

Some combinatorics involving ultrafilters

by

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Abstract. This paper briefly discusses the following property of ultrafilters: if μ and λ are cardinals, an ultrafilter U is (μ, λ) -cohesive iff given μ sets in U there are λ of them whose intersection is in U . Among other things, it is shown that a p -point over ω is (ω_1, ω) -cohesive, but that this property does not characterize p -points. We can in fact prove the following: if U is a p -point over ω and $\{X_\alpha \mid \alpha < \omega_1\} \subseteq U$, then for any $\delta < \omega_1$ there is an $SC\omega_1$ of order type δ so that $\bigcap \{X_\alpha \mid \alpha \in S\} \in U$. A polarized partition relation is strengthened using this fact. These results have direct generalizations to measurable cardinals, and indeed, the paper is written in this general context.

§ 0. Introduction. In this paper, the following rather general combinatorial property of ultrafilters is considered, mainly in connection with ω and measurable cardinals.

DEFINITION. If μ and λ are cardinals, an ultrafilter \mathcal{U} is (μ, λ) -cohesive iff given μ sets in \mathcal{U} , there are λ of them whose intersection is still in \mathcal{U} .

For those familiar with regularity of ultrafilters, notice that (μ, λ) -cohesion is a strong negation of (μ, λ) -regularity. It is shown that if \mathcal{U} is a p -point in the space βN , then \mathcal{U} is (ω_1, ω) -cohesive. The analogous result for measurable cardinals holds by the same proof. Product ultrafilters are considered in this context, and the situations under various set theoretical hypotheses are also discussed. Finally, a new proof and strengthening of a polarized partition relation is derived.

My set theory is ZFC, and the notation is standard, but I do mention the following: $\alpha, \beta, \gamma, \dots$ denote ordinals, but κ, λ , and μ are reserved for cardinals. If x is a set, $|x|$ denotes its cardinality and $\mathcal{P}x$ its power set; if y is also a set, xy denotes the set of functions from x into y ; finally, if n is an integer, $[x]^n$ denotes the collection of n -element subsets of x . If z is a set of ordinals, \bar{z} denotes its order type. An ultrafilter over a set I is actually one in the Boolean algebra $\mathcal{P}I$, and is uniform if each of its members has cardinality $|I|$. Finally, \blacksquare indicates the end of a proof.

While working on this paper the author has profited from discussions with Mathias and Prikry.

§ 1. Preliminaries and P -points. Under the GCH, we can make some initial deductions from the following classical result of Sierpiński (which I state in a slightly weakened version relevant to our purposes) — see P_4 of [Si].

1.1. PROPOSITION (Sierpiński). If $2^\lambda = \lambda^+$, there are functions $f_\alpha: \lambda^+ \rightarrow 2$ for $\alpha < \lambda^+$ so that: whenever $X \subseteq \lambda^+$ and $|X| = \lambda^+$,

$$|\{\alpha < \lambda^+ \mid f_\alpha''X \neq 2\}| < \lambda.$$

Prikry showed that the first part of the next theorem follows from Sierpiński's result, and the second is easy enough to see.

1.2. THEOREM. Suppose $2^\lambda = \lambda^+$.

- (i) (Prikry) If \mathcal{U} is a uniform ultrafilter over λ^+ , then \mathcal{U} is not (λ^+, λ) -cohesive.
- (ii) If \mathcal{V} is a uniform ultrafilter over λ , then \mathcal{V} is not (λ^+, λ^+) -cohesive.

Proof. Let $\{f_\alpha \mid \alpha < \lambda^+\}$ be as in 1.1. For (i), set $A_\alpha^k = \{\xi \mid f_\alpha(\xi) = k\}$ for $k < 2$ and $\alpha < \lambda^+$. Suppose that $k < 2$ and $S \subseteq \lambda^+$ with $|S| = \lambda$. If $X = \bigcap \{A_\alpha^k \mid \alpha \in S\}$, then $\{x \mid f_\alpha''X = \{k\}\} \supseteq S$. Hence $|X| \leq \lambda$ and $X \notin \mathcal{U}$ as \mathcal{U} is uniform. The result now follows, since there must be a $k < 2$ for which there are λ^+ 's so that $A_\alpha^k \in \mathcal{U}$.

To show (ii), set $B_\xi^k = \{\alpha < \lambda \mid f_\alpha(\xi) = k\}$ for $k < 2$ and $\xi < \lambda^+$. Suppose that $k < 2$ and $Y \subseteq \lambda^+$ with $|Y| = \lambda^+$. Then

$$T = \bigcap \{B_\xi^k \mid \xi \in Y\} \subseteq \{\alpha < \lambda \mid f_\alpha''Y = \{k\}\},$$

and hence $|T| < \lambda$, i.e. $T \notin \mathcal{V}$ as \mathcal{V} is uniform. The result now follows as for (i). ■

Having made these initial remarks, I now turn to my main concern, the consideration of measurable cardinals κ and the non-trivial cases involving κ^+ and κ .

1.3. DEFINITION.

- (i) \mathcal{U} is a κ -ultrafilter iff \mathcal{U} is a non-principal, κ -complete ultrafilter over κ .
- (ii) κ is a measurable cardinal iff there is a κ -ultrafilter.
- (iii) κ is κ -compact iff every κ -complete filter over κ can be extended to a κ -complete ultrafilter over κ .

The non-principal ultrafilters over ω are precisely the ω -ultrafilters, and thus, in this paper I regard ω to be both measurable and ω -compact. Note that κ -compactness is just a restricted version of the usual concept of strong compactness, and that it obviously implies the measurability of κ . The following is another observation of a negative kind.

1.4. PROPOSITION. If κ is κ -compact, there is a κ -ultrafilter which is not $(2^\kappa, \kappa)$ -cohesive.

Proof. Let $\mathcal{S} \subseteq \mathcal{P}_\kappa$ be a family of 2^κ κ -independent sets (see Kunen [Ku 3] for details; the existence of such a family only depends on the fact that $\kappa^{\kappa^+} = \kappa$); that is, given any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ so that $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $|\mathcal{A}|, |\mathcal{B}| < \kappa$,

$$|\bigcap \mathcal{A} \cap \bigcap \{\kappa - X \mid X \in \mathcal{B}\}| = \kappa.$$

Then

$$\mathcal{S} \cup \{\kappa - \bigcap \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S} \text{ and } |\mathcal{T}| = \kappa\}$$

generates a uniform κ -complete filter:

Let $\mathcal{A} \subseteq \mathcal{S}$ with $|\mathcal{A}| < \kappa$, and suppose also that $\lambda < \kappa$ and for $\alpha < \lambda$, $\mathcal{T}_\alpha \subseteq \mathcal{S}$ are such that $|\mathcal{T}_\alpha| = \kappa$. It must be shown that

$$(*) \quad |\bigcap \mathcal{A} \cap \bigcap \{\kappa - \bigcap \mathcal{T}_\alpha \mid \alpha < \lambda\}| = \kappa.$$

By the cardinality assumptions, we can inductively choose $X_\alpha \in \mathcal{T}_\alpha - \mathcal{A}$ so that $\alpha < \beta < \lambda$ implies that $X_\alpha \neq X_\beta$. Note that for each $\alpha < \lambda$, $\kappa - \bigcap \mathcal{T}_\alpha \supseteq \kappa - X_\alpha$. Thus,

$$\bigcap \mathcal{A} \cap \bigcap \{\kappa - \bigcap \mathcal{T}_\alpha \mid \alpha < \lambda\} \supseteq \bigcap \mathcal{A} \cap \bigcap \{\kappa - X_\alpha \mid \alpha < \lambda\},$$

and the set on the right has cardinality κ , as \mathcal{S} is independent. Hence, we get (*).

Now by κ -compactness of κ , let \mathcal{U} be any κ -ultrafilter extending the above filter. Clearly the family $\mathcal{S} \subseteq \mathcal{U}$ is a counterexample to the $(2^\kappa, \kappa)$ -cohesion of \mathcal{U} , and we are done. (This example for $\kappa = \omega$ is just Kunen's example (see 2.8 of [Ku 3]) of an ω -ultrafilter of character 2^ω , i.e. one not generated by less than 2^ω sets of integers.) ■

Thus, we see that the negative results 1.2 and 1.4 can be culled from previous set theoretical experience; I now turn to the positive results. The following concepts first arose in the study of βN , the Stone-Čech compactification of the integers, which is identifiable with the set of ultrafilters over ω .

1.5. DEFINITIONS.

- (i) The Rudin-Keisler ordering (RK) on ultrafilters is defined as follows: If \mathcal{U} is an ultrafilter over a set I and \mathcal{V} over J , $\mathcal{V} \leq_{\text{RK}} \mathcal{U}$ iff there is a function $f: I \rightarrow J$ so that $\mathcal{V} = f_*(\mathcal{U})$, where

$$f_*(\mathcal{U}) = \{X \subseteq J \mid f^{-1}(X) \in \mathcal{U}\}.$$

If $\mathcal{V} \leq_{\text{RK}} \mathcal{U}$, then $\mathcal{V} \approx_{\text{RK}} \mathcal{U}$ (\mathcal{V} and \mathcal{U} are isomorphic) iff $\mathcal{U} \leq_{\text{RK}} \mathcal{V}$, and $\mathcal{V} <_{\text{RK}} \mathcal{U}$ iff $\mathcal{U} \not\leq_{\text{RK}} \mathcal{V}$.

- (ii) A κ -ultrafilter \mathcal{U} is minimal iff it is minimal in the RK ordering, i.e. there is no (non-principal) ultrafilter $\mathcal{V} <_{\text{RK}} \mathcal{U}$.

- (iii) A κ -ultrafilter \mathcal{U} is a p -point iff whenever $\{X_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{U}$, there is a $Y \in \mathcal{U}$ so that $|Y - X_\alpha| < \kappa$ for each $\alpha < \kappa$.

See the reference work Comfort-Negreponitis ([CN], especially § 16) for details on these concepts and the general development of the theory of βN . For an analogous development of the theory of κ -ultrafilters for $\kappa > \omega$ with attention to distinctive features and new factors, see Kanamori [Ka]. In the present context, it is not hard to show that if \mathcal{U} is (μ, λ) -cohesive and $\mathcal{V} \leq_{\text{RK}} \mathcal{U}$, then \mathcal{V} is (μ, λ) -cohesive. For future reference, I collect some known characterizations in the next proposition.

1.6. PROPOSITION.

- (i) The following are equivalent for a κ -ultrafilter \mathcal{U} :

- (a) \mathcal{U} is minimal.

- (b) \mathcal{U} is Ramsey: for any $n < \omega$ and $\lambda < \kappa$, if $f: [\kappa]^n \rightarrow \lambda$, there is an $X \in \mathcal{U}$ so that $|f''[X]^n| = 1$.

■

(c) \mathcal{U} is selective: if $f \in {}^*\kappa$ so that $f^{-1}(\{\alpha\}) \notin \mathcal{U}$ for each α , then there is an $X \in \mathcal{U}$ so that $|X \cap f^{-1}(\{\alpha\})| \leq 1$ for each α .

When $\kappa > \omega$, we can also add:

(d) There is a normal κ -ultrafilter \mathcal{N} so that $\mathcal{N} \approx_{\text{RK}} \mathcal{U}$.

(ii) The following are equivalent for a κ -ultrafilter \mathcal{U} :

(a) \mathcal{U} is a p -point.

(b) \mathcal{U} is almost selective: if $f \in {}^*\kappa$ so that $f^{-1}(\{\alpha\}) \notin \mathcal{U}$ for each α , then f is almost 1-1 (mod \mathcal{U}), i.e. there is an $X \in \mathcal{U}$ so that $|X \cap f^{-1}(\{\alpha\})| < \kappa$ for each α .

Hence, minimal κ -ultrafilters are always p -points. When $\kappa = \omega$, the converse is not true under CH or Martin's Axiom (MA), but it is not even known whether p -points exist if we do not assume either of these hypotheses. However, Kunen [Ku 1] has shown that there is a model of ZFC without any minimal ω -ultrafilters. When $\kappa > \omega$, minimal κ -ultrafilters always exist (Scott), but it is consistent that all p -points are minimal, and, in fact, all RK-isomorphic to each other (Kunen — see after 2.2 below). Non-minimal p -point κ -ultrafilters exist if κ is measurable and a limit of measurable cardinals, but it is still open whether such κ -ultrafilters exist when κ is κ -compact.

I now proceed to show that if \mathcal{U} is a p -point κ -ultrafilter, then \mathcal{U} is (κ^+, κ) -cohesive, and, toward this goal, provide a new characterization of p -points which may be of independent interest.

1.7. DEFINITION. If \mathcal{D} is an ultrafilter over some cardinal λ , \mathcal{D} is coherent iff whenever $X \in \mathcal{D}$ and $\mathcal{A} \subseteq \mathcal{D}$ so that for each $\alpha < \lambda$,

$$|\{A \in \mathcal{A} \mid X \cap \alpha = A \cap \alpha\}| \geq \lambda,$$

then there is a $\mathcal{B} \subseteq \mathcal{A}$ so that $|\mathcal{B}| = \lambda$ and $\bigcap \mathcal{B} \in \mathcal{D}$.

If \mathcal{D} and \mathcal{E} are ultrafilters over λ so that $\mathcal{E} \leq_{\text{RK}} \mathcal{D}$, then if \mathcal{D} is coherent, so is \mathcal{E} . Note that coherence makes sense for an ultrafilter \mathcal{U} over an arbitrary set I , by considering some $\varphi: I \rightarrow \lambda$ and formulating the property for $\varphi_*(\mathcal{U})$ instead.

1.8. PROPOSITION. If $2^{<\lambda} = \lambda$ and $\mathcal{A} \subseteq \mathcal{P}\lambda$ with $|\mathcal{A}| > \lambda$, then there is an $X \in \mathcal{A}$ so that $|\{A \in \mathcal{A} \mid A \cap \alpha = X \cap \alpha\}| = |\mathcal{A}|$ for every $\alpha < \lambda$.

Proof. Argue by contradiction, and assume that for each $X \in \mathcal{A}$, there is an $\alpha_X < \lambda$ so that $|\{A \in \mathcal{A} \mid A \cap \alpha_X = X \cap \alpha_X\}| < |\mathcal{A}|$. Surely, there is a $\beta < \lambda$ and a $\mathcal{A}_1 \subseteq \mathcal{A}$ with $|\mathcal{A}_1| = |\mathcal{A}|$ so that $X \in \mathcal{A}_1$ implies $\alpha_X = \beta$. But as $2^\beta < |\mathcal{A}|$, there is an $\mathcal{A}_2 \subseteq \mathcal{A}_1$ with $|\mathcal{A}_2| = |\mathcal{A}|$ so that $X, Y \in \mathcal{A}_2$ imply $X \cap \beta = Y \cap \beta$. This is a contradiction. ■

The following is now immediate from the definitions and 1.8:

1.9. COROLLARY. If $2^{<\lambda} = \lambda$ and \mathcal{D} over λ is coherent, then it is (λ^+, λ) -cohesive.

With these preliminaries, I now prove the main result. The (κ^+, κ) -cohesion of normal κ -ultrafilters for $\kappa > \omega$ was first proved by Solovay.

1.10. THEOREM. If \mathcal{U} is a κ -ultrafilter, then \mathcal{U} is a p -point iff \mathcal{U} is coherent.

Thus, p -point κ -ultrafilters are (κ^+, κ) -cohesive, and in particular, p -points in βN are (ω_1, ω) -cohesive.

Proof. Suppose first that \mathcal{U} is coherent and $\{X_\xi \mid \xi < \kappa\} \subseteq \mathcal{U}$. We must find a $Y \in \mathcal{U}$ so that $|Y - X_\xi| < \kappa$ for each $\xi < \kappa$. By taking successive intersections, we can assume henceforth that $\xi < \zeta < \kappa$ implies $X_\xi \subseteq X_\zeta$.

Set $Y_\xi = X_\xi \cup \xi$ for $\xi < \kappa$. Then for each $\alpha < \kappa$,

$$|\{\xi < \kappa \mid Y_\xi \cap \alpha = \alpha\}| = \kappa$$

and so by coherence, there is a $T \subseteq \kappa$ with $|T| = \kappa$ so that $Y = \bigcap \{Y_\xi \mid \xi \in T\} \in \mathcal{U}$. Now given any $\gamma < \kappa$, let $\delta \geq \gamma$ so that $\delta \in T$. By the definition of the Y_ξ 's and the fact that the X_ξ 's were descending, we have $|Y_\delta - Y_\gamma| < \kappa$. Hence, $|Y - Y_\gamma| < \kappa$ and the result follows.

Conversely, suppose that \mathcal{U} is a p -point, and $X \in \mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{U}$ with $|\mathcal{A}| = \kappa$ so that for each $\alpha < \kappa$,

$$(*) \quad |\{A \in \mathcal{A} \mid X \cap \alpha = A \cap \alpha\}| = \kappa.$$

We must establish the existence of a $\mathcal{B} \subseteq \mathcal{A}$ so that $|\mathcal{B}| = \kappa$ and $\bigcap \mathcal{B} \in \mathcal{U}$.

Since \mathcal{U} is a p -point, there is a $Y \in \mathcal{U}$ so that $|Y - A| < \kappa$ for every $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, with $A \neq X$, let I_A be the half-open interval of ordinals $[\gamma_A, \delta_A)$, where

$$\gamma_A = \bigcup \{\alpha \mid X \cap \alpha = A \cap \alpha\},$$

and

$$\delta_A = \text{least } \delta \geq \gamma_A \text{ so that } Y - \delta \in A.$$

Notice that I_A may be empty; in any case, $|I_A| < \kappa$.

By (*) for every $\rho < \kappa$ there is an $A \in \mathcal{A}$ so that $\rho < \gamma_A$. Hence, by induction we can choose an $\mathcal{A}' \subseteq \mathcal{A}$ so that $|\mathcal{A}'| = \kappa$ and if $A, B \in \mathcal{A}'$ with $A \neq B$, then $I_A \cap I_B = \emptyset$. Now we can find some $\mathcal{B} \subseteq \mathcal{A}'$ so that $|\mathcal{B}| = \kappa$ and

$$Z = \bigcup \{I_A \mid A \in \mathcal{B}\} \notin \mathcal{U}.$$

Thus, $X \cap Y \cap (\kappa - Z) \in \mathcal{U}$.

Suppose now that $\beta \in X \cap Y \cap (\kappa - Z)$, and $A \in \mathcal{B}$. As $\beta \notin I_A$, either $\beta < \gamma_A$ or $\delta_A \leq \beta$. If $\beta < \gamma_A$, then $\beta \in X$ implies $\beta \in A$ by the definition of γ_A . If $\delta_A \leq \beta$, then $\beta \in Y$ implies $\beta \in A$ by the definition of δ_A . Hence, in either case, $\beta \in A$. We have thus shown that $X \cap Y \cap (\kappa - Z) \subseteq \bigcap \mathcal{B}$. This establishes that $\bigcap \mathcal{B} \in \mathcal{U}$, and the proof is complete. ■

In § 2, it is shown that (κ^+, κ) -cohesion does not characterize p -points, and in § 3, a refinement of the argument for 1.10 is given.

§ 2. Product ultrafilters. Let us first recall some further definitions.

2.1. DEFINITIONS. Let \mathcal{D} be an ultrafilter over I , and \mathcal{E}_i ultrafilters over J for $i \in I$.

(i) The \mathcal{D} -sum of $\langle \mathcal{E}_i \mid i \in I \rangle$ is the ultrafilter $\mathcal{D} \sum \mathcal{E}_i$ over $I \times J$ defined by

$$X \in \mathcal{D} \sum \mathcal{E}_i \quad \text{iff} \quad \{i \mid \{j \mid \langle i, j \rangle \in X\} \in \mathcal{E}_i\} \in \mathcal{D}.$$

(ii) When each $\mathcal{E}_i = \mathcal{A}$ fixed \mathcal{E} in (i), we get the product of \mathcal{D} and \mathcal{E} , denoted $\mathcal{D} \times \mathcal{E}$. For $0 < n < \omega$, \mathcal{U}^n is defined by induction: $\mathcal{U}^1 = \mathcal{U}$ and $\mathcal{U}^{n+1} = \mathcal{U} \times \mathcal{U}^n$.

Notice that if \mathcal{D} and \mathcal{E}_α for $\alpha < \kappa$ are all κ -ultrafilters, then $\mathcal{U} = \mathcal{D} \sum \mathcal{E}_\alpha$ is RK-isomorphic to a κ -ultrafilter, but not a p -point, since $\pi: \kappa \times \kappa \rightarrow \kappa$, the projection onto the first coordinate, cannot be almost 1-1 (mod \mathcal{U}). The next propositions show that cohesion is preserved under the taking of sums and products of κ -ultrafilters under suitable conditions, and thus, that this concept does not characterize p -points.

2.2. PROPOSITION. Suppose \mathcal{U} is a minimal κ -ultrafilter. If \mathcal{U} is (μ, λ) -cohesive, then \mathcal{U}^n is (μ, λ) -cohesive for each $n < \omega$. Hence, each \mathcal{U}^n is always (κ^+, κ) -cohesive.

Proof. Let $A_n = \{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle \mid \alpha_1 < \alpha_2 < \dots < \alpha_n < \kappa\}$. It is not hard to establish the following characterization of minimal κ -ultrafilters, using the Ramsey condition:

A κ -ultrafilter \mathcal{D} is minimal iff for any n , $\{X^n \mid X \in \mathcal{D}\} \cup \{A_n\}$ generates \mathcal{D}^n , i.e. for any $A \in \mathcal{D}^n$, there is an $X \in \mathcal{D}$ such that $X^n \cap A_n \subseteq A$.

Hence, that \mathcal{U} is (μ, λ) -cohesive certainly implies that \mathcal{U}^n is (μ, λ) -cohesive for each $n < \omega$. An appeal to 1.9 now yields the full conclusion of the proposition. ■

Kunen [Ku2] showed that in $L[\mathcal{U}]$, the inner model constructed from a normal κ -ultrafilter over $\kappa > \omega$, each κ -ultrafilter is RK-isomorphic to $(\mathcal{U} \cap L[\mathcal{U}])^n$ for some $n < \omega$. Hence, 2.2 immediately shows that if it is consistent that there is a measurable cardinal $\kappa > \omega$, then it is consistent that such a cardinal κ exists and every κ -ultrafilter is (κ^+, κ) -cohesive. Thus, 1.4 is yet another way of showing that κ cannot be κ -compact in $L[\mathcal{U}]$.

The proof of the following result does not generalize for $\kappa > \omega$.

2.3. PROPOSITION. Suppose that \mathcal{U} and \mathcal{V}_n for $n < \omega$ are all (ω_1, ω_1) -cohesive ω -ultrafilters. Then $\mathcal{U} \sum \mathcal{V}_n$ is (ω_1, ω) -cohesive.

Proof. For any $S \subseteq \omega \times \omega$ and $n < \omega$, set $(S)_n = \{i \mid \langle n, i \rangle \in S\}$ for the purposes of this proof. Also, if $S \in \mathcal{U} \sum \mathcal{V}_n$, let $S^* = \{n \mid (S)_n \in \mathcal{V}_n\}$. Thus, $S^* \in \mathcal{U}$.

Now let $\mathcal{A} \subseteq \mathcal{U} \sum \mathcal{V}_n$ with $|\mathcal{A}| = \omega_1$. By the (ω_1, ω_1) -cohesion of \mathcal{U} , there is an $\mathcal{A}' \subseteq \mathcal{A}$ so that $|\mathcal{A}'| = \omega_1$ and $K = \bigcap \{A^* \mid A \in \mathcal{A}'\} \in \mathcal{U}$.

By induction on the ascending enumeration of K , we can define $\mathcal{B}_n \subseteq \mathcal{A}'$ for $n \in K$ with the following properties:

- (a) $m < n$ implies $\mathcal{B}_m \subseteq \mathcal{B}_n$,
- (b) $R_n = \bigcap \{(A)_n \mid A \in \mathcal{B}_n\} \in \mathcal{V}_n$, and
- (c) $|\mathcal{B}_n| = \omega_1$.

Choose $S_n \in \mathcal{B}_n$ for $n \in K$ so that $m < n$ and $m, n \in K$ imply $S_m \neq S_n$. For each $n \in K$, we have

$$T_n = R_n \cap \bigcap \{(S_m)_n \mid m < n \text{ and } m \in K\} \in \mathcal{U}_n.$$

Hence, by the construction,

$$\bigcup_{n \in K} \{n\} \times T_n \subseteq \bigcap \{S_n \mid n \in K\} \in \mathcal{U} \sum \mathcal{V}_n.$$

The proof is complete. ■

We know from 1.2(ii) that CH implies that no ω -ultrafilter is (ω_1, ω_1) -cohesive. However, the previous proposition is not vacuous under Martin's Axiom. Booth [Bo] showed that MA implies the existence of minimal ω -ultrafilters \mathcal{U} with the following property: for any $\mu < 2^\omega$ and $\mathcal{A} \subseteq \mathcal{U}$ so that $|\mathcal{A}| = \mu$, there is a $Y \in \mathcal{U}$ so that $|Y - X| < \omega$ for every $X \in \mathcal{A}$. Thus, when μ is uncountable, there is a finite set s so that $Y - s$ is contained in μ members of \mathcal{A} , and hence, \mathcal{U} is (μ, μ) -cohesive. It is also clear from Booth's work how to get non-minimal p -points under MA which are still (μ, μ) -cohesive for $\omega_1 \leq \mu < 2^\omega$. On the other hand, Solomon [So] showed that MA and $2^\omega > \omega_1$ also imply the existence of minimal ω -ultrafilters which are not (ω_1, ω_1) -cohesive. ■

§ 3. Polarized partition relations. This section is devoted to showing that a refinement of the proof of 1.10 yields a strengthened, ultrafilter related, version of a known polarized partition relation for measurable cardinals. Let us first recall the definitions of the relevant versions of the polarized partition symbol of Erdős and Hajnal, and also specify a modification. Recall that if κ is a set of ordinals, $\bar{\kappa}$ denotes its order type.

3.1. DEFINITIONS.

(i) The polarized partition symbol

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{\lambda}^{m, n}$$

where $m, n < \omega$, denotes the following statement: whenever $F: [X]^m \times [Y]^n \rightarrow \lambda$, there are $A \subseteq X$ and $B \subseteq Y$ so that $\bar{A} = \gamma$ and $\bar{B} = \delta$, and $|F''([A]^m \times [B]^n)| = 1$.

(ii) When " n " in the symbol is replaced by " $< \omega$ ", we mean the following statement: whenever $F_n: [X]^m \times [Y]^n \rightarrow \lambda$ for each $n < \omega$, there are $A \subseteq X$ and $B \subseteq Y$ so that $\bar{A} = \gamma$ and $\bar{B} = \delta$, and for all $n < \omega$, $|F_n''([A]^m \times [B]^n)| = 1$.

(iii) When " δ " in the symbol (either in context (i) or (ii)) is replaced by " $\in \mathcal{A}$ " where \mathcal{A} is a set, we mean that the Y specified is a member of \mathcal{A} (instead of $\bar{Y} = \delta$).

The following result strengthens a known polarized partition relation. The reader is referred to Hajnal [H] and Choodnovsky [Ch] for the previous efforts in this direction. In particular, a question asked in passing in [H] (top of p. 44) is now answered positively.

3.2. THEOREM. Let $\kappa \geq \omega$ be a measurable cardinal.

(i) If \mathcal{U} is a p -point κ -ultrafilter, then

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \rightarrow \begin{pmatrix} \eta \\ \in \mathcal{U} \end{pmatrix}_{\lambda}^{1, 1}$$

for any $\eta < \kappa^+$ and $\lambda < \kappa$.

(ii) If \mathcal{U} is a minimal κ -ultrafilter, then

$$\binom{\kappa^+}{\kappa} \rightarrow \left(\binom{\eta}{\in \mathcal{U}} \right)_\lambda^{1, n}$$

for any $\eta < \kappa^+$, $\lambda < \kappa$, and $n < \omega$. When $\kappa > \omega$, the " n " can be replaced by " $< \omega$ ".

Proof. If \mathcal{U} is a κ -ultrafilter, $\lambda < \kappa$, and $F: \kappa^+ \times \kappa \rightarrow \lambda$, for each $\xi < \kappa^+$ there is an $X_\xi \in \mathcal{U}$ and a $\beta_\xi < \lambda$ so that $F''(\{\xi\} \times X_\xi) = \beta_\xi$. Also, $\beta_\xi = \text{fixed } \beta \text{ for } \kappa^+ \text{ } \xi$'s. Hence, to show (i), it suffices to show the following: If \mathcal{U} is a p -point and $\{X_\xi \mid \xi < \kappa^+\} \subseteq \mathcal{U}$, then for any $\eta < \kappa^+$ there is a $B \subseteq \kappa^+$ with $\bar{B} = \eta$ and $\bigcap \{X_\xi \mid \xi \in B\} \in \mathcal{U}$.

The refinement to get (ii) is just an initial application of the Ramsey property of minimal κ -ultrafilters in the above argument, and the final remark in (ii) follows from an application next, for each $\xi < \kappa^+$, of the countable completeness of κ -ultrafilters for $\kappa > \omega$.

Thus, suppose that \mathcal{U} is a p -point and $\{X_\xi \mid \xi < \kappa^+\} \subseteq \mathcal{U}$. By Proposition 1.8, there is a $Y \in \mathcal{U}$ so that for each $\alpha < \kappa$,

$$|\{\xi < \kappa^+ \mid Y \cap \alpha = X_\xi \cap \alpha\}| = \kappa^+.$$

We can surely define ordinals $f(\zeta) < \kappa^+$ for $\zeta < \kappa^+$ by induction so that the following are satisfied:

(i) f is a normal function, i.e. f is strictly increasing and continuous at limits.

(ii) For any $\zeta < \kappa^+$ and $\alpha < \kappa$, $|\{\xi \mid f(\zeta) \leq \xi < f(\zeta+1) \text{ and } Y \cap \alpha = X_\xi \cap \alpha\}| = \kappa$.

Now fix an $\eta < \kappa^+$, where, to avoid trivialities, we assume $\kappa \leq \eta$. Since \mathcal{U} is a p -point, there is a $Z \in \mathcal{U}$ so that $|Z - X_\xi| < \kappa$ for any $\xi < f(\eta + \eta + 1)$. Define (possibly empty) intervals I_ξ for $\xi < f(\eta + \eta + 1)$ as in the proof of 1.9: $I_\xi = [\gamma_\xi, \delta_\xi]$, where

$$\gamma_\xi = \bigcup \{\alpha \mid Y \cap \alpha = X_\xi \cap \alpha\},$$

and

$$\delta_\xi = \text{least } \delta \geq \gamma_\xi \text{ so that } Z - \delta \subseteq X_\xi.$$

Let $\varphi: \kappa \rightarrow \eta + \eta$ be a bijection. By induction, we can choose $\xi_\alpha < \kappa^+$ for $\alpha < \kappa$ as follows: If ξ_β for $\beta < \alpha$ have been chosen, let ξ_α be such that:

(a) $f(\varphi(\alpha)) \leq \xi_\alpha < f(\varphi(\alpha) + 1)$, and

(b) $I_{\xi_\alpha} \cap I_{\xi_\beta} = \emptyset$ for $\beta < \alpha$.

By the definition of the intervals I_ξ , the condition (b) can always be met because of the property (ii) of the function f .

Clearly, $\{\xi_\alpha \mid \alpha < \kappa\}$ has order type $\eta + \eta$. By splitting it into two parts each of type η , it is seen that there must be a $B \subseteq \{\xi_\alpha \mid \alpha < \kappa\}$ so that $\bar{B} = \eta$ and

$$T = \bigcup \{I_\xi \mid \xi \in B\} \notin \mathcal{U}.$$

Hence, like in the proof of 1.10,

$$Y \cap Z \cap (\kappa - T) \subseteq \bigcap \{X_\xi \mid \xi \in B\},$$

and so since this last set is in \mathcal{U} , the proof is complete. ■

3.3. COROLLARY (Galvin for $\kappa = \omega$, unpublished Choodnovsky [Ch]). If $\kappa \geq \omega$ is measurable, then

$$\binom{\kappa^+}{\kappa} \rightarrow \left(\binom{\eta}{\kappa} \right)_\lambda^{1, n}$$

for any $\eta < \kappa^+$, $\lambda < \kappa$, and $n < \omega$. When $\kappa > \omega$, the " n " can be replaced by " $< \omega$ ".

Proof. For $\kappa > \omega$, the result is immediate from 3.2, since normal κ -ultrafilters always exist. But, as remarked after 1.6 there are models of ZFC without any minimal ω -ultrafilters. However, the following stratagem is available:

A minimal ω -ultrafilter \mathcal{U} can always be added to any model of ZFC by an ω -closed notion of forcing. (For example, say that p is a condition iff p is a countable collection of infinite subsets of ω with the finite intersection property and that a condition q is stronger than p iff $q \supseteq p$. Notice that if $m, n < \omega$ and $f: [\omega]^n \rightarrow m$, any condition p can be extended to one which contains a homogeneous set for f : first let $Y \subseteq \omega$ be infinite so that $|Y - X|$ is finite for every $X \in p$, and use Ramsey's theorem to get an infinite homogeneous subset Z of Y for f . Then $p \cup \{Z\}$ is stronger than p .)

Thus, the forcing adds no new countable sequences of ordinals, and ω_1 is preserved as a cardinal. Hence, for any $F: \omega_1 \times [\omega]^\omega \rightarrow m$ with $m, n < \omega$, 3.2 can be applied in the extension using \mathcal{U} , and any resultant "homogeneous" set for F , being countable, must already exist in the ground model. ■

Galvin's proof of 3.3 for $\kappa = \omega$ apparently did not generalize, and both the proof of Choodnovsky [Ch], and of Hajnal [H] for the weaker statement with η replaced by κ , relied on developing a tree and showing that a long branch exists. The present proof yields more information, being a thinning process which works by keeping the needed large sets in an ultrafilter. In the paper Baumgartner and Hajnal [BH] another proof of 3.3 for $\kappa = \omega$ is outlined which, like the one I give, depends on a forcing and absoluteness argument. But their forcing is one to make MA true in the extension, and hence not ω -closed, and thus a more involved argument was needed to show absoluteness.

3.4. Interestingly enough, when $\kappa > \omega$ the well-foundedness of ultrapowers can be used to yield a simpler proof of the main assertion of 3.2 (and hence, 1.10):

Let $\kappa > \omega$, and again, \mathcal{U} a p -point κ -ultrafilter with $\{X_\xi \mid \xi < \kappa^+\} \subseteq \mathcal{U}$. For a fixed η , $\kappa \leq \eta < \kappa^+$, we want to find a $B \subseteq \kappa^+$ with $\bar{B} = \eta$ so that $\bigcap \{X_\xi \mid \xi \in B\} \in \mathcal{U}$. Just as before, we can suppose that there is a $Y \in \mathcal{U}$ so that

$$|\{\xi < \kappa^+ \mid Y \cap \alpha = X_\xi \cap \alpha\}| = \kappa^+$$

for every $\alpha < \kappa$.

By well-foundedness, let $h \in {}^*\kappa$ be a "least" non-constant function, i.e. one so that for any $\alpha < \kappa$, $h^{-1}(\{\alpha\}) \notin \mathcal{U}$, but so that if $g \in {}^*\kappa$ and $\{\xi < \kappa \mid g(\xi) < h(\xi)\} \in \mathcal{U}$,

then $g^{-1}(\{\beta\}) \in \mathcal{U}$ for some $\beta < \kappa$. Since \mathcal{U} is a p -point, we can assume that h is almost 1-1, i.e. for each $\alpha < \kappa$, $|h^{-1}(\{\alpha\})| < \kappa$.

Let $\tau: \kappa \rightarrow \eta$ be a bijection. We can define ordinals $f(\zeta) < \kappa^+$ for $\zeta < \eta$ by induction so that the following are satisfied:

- (i) f is strictly increasing.
- (ii) $Y \cap (\alpha+1) = X_{f(\zeta)} \cap (\alpha+1)$ for any α so that $h(\alpha) \leq \tau^{-1}(\zeta)$. (Recall h is almost 1-1.)

It now suffices to show that $T = \bigcap \{X_{f(\zeta)} \mid \zeta < \eta\} \in \mathcal{U}$. If not, $Z = Y \cap (\kappa - T) \in \mathcal{U}$. On Z we can then define a function g by

$$g(\alpha) = \tau^{-1} \text{ of the least } \zeta \text{ so that } \alpha \notin X_{f(\zeta)}.$$

If $\alpha \in Z$ and $h(\alpha) \leq g(\alpha)$, then $Y \cap (\alpha+1) = X_{f(\zeta)} \cap (\alpha+1)$ where $\tau^{-1}(\zeta) = g(\alpha)$. But $\alpha \in Y$ so that $\alpha \in X_{f(\zeta)}$, contradicting the definition of g . Hence, $\alpha \in Z$ implies $g(\alpha) < h(\alpha)$, and thus $g^{-1}(\{\gamma\}) \in \mathcal{U}$ for some $\gamma < \kappa$. But this set is disjoint from $X_{f(\tau(\gamma))} \in \mathcal{U}$, an evident contradiction. Thus, this proof is complete. ■

This argument enables us to make the following observation about closed unbounded sets.

3.5. PROPOSITION. Suppose $\lambda^{<\lambda} = \lambda$ and C_α for $\alpha < \lambda^+$ are closed unbounded subsets of λ . Then for any $\eta < \lambda^+$, there is a $B \subset \lambda^+$ with $\bar{B} = \eta$ so that $\bigcap \{C_\alpha \mid \alpha \in B\}$ is still closed unbounded in λ .

Proof. Mimic the argument of 3.4 with the identity function: $\lambda \rightarrow \lambda$ in the role of h , and use the normality of the ideal of non-stationary subsets of λ at the appropriate places. ■

§ 4. Open questions. I conclude the paper with two typical open questions.

4.1. QUESTION. Is it provable in ZFC alone that there is a (ω_1, ω) -cohesive ω -ultrafilter?

4.2. QUESTION. Is it consistent that there is a $\kappa > \omega$ and a κ -ultrafilter \mathcal{U} which is (κ^+, κ^+) -cohesive? $2^\kappa > \kappa^+$. If there were such a κ -ultrafilter, then by 1.2(ii), Silver first showed that the consistency of the existence of a measurable cardinal $\kappa > \omega$ so that $2^\kappa > \kappa^+$ follows from a large cardinal assumption (2-extendibility).

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