^\kappa)^M = 2^\kappa$ and $(2^\kappa)^{+M} = (2^\kappa)^+$

Note also for $\kappa \leq \xi < j(\kappa)$, if

$$\mathcal{U}_\xi = \{X \subseteq \kappa \mid \xi \in j(X)\} ,$$

$\mathcal{U}_\xi$ is a κ-ultrafilter. The proof now depends on two lemmas.

Lemma 1. *If $\alpha, \beta < 2^\kappa$, there is a* $1-1$ *function $f \in {}^\kappa\kappa$ such that $j(f)(\beta) = \alpha$.*

Proof. Same as for 3.10. To show that f can be taken $1-1$, let $g \in {}^\kappa\kappa$ so that $j(g)(\alpha) = \beta$ and use the idea of 2.4.

Lemma 2. *If $\alpha < \beta < (2^\kappa)^+$, there is an almost* $1-1$ *function $f \in {}^\kappa\kappa$ such that $j(f)(\beta) = \alpha$*

Proof. Let $G \in {}^\kappa\kappa$ be defined by $G(\delta) = |\delta|$, and let $F : \{\langle \delta, \eta\rangle \mid \eta < \delta < \kappa\} \to \kappa$ be such that:

(i) for $\delta < \kappa$, F_δ defined by $F_\delta(\eta) = F(\langle \delta, \eta\rangle)$ is an injective function: $\delta \to |\delta|$.

(ii) for $\eta < \kappa$, there is a ν_η and ρ_η such that $\nu_\eta \leq \delta \to F_\delta(\eta) = \rho_\eta$.

By Lemma 1, it is only necessary to consider β such that $2^\kappa < \beta < (2^\kappa)^+$, so that $j(G)(\beta) = 2^\kappa$. Suppose $j(F)(\langle\beta, \alpha\rangle) = \gamma < 2^\kappa$. By Lemma 1, there is a $1-1$ g so that $j(g)(2^\kappa) = \gamma$. Then if $f \in {}^\kappa\kappa$ is defined by:

$$f(\delta) = F_\delta^{-1} gG(\delta) \text{ on } S = \{\delta \mid F_\delta^{-1} gG(\delta) \text{ is defined}\}$$

and $f \mid \kappa - S$ is arbitrary but $1-1$, then $j(f)(\beta) = \alpha$. To see that f is almost $1-1$, note that for any $\eta < \kappa$,

$$S \cap f^{-1}(\{\eta\}) \subseteq \nu_\eta \cup \{\delta \mid gG(\delta) = \rho_\eta\} ,$$

where g is $1-1$ and G is almost $1-1$, so that $|f^{-1}(\{\eta\})| < \kappa$. The proof of the lemma is complete.

To prove the theorem, define a sequence of ordinals θ_α for $\alpha < (2^\kappa)^+$ as follows:

$$\theta_0 = \kappa$$

$$\theta_{\alpha+1} = \text{least } \delta > \theta_\alpha \text{ such that for all } f \in {}^\kappa\kappa,\ j(f)(\theta_\alpha) \neq \delta.$$

$$\theta_\gamma = \sup\{\theta_\alpha \mid \alpha < \gamma\}, \quad \gamma \text{ a limit.}$$

By the lemmas, $\theta_1 > 2^\kappa$ and the θ_α's are just the beginnings of constellations $< (2^\kappa)^+$. If we set $\mathcal{X}_\alpha = \mathcal{U}_{\theta_\alpha}$, $\langle \mathcal{X}_\alpha \mid \alpha < (2^\kappa)^+\rangle$ is as required by the theorem:

(i) They are p-points, since if $j(f)(\theta_\alpha) = \kappa$, $[f]_{\mathcal{X}_\alpha} = \kappa$ and f can be taken almost $1-1$.

(ii) If $\mathcal{U} < \mathcal{X}_\alpha$, by Solovay's Lemma 4.1 we can assume $f_*(\mathcal{X}_\alpha) = \mathcal{U}$ and $[f]_{\mathcal{X}_\alpha} < [\mathrm{id}]_{\mathcal{X}_\alpha}$. But then, $j(f)(\theta_\alpha) < \theta_\alpha$ so that $j(f)(\theta_\alpha)$ is in the constellation of some θ_β for $\beta < \alpha$, i.e. $\mathcal{U} \cong \mathcal{X}_\beta$.

The proof of the theorem is now complete.

Question 4.12. Can Kunen's $\mathcal{X}_n$ for $n < \omega$ be strictly n-Ramsey? In general, is a two constellation p-point 2-Ramsey?

5. Sums and limits of ultrafilters

This section contains several results on sum and limit constructions. The following notational convenience will be used throughout.

Notation 5.1. If $f : \kappa \times \kappa \to \kappa$, then $f^\alpha : \kappa \to \kappa$ for $\alpha < \kappa$ is the function defined by $f^\alpha(\beta) = f(\langle \alpha, \beta \rangle)$.

If $\mathcal{D}$ and $\mathcal{E}_\alpha$ for $\alpha < \kappa$ are κ-ultrafilters, $\mathcal{D} \Sigma \mathcal{E}_\alpha$ is a κ-ultrafilter over $\kappa \times \kappa$ such that $\mathcal{D} < \mathcal{D} \Sigma \mathcal{E}_\alpha$ via the projection to the first coordinate, π_1. In fact, $\mathrm{er}([\pi_1])$ constitutes an initial segment of ordinals $\geq \kappa$, and $[f] \in \mathrm{er}([\pi_1])$ *iff* f^α is constant (mod $\mathcal{E}_\alpha$) for α in a set in $\mathcal{D}$. Note also that when $[f] \notin \mathrm{er}([\pi_1])$,

$$f_*(\mathcal{D} \Sigma \mathcal{E}_\alpha) = \mathcal{D}\text{-}\lim f^\alpha_*(\mathcal{E}_\alpha),$$

and that in particular,

$$\pi_{2*}(\mathcal{D} \Sigma \mathcal{E}_\alpha) = \mathcal{D}\text{-}\lim \mathcal{E}_\alpha .$$

The following formula is essentially due to Puritz.

Theorem 5.2. (Puritz [26]) *If $\mathcal{D}$, $\mathcal{E}_\alpha$ for $\alpha < \kappa$ are κ-ultrafilters,*

$$\tau(\mathcal{D} \Sigma \mathcal{E}_\alpha) = \tau(\mathcal{D}) + \prod_{\alpha<\kappa} \tau(\mathcal{E}_\alpha)/\mathcal{D} .$$

Proof. Let $\mathcal{U} = \mathcal{D} \Sigma \mathcal{E}_\alpha$. The first contribution to the ordinal sum on the right is due to the fact that $\mathrm{er}([\pi_1]_u)$ is an initial segment of the interval $[\kappa, i_u(\kappa))$. For the second, because of the basic relationship of skies to the τ function (see 3.5), it suffices to show the following: if $[f]_u$, $[g]_u \notin$

er($[\pi_1]_u$),

$$[f]_u \sim [g]_u \quad \textit{iff} \quad [f^\alpha]_{\mathcal{E}_\alpha} \sim [g^\alpha]_{\mathcal{E}_\alpha} \quad \text{for } \alpha \text{ is a set in } \mathcal{D}\,.$$

One direction is easy. For the other, assume for example that $X \in \mathcal{D}$ and for $\alpha \in X$, $[h_\alpha f^\alpha]_{\mathcal{E}_\alpha} > [g^\alpha]_{\mathcal{E}_\alpha}$ for some function h_α. If h is any function so that $h(\beta) > h_\alpha(\beta)$ for every $\alpha \leq \beta < \kappa$, then $[hf]_u > [g]_u$. An analogous argument with the f and g interchanged shows that $[f]_u \sim [g]_u$. The proof is complete.

Remark 5.3. This theorem can be used to construct "good" examples of many-skied ultrapowers, as heralded in 3.9. For instance, let $\mathcal{E}_\mu$ for μ regular $< \kappa$ be such that $\tau(\mathcal{E}_\mu) = \mu + 1$, as in 3.7. If $\mathcal{D}$ is any κ-ultrafilter and $f \in {}^\kappa\kappa$ is such that if $[f]_d = \gamma$, $M \cong V^\kappa/\mathcal{D}$ satisfies "$\kappa \leq \gamma$ and γ is a regular cardinal", then

$$\tau(\mathcal{D}\,\Sigma\,\mathcal{E}_{f(\alpha)}) = \tau(\mathcal{D}) + \gamma + 1\,.$$

Puritz goes on to establish conditions for when skies in the ultrapower by a sum can be just one constellation. As a corollary, he gets a result proved also by several others. For κ-ultrafilters, it states that given κ distinct normal κ-ultrafilters $\mathcal{N}_\alpha$, $\alpha < \kappa$, and any κ-ultrafilter $\mathcal{D}$, then $\mathcal{D}\,\Sigma\,\mathcal{N}_\alpha$ is a q-point such that $\tau(\mathcal{D}\,\Sigma\,\mathcal{N}_\alpha) = \tau(\mathcal{D}) + 1$.

I now prove a theorem on product ultrafilters which provides quite a useful characterization; for applications, see also Glazer [12]. It is implicit in Ketonen [17], and I derived this formulation independently of Puritz [27], Theorem 3.4.

Theorem 5.4. *Let $f, g \in {}^\kappa\kappa$ and $h : \kappa \to \kappa \times \kappa$ be defined by $h(\alpha) = \langle f(\alpha), g(\alpha)\rangle$. If $\mathcal{U}$, $\mathcal{D}$, and $\mathcal{E}$ are κ-ultrafilters, $h_*(\mathcal{U}) = \mathcal{D} \times \mathcal{E}$ iff*

(i) $f_*(\mathcal{U}) = \mathcal{D}$ and $g_*(\mathcal{U}) = \mathcal{E}$.

(ii) $[f]_u < \bigcap \mathrm{er}([g]_u)$.

($\bigcap \mathrm{er}([g]_u)$ is, of course, the least element of er($[g]_u$).*)*

Proof. Suppose that $h_*(\mathcal{U}) = \mathcal{D} \times \mathcal{E}$. Then (i) is straightforward, and for (ii), let $[kg]_u$ be the least element of er($[g]_u$). For $\alpha < \kappa$ $X_\alpha = \{\beta < \kappa \mid \alpha < k(\beta)\} \in \mathcal{E}$, so $Y = \bigcup_{\alpha<\kappa} \{\alpha\} \times X_\alpha \in \mathcal{D} \times \mathcal{E}$. Thus $h^{-1}(Y) \in \mathcal{U}$ and $\alpha \in h^{-1}(Y) \to kg(\alpha) > f(\alpha)$.

Conversely, suppose $X \in \mathcal{D} \times \mathcal{E}$. If $X \mid \alpha = \{\beta \mid \langle \alpha, \beta\rangle \in X\}$, $Y = \{\alpha \mid X \mid \alpha \in \mathcal{E}\} \in \mathcal{D}$. One can assume that $X \mid \alpha \subseteq (\alpha, \kappa)$ for every α. Define a function $t \in {}^{\kappa}\kappa$ by

$$t(\beta) = \text{least } \alpha \in Y \quad \text{such that } \beta \notin X \mid \alpha\,.$$

If $[tg]_u = \eta < \kappa$, $X \mid \eta \in \mathcal{E}$ but $\{\alpha \mid g(\alpha) \notin X \mid \eta\} \in \mathcal{U}$, contradicting $g_*(\mathcal{U}) = \mathcal{E}$. Hence by hypothesis, $Z = \{\alpha \mid f(\alpha) < tg(\alpha)\} \in \mathcal{U}$. But then $Z \cap f^{-1}(Y) \in \mathcal{U}$ and $\alpha \in Z \cap f^{-1}(Y) \to g(\alpha) \in X \mid f(\alpha)$. Hence $h^{-1}(X) \in \mathcal{U}$, which was to be proved.

The next theorem is also useful. Due to M.E. Rudin in the ω case, it says that the Rudin–Frolík ordering is a tree, i.e. the predecessors of any ultrafilter are linearly ordered.

Theorem 5.5. (Linearity of RF) (M.E. Rudin) *For κ-ultrafilters such that $\mathcal{D}$-lim $\mathcal{E}_\alpha = \mathcal{U}$-lim $\mathcal{V}_\alpha$, where $\{\mathcal{E}_\alpha \mid \alpha < \kappa\}$ and $\{\mathcal{V}_\alpha \mid \alpha < \kappa\}$ are discrete families, one of the following occurs:*

(i) $\mathcal{D} \cong \mathcal{U}$, $f_*(\mathcal{D}) = \mathcal{U}$, *and* $\mathcal{E}_\alpha = \mathcal{V}_{f(\alpha)}$ *for α in a set in $\mathcal{D}$.*

(ii) *There is a discrete family $\{\mathcal{F}_\beta \mid \beta < \kappa\}$ so that $\mathcal{D} = \mathcal{U}$-lim $\mathcal{F}_\beta$, and for β in a set in $\mathcal{U}$, $\mathcal{V}_\beta = \mathcal{F}_\beta$-lim $\mathcal{E}_\alpha$.*

(iii) *There is a discrete family $\{\mathcal{G}_\beta \mid \beta < \kappa\}$ so that $\mathcal{U} = \mathcal{D}$-lim $\mathcal{G}_\beta$, and for β in a set in $\mathcal{D}$, $\mathcal{E}_\beta = \mathcal{G}_\beta$-lim$\mathcal{V}_\alpha$.*

Corollary 5.6. *For κ-ultrafilters such that $\mathcal{D} \times \mathcal{E} \cong \mathcal{U} \times \mathcal{V}$, one of the following occurs:*

(i) $\mathcal{D} \cong \mathcal{U}$ *and* $\mathcal{E} \cong \mathcal{V}$.

(ii) *For some κ-ultrafilter $\mathcal{F}$, $\mathcal{D} \cong \mathcal{U} \times \mathcal{F}$ and $\mathcal{V} \cong \mathcal{F} \times \mathcal{E}$.*

(iii) *For some κ-ultrafilter $\mathcal{G}$, $\mathcal{U} \cong \mathcal{D} \times \mathcal{G}$ and $\mathcal{E} \cong \mathcal{G} \times \mathcal{V}$.*

Proofs. See for example Blass [1] or Booth [5]. No modifications are needed to get the measurable cardinal case.

Some applications of the two previous theorems are now made. The following interesting result was first discovered by Solovay; the analogue for $\beta_u\omega$ is not known. By using Rudin's Theorem a short proof is possible; the original proof was presumably more involved.

Theorem 5.7. (Solovay) *If $\mathcal{U}$ and $\mathcal{V}$ are κ-ultrafilters such that*

$$\mathcal{U} \times \mathcal{V} \cong \mathcal{V} \times \mathcal{U}\,,$$

then there is a κ-ultrafilter $\mathcal{W}$ and integers n and m so that $\mathcal{U} \cong \mathcal{W}^n$ and $\mathcal{V} \cong \mathcal{W}^m$.

Proof. Use 5.6. If (i) of 5.6 occurs, we are done. Otherwise, for example, there is a $\mathcal{D}$ such that $\mathcal{U} \cong \mathcal{V} \times \mathcal{D}$ and $\mathcal{U} \cong \mathcal{D} \times \mathcal{V}$. Since $\mathcal{U} < \mathcal{U} \times \mathcal{V}$, the problem has been reduced one step down in the RK order. Repeating this process and using the well-foundedness of the order, we see that (i) will eventually occur, and thus, the required $\mathcal{W}$ will emerge.

So, for example, if $\mathcal{U}$ and $\mathcal{V}$ are κ-ultrafilters so that $\tau(\mathcal{U}) < \omega$ and $\tau(\mathcal{V}) \geq \omega$, then $\mathcal{U} \times \mathcal{V} \not\cong \mathcal{V} \times \mathcal{U}$.

Question 5.8. If $\mathcal{U}$ and $\mathcal{V}$ are κ-ultrafilters such that $\mathcal{U} \times \mathcal{V} \leq \mathcal{V} \times \mathcal{U}$, does the conclusion to 5.7 still hold?

Kunen [21] showed by an elegant argument (without CH) that the RK ordering on $\beta_u \omega$ is not a linear ordering. However, note that RK on κ-ultrafilters in $L[\mathcal{U}]$ is (trivially) linear. On the other hand, distinct normal κ-ultrafilters are, of course, RK incomparable. Whether there is more than one normal κ-ultrafilter or not, the following theorem will show that RK on κ-ultrafilters is not linear in most cases, e.g. when κ is κ-compact. It is a solvable case of 5.8.

Theorem 5.9. *If $\mathcal{U}$ and $\mathcal{V}$ are κ-ultrafilters, $\mathcal{U}$ is a p-point, and*

$$\mathcal{U} \times \mathcal{V} \leq \mathcal{V} \times \mathcal{U},$$

then there is an integer n such that $\mathcal{V} \cong \mathcal{U}^n$, so $\mathcal{U} \times \mathcal{V} \cong \mathcal{V} \times \mathcal{U}$.

Remark. If $\mathcal{U}$ is a p-point and $\mathcal{V}$ is such that $\tau(\mathcal{V}) \geq \omega$, then by the theorem, $\mathcal{U} \times \mathcal{V} \not\leq \mathcal{V} \times \mathcal{U}$. Also, since $\tau(\mathcal{U} \times \mathcal{V}) = \tau(\mathcal{V})$ and $\tau(\mathcal{V} \times \mathcal{U}) = \tau(\mathcal{V}) + 1$, $\mathcal{V} \times \mathcal{U} \not\leq \mathcal{U} \times \mathcal{V}$. Hence, $\mathcal{U} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{U}$ are RK incomparable.

Proof of Theorem. By 5.4, let $h_*(\mathcal{V} \times \mathcal{U}) = \mathcal{U} \times \mathcal{V}$, $h(x) = \langle f(x), g(x) \rangle$, $f_*(\mathcal{V} \times \mathcal{U}) = \mathcal{U}$, $g_*(\mathcal{V} \times \mathcal{U}) = \mathcal{V}$, and $[f] < \cap\, \mathrm{er}([g])$. (All equivalence classes of functions in this proof are mod $\mathcal{V} \times \mathcal{U}$, unless otherwise subscripted.) I first show that: (a) $[f] \in \mathrm{er}([\pi_1])$, and (b) $[g] \notin \mathrm{er}([\pi_1])$.

For (a), if $[f] \notin \mathrm{er}([\pi_1])$, then by 5.2, and the fact that $\mathcal{U}$ is a p-point, $[f]$ would have to be in the highest sky. But this violates $[f] < \cap\, \mathrm{er}([g])$. Now to show (b), suppose otherwise and let $[g] = [k\pi_1]$. Then $k_*(\mathcal{V}) = \mathcal{V}$ so k is the identity, and $[g] = [\pi_1]$. But $\mathrm{er}([\pi_1])$ is an initial segment of the ultrapower, so that $[f] \in \mathrm{er}([g])$, a contradiction.

Now to proceed with the proof. By (b), $\mathcal{V} = \mathcal{V}\text{-}\lim g^\alpha_*(\mathcal{U})$. Set $\mathcal{U}_\alpha = g^\alpha_*(\mathcal{U})$. If $\mathcal{U}_\alpha = \mathcal{V}$ for α in a set in $\mathcal{V}$, $\mathcal{V} \leq \mathcal{U}$, and since from (a) it follows that $\mathcal{U} \leqslant \mathcal{V}$, $\mathcal{V} \cong \mathcal{U}$ and we are done. So, we can assume that this is not the case. Let $t \in {}^\kappa\kappa$ be any function such that $t(\alpha) = t(\beta)$ *iff* $\mathcal{U}_\alpha = \mathcal{U}_\beta$. t is thus non-constant (mod $\mathcal{V}$), and if we set $\mathcal{W}_{t(\alpha)} = \mathcal{U}_\alpha$, then

$$\mathcal{V} = \mathcal{V}\text{-lim } \mathcal{U}_\alpha = \mathcal{V}\text{-lim } \mathcal{W}_{t(\alpha)} = t_*(\mathcal{V})\text{-lim } \mathcal{W}_\alpha .$$

The $\mathcal{W}_\alpha$'s are now distinct (and p-points, being $\leqslant \mathcal{U}$) so they are discrete by 1.12, i.e.

$$(*) \qquad \mathcal{V} \cong t_*(\mathcal{V}) \Sigma \mathcal{W}_\alpha .$$

Since $\mathcal{W}_\alpha \leqslant \mathcal{U}$ for $\alpha < \kappa$, it is easy to show $t_*(\mathcal{V}) \Sigma \mathcal{W}_\alpha \leqslant t_*(\mathcal{V}) \times \mathcal{U}$. Hence, it suffices to prove $\mathcal{U} \times t_*(\mathcal{V}) \leqslant \mathcal{V}$, for then

$$(**) \qquad \mathcal{U} \times t_*(\mathcal{V}) \leqslant t_*(\mathcal{V}) \Sigma \mathcal{W}_\alpha \leqslant t_*(\mathcal{V}) \times \mathcal{U} .$$

Thus, as $t_*(\mathcal{V}) < \mathcal{V}$, an RK reduction would have been achieved, and we can conclude that if $t_*(\mathcal{V}) \cong \mathcal{U}^n$, then $\mathcal{U}^{n+1} \cong \mathcal{V}$ by (*) and (**).

Finally, to prove $\mathcal{U} \times t_*(\mathcal{V}) \leqslant \mathcal{V}$ it is sufficient to show $[f] < \bigcap \mathrm{er}([t\pi_1])$. Since by (a) $[k\pi_1] = [f]$ for some k, it would follow that $[k]_\mathrm{v} < \bigcap \mathrm{er}([t]_\mathrm{v})$ and $k_*(\mathcal{V}) = \mathcal{U}$. Hence, 5.4 would be applicable.

That $[f] < \bigcap \mathrm{er}([t\pi_1])$ follows from $[f] < \bigcap \mathrm{er}([g])$ and the next claim:

Claim. $[t\pi_1] = [sg]$ for some $s \in {}^\kappa\kappa$.

To prove this, let $\langle P_\alpha \mid \alpha < \kappa \rangle$ be a partition of κ so that $P_\alpha \in \mathcal{W}_\alpha$. Define s by $s(\delta) = \alpha$ *iff* $\delta \in P_\alpha$. Then on $\bigcup_{\alpha<\kappa} \{\alpha\} \times (g^\alpha)^{-1}(P_{t(\alpha)}) \in \mathcal{V} \times \mathcal{U}$,

$$t\pi_1(\langle \alpha, \beta \rangle) = t(\alpha) = s(g^\alpha(\beta)) = sg(\langle \alpha, \beta \rangle) .$$

The proof of the theorem is now complete.

The next theorem has to do with the Rudin–Frolik ordering on κ-ultrafilters; it shows that the tree ordering cannot be very high.

Theorem 5.10. *No κ-ultrafilter can have κ Rudin–Frolik predecessors.*

Proof. Argue by contradiction, and let $\mathcal{U}$ be a counterexample. By well-foundedness of the RF order on κ-ultrafilters, we can assume that there are $\mathcal{D}_\alpha$ for $\alpha < \kappa$ so that

$$\alpha < \beta < \kappa \to \mathcal{D}_\alpha <_{\mathrm{RF}} \mathcal{D}_\beta <_{\mathrm{RF}} \mathcal{U} ,$$

$$\mathcal{E} <_{\mathrm{RF}} \mathcal{U} \to \mathcal{E} \cong \mathcal{D}_\alpha \quad \text{for some } \alpha < \kappa,$$

and that $\mathcal{U}$ is an RF-least upper bound for the $\mathcal{D}_\alpha$'s. To get a contradiction, a $\mathcal{V}$ will be found so that $\mathcal{V} <_{RF} \mathcal{U}$ and $\mathcal{V}$ is still an RF-upper bound for the $\mathcal{D}_\alpha$'s. The argument is based on a diagonalization process.

For $\alpha < \beta < \kappa$, let $\{\mathcal{E}_\xi^\alpha \mid \xi < \kappa\}$, $\{\mathcal{F}_\xi^{\alpha\beta} \mid \xi < \kappa\}$ be discrete families so that

$$\mathcal{D}_\alpha\text{-lim } \mathcal{E}_\xi^\alpha = \mathcal{U},$$

$$\mathcal{E}_\xi^\alpha = \mathcal{F}_\xi^{\alpha\beta}\text{-lim } \mathcal{E}_\eta^\beta \quad \text{for } \xi \text{ in a set in } \mathcal{D}_\alpha.$$

$$\mathcal{D}_\beta = \mathcal{D}_\alpha\text{-lim } \mathcal{F}_\xi^{\alpha\beta}.$$

The existence of these various families and relationships follows from the definition and linearity of the RF order.

Suppose first that for some $\alpha < \beta < \kappa$, $T = \{\eta \mid \mathcal{E}_\eta^\beta = \mathcal{E}_\xi^\alpha$ for some $\xi < \kappa\} \in \mathcal{D}_\beta$. Then, $T \in \mathcal{F}_\xi^{\alpha\beta}$ and $\mathcal{E}_\xi^\alpha = \mathcal{F}_\xi^{\alpha\beta}\text{-lim } \mathcal{E}_\eta^\beta$ for ξ in a set in $\mathcal{D}_\alpha$. But this leads to a contradiction, since for such ξ, $\mathcal{E}_\xi^\alpha$ is a limit of others in $\{\mathcal{E}_\delta^\alpha \mid \delta < \kappa\}$. Thus, by an inductive argument on β, we can suppose in what follows that $\mathcal{E}_\eta^\beta$ is different from any $\mathcal{E}_\xi^\alpha$ for $\alpha < \beta < \kappa$.

Now define a function f inductively so that: (a) $\mathcal{E}_\xi^0 = \mathcal{F}_\xi^{0f(\xi)}\text{-lim } \mathcal{E}_\eta^{f(\xi)}$, and (b) whenever possible, $f(\xi) > f(\xi')$ for any $\xi' < \xi$. It is not hard to see that f is defined (mod $\mathcal{D}_0$) and cannot be constant (mod $\mathcal{D}_0$). Let $\langle X_\xi \mid \xi < \kappa\rangle$ be a partition of κ so that $X_\xi \in \mathcal{E}_\xi^0$. Consider now the family

$$\Delta = \{\mathcal{E}_\eta^{f(\xi)} \mid \xi < \kappa \text{ and } X_\xi \in \mathcal{E}_\eta^{f(\xi)}\}.$$

Δ is clearly a discrete family. By limit considerations, if $X \in \mathcal{U}$, $X \in \mathcal{E}_\xi^0$ for a ξ so that for some η, X and X_ξ are in $\mathcal{E}_\eta^{f(\xi)}$. Thus, $\mathcal{U}$ is a limit of Δ. Let $\{\mathcal{V}_\delta \mid \delta < \kappa\}$ be an enumeration of Δ, and set

$$\mathcal{V} = \{Y \subseteq \kappa \mid Y = \{\delta \mid X \in \mathcal{V}_\delta\} \text{ and } X \in \mathcal{U}\}.$$

By straightforward arguments (as in the ω case) $\mathcal{V}$ is a κ-ultrafilter so that $\mathcal{V}\text{-lim } \mathcal{V}_\delta = \mathcal{U}$. Hence, $\mathcal{V} <_{RF} \mathcal{U}$.

To complete the proof, it suffices to show that $\mathcal{D}_\alpha <_{RF} \mathcal{V}$ for every $\alpha < \kappa$. If not, then for some $\beta < \kappa$ we must have $\mathcal{D}_\beta \cong \mathcal{V}$, a function π so that $\pi_*(\mathcal{D}_\beta) = \mathcal{V}$, and $\mathcal{E}_\eta^\beta = \mathcal{V}_{\pi(\eta)}$ for η in a set in $\mathcal{D}_\beta$. For such η, by definition of Δ and the distinctness assumptions on the $\mathcal{E}_\xi^\alpha$'s, $X_\xi \in \mathcal{E}_\eta^\beta$ for a ξ such that $f(\xi) = \beta$. Hence,

$$\bigcup\{X_\xi \mid f(\xi) = \beta\} \in \mathcal{D}_\beta\text{-lim } \mathcal{E}_\eta^\beta = \mathcal{U}.$$

This contradicts the fact that f is not constant (mod $\mathcal{D}_0$), and the proof is now complete.

I am indebted to J. Paris for pointing out an error in a previous version of the above proof.

It might be appropriate here to state perhaps the two most important open questions in the structure theory of κ-ultrafilters:

Question 5.11. Is there a κ-ultrafilter with an infinite number of Rudin–Frolik predecessors?

Question 5.12. If $\{\mathcal{D}_\alpha \mid \alpha < \kappa\}$ is a family of distinct κ-ultrafilters and $\mathcal{U}$ any κ-ultrafilter, is there an $X \in \mathcal{U}$ so that $\{\mathcal{D}_\alpha \mid \alpha \in X\}$ is a discrete family?

For the ω case, Booth [5] constructs an example to answer the first question in the affirmative, and Kunen [19] constructs with CH a counterexample (in a strong sense) to the second question.

There are no κ-ultrafilters as hypothesized by 5.11 in the Kunen–Paris model [22] nor in the Mitchell model [24], and a negative answer to the question in general would be very interesting. It would follow, for example, that there is an RF-minimal non-p-point by a simple argument. Solovay showed that such a κ-ultrafilter exists, but from the assertion that κ is 2^κ-supercompact and by an involved argument. In any case, the following observation can be made.

Proposition 5.13. *If there is a κ-ultrafilter with an infinite number of Rudin–Frolik predecessors, then Solovay's $0^\dagger$ exists.* ($0^\dagger$ is the analogue of $0^\#$ for the model $L[\mathcal{U}]$; see Kunen [20] for some results concerning it.)

Proof. Suppose that

$$\mathcal{D}_0 <_{RF} \mathcal{D}_1 <_{RF} \mathcal{D}_2 <_{RF} \cdots <_{RF} \mathcal{D} .$$

For each $n \in \omega$, there is a discrete family $\{\mathcal{E}^n_\alpha \mid \alpha < \kappa\}$ so that $\mathcal{D}_n\text{-lim}\, \mathcal{E}^n_\alpha = \mathcal{D}$. By the linearity of the RF order, if $n < m < \omega$, each $\mathcal{E}^n_\alpha$ is a limit of $\{\mathcal{E}^m_\beta \mid \beta < \kappa\}$. Thus, an easy well-foundedness argument shows that there must exist a situation of the following kind:

$$\mathcal{U} = \mathcal{V}\text{-lim}\ \mathcal{V}_\alpha \cong \mathcal{V}\, \Sigma\, \mathcal{V}_\alpha ,$$

and

$$\{\alpha \mid \mathrm{i}_{\mathrm{v}_\alpha}(\kappa) = \mathrm{i}_{\mathrm{u}}(\kappa)\} \in \mathcal{V} .$$

Set $\gamma = \mathrm{i}_u(\kappa)$. Then

$$\begin{aligned}\gamma &= \text{order type}\,\Big(\prod_{\alpha<\kappa} \mathrm{i}_{v_\alpha}(\kappa)/\mathcal{V}\Big)\\ &= \text{order type}\,(\gamma^\kappa/\mathcal{V})\\ &= \mathrm{i}_v(\gamma)\,.\end{aligned}$$

Finally, define κ_n for $n < \omega$ by induction as follows: $\kappa_0 = \kappa$, $\kappa_{n+1} = \mathrm{i}_v(\kappa_n)$, and $\kappa_\omega = \sup\{\kappa_n \mid n \in \omega\}$. Then κ_ω is the least element $> \kappa$ fixed by i_v, so that $\kappa_\omega < \gamma = \mathrm{i}_u(\kappa)$. By Theorem 9.4 of Kunen [20], $0^\dagger$ exists.

Actually, by methods of [20] an inner model with two measurable cardinals can be constructed from the fact that $\kappa_\omega < \mathrm{i}_u(\kappa)$. Concerning the question 5.12, the following proposition is relevant.

Proposition 5.14. *Let $\mathcal{D}$, $\mathcal{E}_\alpha$ for $\alpha < \kappa$, be distinct κ-ultrafilters and consider $\mathcal{U} = \mathcal{D}\ \Sigma\ \mathcal{E}_\alpha$. The following are then equivalent.*

(i) *There is an $X \in \mathcal{D}$ so that $\{\mathcal{E}_\alpha \mid \alpha \in X\}$ is a discrete collection.*

(ii) *π_2 is* 1–1 (mod $\mathcal{U}$).

(iii) $[\pi_1] \in \mathrm{er}([\pi_2])$.

(iv) $\mathrm{sk}([\pi_1]) \cap \mathrm{er}([\pi_2]) \neq \emptyset$.

Proof. The logical progression is (i) → (ii) → (iii) → (iv) → (i). I only prove the last implication, as the others are evident.

Suppose that $[f\pi_1] = [g\pi_2]$, where f is an almost 1–1 function. If $X_\alpha = \{\beta \mid f(\alpha) = g(\beta)\}$ for $\alpha < \kappa$, then $Y = \{\alpha < \kappa \mid X_\alpha \in \mathcal{E}_\alpha\} \in \mathcal{D}$. Observe that $X_\alpha \cap X_\beta \neq \emptyset$ implies $f(\alpha) = f(\beta)$. But f was almost 1–1, so for each $\gamma < \kappa$: the X_α's for those α so that $f(\alpha) = \gamma$ can certainly be made mutually disjoint so that (calling these new sets again X_α) each X_α is still a member of $\mathcal{E}_\alpha$ for $\alpha \in Y$. Hence, $\{\mathcal{E}_\alpha \mid \alpha \in Y\}$ is a discrete family.

The following corollary is known, but usually via a different proof, e.g. Blass [1].

Corollary 5.15. *Let $\mathcal{D}$ be a p-point and $\mathcal{E}_\alpha$ for $\alpha < \kappa$ distinct κ-ultrafilters. Then there is a $Y \in \mathcal{D}$ such that $\{\mathcal{E}_\alpha \mid \alpha \in Y\}$ is discrete.*

Proof. Consider $\mathcal{U} = \mathcal{D}\ \Sigma\ \mathcal{E}_\alpha$. If $[\pi_1] < \cap\, \mathrm{er}([\pi_2])$, then by 5.4,

$$\mathcal{D}\ \Sigma\ \mathcal{E}_\alpha = \mathcal{D} \times \pi_{2*}(\mathcal{D}\ \Sigma\ \mathcal{E}_\alpha) = \mathcal{D} \times \mathcal{D}\text{-lim}\ \mathcal{E}_\alpha\,.$$

It follows that $\mathcal{E}_\beta = \mathcal{D}\text{-lim}\ \mathcal{E}_\alpha$ for β in a set in $\mathcal{D}$, a contradiction since the $\mathcal{E}_\beta$'s were distinct. Thus, for some f, $[f\pi_2] \leqslant [\pi_1]$. Since $\mathcal{D}$ is a p-point, $\mathrm{sk}([\pi_1])$ is the least sky of $\mathcal{U}$, i.e. $\mathrm{sk}([\pi_1]) \cap \mathrm{er}([\pi_2]) \neq \emptyset$. The result now follows from the proposition.

By the reasoning of the corollary, it follows that if there are $\mathcal{D}$, $\mathcal{E}_\alpha$ for $\alpha < \kappa$ which do not satisfy any (and hence all) of the conditions of the proposition, then for $\mathcal{D}\,\Sigma\,\mathcal{E}_\alpha$ we must have $\mathrm{er}([\pi_2])$ not meeting $\mathrm{sk}([\pi_1])$, but containing elements below it – a rather peculiar situation. For more on 5.12 and related topics, see Glazer [12].

6. Jonsson and Rowbottom filters

This final section is concerned with filter related formulations of some well-known concepts and their relationships with the function τ. Some similar notions were independently considered by Ketonen [16].

Definitions 6.1. Let $\mathcal{F}$ be a uniform filter over a cardinal λ. $\mathcal{F}$ is *Jonsson iff* for any $F : [\lambda]^{<\omega} \to \lambda$ there is an $X \in \mathcal{F}$ so that $F''[X]^{<\omega} \neq \lambda$. If $\mu < \lambda$, $\mathcal{F}$ is *μ-Rowbottom iff* for any $F : [\lambda]^{<\omega} \to \nu < \lambda$ there is an $X \in \mathcal{F}$ so that $|F''[X]^{<\omega}| < \mu$.

For the terminology, see e.g. Devlin [10]. Straightforward arguments show that if $\mathcal{U}, \mathcal{V} \in \beta_u\lambda$ and $\mathcal{V} \leqslant \mathcal{U}$, $\mathcal{U}$ Jonsson $\to$ $\mathcal{V}$ Jonsson, and $\mathcal{U}$ μ-Rowbottom $\to$ $\mathcal{V}$ μ-Rowbottom. The following auxiliary notion, somewhat akin to that of indecomposability in Prikry [25], will be useful in the discussion.

Definition 6.2. If $\mathcal{F}$ is a uniform filter over a cardinal λ and $\alpha < \lambda$, $\mathcal{F}$ is *α-strongly indecomposable* (abbreviated α-str. indec.) *iff* for any $G : [\lambda]^{<\omega} \to \alpha$ there is an $X \in \mathcal{F}$ so that $|G''[X]^{<\omega}| < |\alpha|$.

The main interest here is in κ-ultrafilters, but the general situation will be discussed briefly at the end of the section. The following theorem applies only to κ-ultrafilters.

Theorem 6.3. *If $\mathcal{U}$ is a κ-ultrafilter, $\mathcal{U}$ is Jonsson iff $\mathcal{U}$ is μ-Rowbottom for some $\mu < \kappa$.*

Proof. One direction is standard: if $\mathcal{U}$ is μ-Rowbottom and $F : [\kappa]^{<\omega} \to \kappa$, define $G : [\kappa]^{<\omega} \to \mu$ by

$$G(s) = \begin{cases} F(s) & \text{if } F(s) < \mu, \\ 0 & \text{otherwise.} \end{cases}$$

If $X \in \mathcal{U}$ and $|G''[X]^{<\omega}| < \mu$, then $F''[X]^{<\omega} \neq \kappa$.

The proof of the converse is in two steps. For an ordinal δ, say that $\phi_u(\delta)$ is satisfied *iff* for any $F : [\kappa]^{<\omega} \to \delta$ there is an $X \in \mathcal{U}$ so that $F''[X]^{<\omega} \neq \delta$. I will show: (a) there is $\gamma < \kappa$ so that $\phi_u(\gamma)$, and (b) if $\phi_u(\delta)$ for $\delta < \kappa$, then in fact $\mathcal{U}$ is δ-str. indec. Since $\phi_u(\delta)$ and $\delta < \eta < \kappa$ implies $\phi_u(\eta)$ by an argument like in the preceding paragraph, the result would follow by repeated use of (b).

The argument to show (a) is due to Kleinberg [18]. If no $\gamma < \kappa$ satisfies $\phi_u(\gamma)$, let $G_\gamma : [\kappa]^{<\omega} \to \gamma$ be counterexamples for every $\gamma < \kappa$. Define $F: [\kappa]^{<\omega} \to \kappa$ by

$$F(\{\alpha_1, ..., \alpha_n\}) = G_{\alpha_1}(\{\alpha_2, ..., \alpha_n\})$$

where $\alpha_1 < ... < \alpha_n$. Then for every $X \in \mathcal{U}$ $F''[X]^{<\omega} = \kappa$, contradicting $\mathcal{U}$ is Jonsson.

To show (b), let $\phi_u(\delta)$ be satisfied for a $\delta < \kappa$ and assume $F: [\kappa]^{<\omega} \to \delta$ is a counterexample to (b). If

$$\mathcal{S} = \{S \subseteq \delta \mid F''[X]^{<\omega} = S \ \text{ for some } X \in \mathcal{U}\},$$

$\mathcal{S}$ is a basis for a uniform filter over δ which is κ-complete. Hence, as $2^\delta < \kappa$, $\mathcal{S}$ has some principal generator $S_0 \subseteq \delta$. Define $H : [\kappa]^{<\omega} \to S_0$ by

$$H(s) = \begin{cases} F(s) & \text{if } F(s) \in S_0, \\ \rho & \text{otherwise,} \end{cases}$$

where ρ is some fixed element of S_0. Then for any $X \in \mathcal{U}$ $H''[X]^{<\omega} = S_0$, and $|S_0| = |\delta|$, contradicting $\phi_u(\delta)$.

The proof is now complete.

The next theorem will be used to get upper bounds on the number of skies Jonsson and Rowbottom κ-ultrafilters can have. But first, a simple proposition; recall that a set of ordinals is *μ-closed* if it is closed under increasing μ sequences in the order topology.

Proposition 6.4. *Let $\mu < \lambda$ be regular cardinals and $\mathcal{U} \in \beta_u \lambda$. Then $\mathcal{U}$ extends the filter generated by the μ-closed, unbounded sets iff $\mathcal{U} \supseteq \mathcal{C}_\lambda \cup \{\{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu\}\}$.*

Proof. Suppose $\mathcal{U} \supseteq \mathcal{C}_\lambda \cup \{\{\alpha \mid \mathrm{cf}(\alpha) = \mu\}\}$ and X is μ-closed unbounded. If $\kappa - X \in \mathcal{U}$, then f defined by $f(\alpha) = \sup(X \cap \alpha)$ is regressive on $(\kappa - X) \cap \{\alpha \mid \mathrm{cf}(\alpha) = \mu\} \in \mathcal{U}$. But every set in $\mathcal{U}$ is stationary, and so f is constant on an unbounded set, a contradiction.

Theorem 6.5. *Let $\mu < \lambda$ be regular cardinals and $\mathcal{U} \in \beta_u\lambda$. If $\mathcal{U}$ is μ-str. indec., then $\mathcal{U}$ does not extend the filter generated by the μ-closed, unbounded sets.*

Proof. For each $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) = \mu$, fix a cofinal sequence $\langle \gamma^\alpha_\xi \mid \xi < \mu \rangle$. Define $F : [\lambda]^2 \to \mu$ by

$$F(\{\alpha,\beta\}) = \begin{cases} \xi & \text{if } \mathrm{cf}(\beta) = \mu,\ \alpha < \beta, \text{ and } \xi \text{ is least such that } \alpha \leq \gamma^\beta_\xi, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{U}$ is μ-str. indec., let $X \in \mathcal{U}$ and $F''[X]^2 \subseteq \delta$, where $\delta < \mu$. Suppose that $\langle \alpha_\xi \mid \xi < \mu \rangle$ is a strictly increasing sequence of elements of X, and let $\alpha = \sup\{\alpha_\xi \mid \xi < \mu\}$. If α were in X, $\alpha_\xi \leq \gamma^\alpha_\delta < \alpha$ for every $\xi < \mu$, a contradiction. Thus, X is disjoint from its μ-closure $- X$, which is obviously μ-closed, unbounded, and the theorem is proved.

Corollary 6.6. *If $\mathcal{U}$ is a κ-ultrafilter,*

(i) *$\mathcal{U}$ is Jonsson $\to \tau(\mathcal{U}) < \kappa$.*

(ii) *$\mathcal{U}$ is μ-Rowbottom where μ is regular $< \kappa \to \tau(\mathcal{U}) < \mu$.*

(iii) *$\mathcal{U}$ is ω_1-Rowbottom $\to$ $\mathcal{U}$ has at most countably many skies.*

Proof. By previous results, $\tau(\mathcal{U}) \geq \mu \to$ there is a $\mathcal{V} \leq \mathcal{U}$ such that $\mathcal{V} \supseteq C_\kappa \cup \{\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}\}$.

Thus, if κ is κ-compact, there are non-Jonsson κ-ultrafilters. But also, if $\mathcal{U}$ is a normal κ-ultrafilter, by arguments like in 4.4, $\mathcal{U}^n$ is ω_1-Rowbottom for every integer n. Hence, it is consistent that every κ-ultrafilter is ω_1-Rowbottom, since this is true in $L[\mathcal{U}]$.

Question 6.7. Are there ω_1-Rowbottom κ-ultrafilters with an infinite number of skies?

The next theorem is also important because of its corollary.

Theorem 6.8. *Let $\mu < \lambda$ be regular cardinals and $\mathcal{U} \in \beta_u\lambda$. Suppose that for $\alpha < \mu$ there are functions $f_\alpha \in {}^\lambda\lambda$ such that each f_α is unbounded (mod $\mathcal{U}$) and $\alpha < \beta < \mu \to [f_\alpha] \notin \mathrm{con}([f_\beta])$, but there is an h so that $hf_\beta(\xi) = f_\alpha(\xi)$ for every $\xi < \lambda$. Then $\mathcal{U}$ is not μ-str. indec.*

Proof. Let $\mathcal{P}$ be the common refinement of the f_α's considered as partitions of λ and let $f \in {}^\lambda\lambda$ be any function such that $f(\alpha) = f(\beta)$ *iff* α and β are in the same partition. Consider $\mathcal{V} = f_*(\mathcal{U})$. We can assume that the hypotheses on $\mathcal{U}$ continue to hold for $\mathcal{V}$, and, in addition, the following: if $\xi < \eta < \lambda$ there is an $\alpha < \mu$ so that $f_\alpha(\xi) \neq f_\alpha(\eta)$. Define $f: [\lambda]^2 \to \mu$ by

$$F(\{\xi, \eta\}) = \text{least } \alpha < \mu(f_\alpha(\xi) \neq f_\alpha(\eta)).$$

If $\mathcal{V}$ were μ-str. indec., there would be an $X \in \mathcal{V}$ such that $F''[X]^2 \subseteq \delta$, where $\delta < \mu$. But then, it is evident that $\xi, \eta \in X$ and $f_\delta(\xi) = f_\delta(\eta) \to \xi = \eta$, i.e. f_δ is $1-1$ (mod $\mathcal{V}$). This contradiction shows that $\mathcal{V}$ is not μ-str. indec., and since $\mathcal{V} \leqslant \mathcal{U}$, $\mathcal{U}$ is not μ-str. indec.

Corollary 6.9. *If κ is 2^κ-supercompact, there is a non-Jonsson p-point κ-ultrafilter.*

Proof. Consider $\mathcal{K}_\kappa$ in the Kunen chain of p-points, 4.11. For every regular $\mu < \kappa$, the hypotheses of the theorem are satisfied for $\mathcal{K}_\kappa$, so $\mathcal{K}_\kappa$ is not μ-Rowbottom. Hence $\mathcal{K}_\kappa$ is not Jonsson.

It might be appropriate to conclude this section with some discussion of Jonsson and Rowbottom filters in general, when the underlying cardinal is not necessarily measurable. It is a well known result of Solovay [31] that any λ-complete normal μ-saturated filter over λ, where μ is an uncountable regular cardinal $< \lambda$, is a μ-Rowbottom filter. For more on questions of existence and relative consistency, see Devlin [10].

In the following, Kleinberg [18] will be very relevant. For example the next theorem is a filter related formulation of some results which appear there.

Theorem 6.10. (Kleinberg [18]) *If $\mathcal{F}$ is a Jonsson filter over λ, then for some $\delta < \lambda$ $\mathcal{F}$ is δ-str. indec., and for μ such that $\delta < \mu < \lambda$: if $\mathcal{F}$ is not μ-str. indec. and $G : [\lambda]^{<\omega} \to \mu$ is a counterexample to this, then*

$$\{G''[X]^{<\omega} \mid X \in \mathcal{F}\}$$

generates a Jonsson filter over μ.

Actually, the least δ as above is the least γ so that $\phi_{\mathcal{F}}(\gamma)$ is satisfied, as in (a) of the proof of 6.3.

Corollary 6.11. (i) (Kleinberg [18]). *If λ_0 is the least cardinal which carries a Jonsson filter and $\mathcal{F}$ over λ_0 is Jonsson, then for some $\mu < \lambda_0$ $\mathcal{F}$ is μ-Rowbottom.*

(ii) (G.C.H.) *If $\mathcal{F}$ over λ is Jonsson, then for sufficiently large $\mu < \lambda$, if μ is a successor cardinal, $\mathcal{F}$ is μ-str. indec.*

Proof. (i) is immediate, and for (ii), by a well known result of Erdős and Hajnal (see Devlin [10]), no successor cardinal can carry a Jonsson filter. Hence λ is a limit cardinal, and the rest follows as well.

Concerning strong indecomposability, there is also:

Proposition 6.12. *Let $\mathcal{F}$ be a filter over λ.*

(i) (Kleinberg [18]). *If $\mathcal{F}$ is μ-str. indec., then $\mathcal{F}$ is μ^+-str. indec.*

(ii) *If $\mathcal{F}$ is μ-str. indec., where μ is regular, and $\gamma < \mu \to 2^\gamma < 2^\mu$, then is 2^μ-str. indec.*

(iii) *If μ is a limit cardinal $> \omega$ such that $2^\mu = \mu^+$ and $\mathcal{F}$ is μ-str. indec., then for sufficiently large $\rho < \mu$ $\mathcal{F}$ is ρ-str. indec.*

Proof. (i) follows directly from the following Lemma in Kleinberg [18]: if $\mu < \lambda$ and $F : [\lambda]^{<\omega} \to \mu^+$, then there is a $G : [\lambda]^{<\omega} \to \mu$ so that if $X \subseteq \lambda$,

$$|F''[X]^{<\omega}| \leq |G''[X]^{<\omega}|^+ .$$

To show (ii), suppose $f : [\lambda]^{<\omega} \to \mathcal{P}(\mu)$. For $s, t \in [\lambda]^{<\omega}$, set

$$\Delta(s,t) = \begin{cases} 0 & \text{if } f(s) = f(t), \\ \text{least } \delta\ (f(s) \cap \delta \neq f(t) \cap \delta), & \text{otherwise.} \end{cases}$$

Now define a function $g : [\lambda]^{<\omega} \to \mu$ by:

$$g(s) = \sup\{\Delta(s_1, \ldots) \mid s_1, s_2 \subseteq s\} .$$

Let $X \in \mathcal{F}$ and $g''[X]^{<\omega} \subseteq \gamma < \mu$. If $s, t \in [X]^{<\omega}$, either $f(s) = f(t)$ or else $f(s) \cap \gamma \neq f(t) \cap \gamma$, since $g(s \cup t) < \gamma$. Hence, $|f''[X]^{<\omega}| \leq 2^\gamma < 2^\mu$.

To show (ii), assume that for cofinally many $\nu < \mu$ there are counterexamples F_ν to ν-str. indec. Viewing each F_ν as a partition of $[\lambda]^{<\omega}$, consider their canonical refinement $\mathcal{R}$. This is a partition of $[\lambda]^{<\omega}$ into at most $2^\mu = \mu^+$ parts, and it is easy to see that for any $X \in \mathcal{F}$, $[X]^{<\omega}$ must intersect at least μ parts. If there is an $X \in \mathcal{F}$ so that $[X]^{<\omega}$ meets exactly μ parts, then $\mathcal{R}$ would be a counterexample to μ-str. indec. Otherwise $\mathcal{R}$ is a counterexample to μ^+-str. indec., and the result follows from (i).

Without assuming $2^\mu = \mu^+$ in (ii) of the proposition, one can still show that the conclusion holds if μ is singular and $\mathcal{F}$ is μ-str. indec. but *not* $\mathrm{cf}(\mu)$-str. indec., using a straightforward modification of an argument in [18].

Corollary 6.13. (G.C.H.) *If $\mathcal{F}$ is a Jonsson filter, the least μ so that $\mathcal{F}$ is μ-str. indec. is a successor cardinal.*

Finally, consider Prikry forcing (see [9]). Let M model ZFC, $\mathcal{F}$ a filter over regular λ in M, and $M[G]$ a generic extension via Prikry forcing with $\mathcal{F}$. It is evident from the work of Devlin [9] that for $\mu < \lambda$:

(a) If $\mathcal{F}$ is $\mathrm{cf}^M(\mu)$-str. indec., $\mathrm{cf}^{M[G]}(\mu) = \mathrm{cf}^M(\mu)$;

(b) if $\mathcal{F}$ is not $\mathrm{cf}^M(\mu)$-str. indec., $\mathrm{cf}^{M[G]}(\mu) = \omega$.

Hence, the following:

Theorem 6.14. (i) *Prikry forcing with $\mathcal{F}$ over a regular limit cardinal $\lambda > \omega$ preserves λ as a cardinal while changing its cofinality to ω iff $\mathcal{F}$ is μ-str. indec. for arbitrarily large $\mu < \lambda$*

(ii) (G.C.H.) *Either, so both, conditions in* (i) *hold if $\mathcal{F}$ is Jonsson. In fact, if $\mathcal{F}$ is ν-str. indec.. every cardinal between ν and λ is preserved by the forcing.*

Proof. (i) If $\mathcal{F}$ is μ-str. indec. for arbitrarily large $\mu < \lambda$, by 6.12(i) we can assume that these μ are regular.

The result follows from (a) above since λ being a limit of cardinals which are preserved must itself be preserved. Conversely, if λ is preserved as a limit cardinal, cofinally many $\mu < \lambda$ must also be preserved.

To show (ii), argue by contradiction and assume μ is the least so that $\nu < \mu < \lambda$ and μ is not preserved as a cardinal. Then μ must be a successor, but as in 6.11(ii), $\mathcal{F}$ is μ-str. indec., contradicting (a) above.

6.14(ii) is somewhat in contradistinction to Theorem 3 of Devlin [9]. To conclude, two questions which naturally suggest themselves.

Question 6.15. Is the least Jonsson cardinal less than the least cardinal which carries a Jonsson filter?

Question 6.16. If $\mathcal{U}$ is a normal κ-ultrafilter, are there any Jonsson filters over $\lambda \neq \kappa$ in $L[\mathcal{U}]$?

7. Open questions

For the reader's convenience, I list here the open questions stated in this paper with their original numbering.

Question 4.8. Can 2-Ramsey κ-ultrafilters always be written as a discrete limit of ultrafilters over smaller cardinals?

Question 4.9. If there is a non-minimal p-point over a measurable cardinal κ, does Solovay's $0^\dagger$ still exist. If κ is κ-compact, is there a non-minimal p-point over κ?

Question 4.12. Can Kunen's $\mathcal{X}_n$ for $n < \omega$ be strictly n-Ramsey? In general, is a two constellation p-point 2-Ramsey?

Question 5.8. If $\mathcal{U}$ and $\mathcal{V}$ are κ-ultrafilters such that $\mathcal{U} \times \mathcal{V} < \mathcal{V} \times \mathcal{U}$, is there a $\mathcal{W}$ and integers n and m so that $\mathcal{U} \cong \mathcal{W}^n$ and $\mathcal{V} \cong \mathcal{W}^m$?

Question 5.11. Is there a κ-ultrafilter with an infinite number of Rudin–Frolik predecessors?

Question 5.12. If $\{\mathcal{D}_\alpha \mid \alpha < \kappa\}$ is a family of distinct κ-ultrafilters and $\mathcal{U}$ any κ-ultrafilter, is there an $X \in \mathcal{U}$ so that $\{\mathcal{D}_\alpha \mid \alpha \in X\}$ is a discrete family?

Question 6.7. Are there ω_1-Rowbottom κ-ultrafilters with an infinite number of skies?

Question 6.15. Is the least Jonsson cardinal less than the least cardinal which carries a Jonsson filter?

Question 6.16. If $\mathcal{U}$ is a normal κ-ultrafilter, are there any Jonsson filters over $\lambda \neq \kappa$ in $L[\mathcal{U}]$?

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