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STRONG AXIOMS OF INFINITY AND ELEMENTARY EMBEDDINGS

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This is the expository paper on strong axioms of infinity and elementary embeddings originally to have been authored by Reinhardt and Solovay. It has been owed for some time and already cited with some frequency in the most recent set theoretical literature. However, for various reasons the paper did not appear in print for several years. The impetus for actual publication came from a series of lectures on the subject by Kanamori (Cambridge, 1975) and a set of notes circulated thereafter. Thus, although this present exposition is a detailed reworking of these notes, the basic conceptual framework was first developed by Reinhardt and Solovay some years ago. One factor which turns this delay in publication to advantage is that a more comprehensive view of the concepts discussed is now possible with the experience of the last few years, particularly in view of recent consistency results and also consequences in the presence of the axiom of determinacy. A projected sequel by Solovay to this paper will deal further with these considerations.

One of the most notable characteristics of the axiom of infinity is that its truth implies its independence of the other axioms. This, of course, is because the (infinite) set of hereditarily finite sets forms a model of the other axioms, in which there is no infinite set. Clearly, accepting an assertion whose truth implies its independence of given axioms requires the acceptance of new axioms. It is not surprising that the axiom of infinity should have this character (one would expect to have to adopt it as an axiom anyway), and moreover one would expect the existence of larger and larger cardinalities to have such character, as indeed it has. The procedures for generating cardinals studied by Mahlo [29] provided a notable example. It is remarkable that the new consequences of the corresponding (generalized) axioms of infinity also include arithmetic statements: this application of Gödel's second theorem is by now quite familiar. It is also remarkable that

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certain properties of cardinals which were originally introduced with little thought of "size" considerations should turn out to have this same character of implying their own independence (see, for example, Ulam [44] and Hanf [6]).

The monumental paper Keisler-Tarski [12] examined in detail three classes of "large cardinals": weakly compact cardinals, measurable cardinals, and strongly compact cardinals. Taking these as typical examples, it is reasonable to say (though we are in no way establishing an absolute criterion) that a property is of "large cardinal" character if it has the following two consequences:

(i) the existence of a cardinal which (at least in some inner model) is essentially "larger" than inaccessible cardinals and other "smaller" large cardinals (in the sense that it is a fixed point of reasonable thinning procedures, like Mahlo's, beginning from these cardinals);

(ii) a discernible new strength in set theory, not only in the provability of more formal statements (like Con(ZFC), etc.) but also in the existence of a richer structure on the cumulative hierarchy itself (for example, new combinatorial properties).

We observe in the preceding the emergence of interesting (and somewhat unexpected) mathematical connections among size, combinatorial properties, and syntactic strength.

It is well known that below each measurable cardinal there are many weakly compact cardinals. In fact, the experience of the last few years indicates that weak compactness is relatively weak, and many interesting train stops lie on the way from measurability to weak compactness (see Devlin [1] for a comprehensive survey). On the other hand, though strong compactness implies the consistency of the existence of any specified number of measurable cardinals, it is now known that the least measurable cardinal can be strongly compact (see Section 4 for a discussion). It is the purpose of this paper to consider even stronger large cardinal properties, and to investigate their various inter-relationships, as well as the effects of their presence on the cumulative hierarchy of sets.

The circumstance that some mathematical problems give rise (unexpectedly) to large cardinal properties raises the question of adopting new axioms. One possibility, which seems a bit like cheating, is to "solve" the problem by adopting its solution as an axiom. Another approach (suggested in the paper [12]) is to attempt to bypass the question by regarding all results showing that $P(\kappa)$ is a large cardinal property (which of course show that $\neg P(\kappa)$ has strong closure properties), as partial results in the direction of showing $\forall\kappa\ \neg P(\kappa)$. If, however, $\exists\kappa\, P(\kappa)$ should be true and have important consequences, this may appear somewhat futile, as comparison with the paradigm case of the axiom of infinity suggests. A third approach is to attempt to formulate new strong axioms of infinity. Ultimately, since this paper is an exposition of mathematics, the issue of whether the large cardinal properties we investigate are to be considered axiomatic or problematic can be left to one side. We do, however, wish to discuss briefly the problem of formulating strong axioms of infinity.

The whole question of what intuitive and set theoretical considerations should lead to the formulation of strong axioms of infinity is rather complicated and merits a systematic analysis (which we do not attempt here). There is some discussion of this in Wang [45], Ch. VI, especially p. 189, which gives a descriptive classification (due to Gödel) of the considerations which have so far led to such axioms. We remark that series of axioms such as $T_0 = \mathrm{ZF}, \ldots, T_n, T_{n+1} = T_n + \mathrm{Con}\, T_n, \ldots$, or those of Mahlo, appear "endless" in that it always seems possible to use the same guiding idea to get yet stronger principles (although it is not clear how to express this precisely). This even seems a desirable characteristic in hierarchies of axioms: if they are given as an r.e. sequence, they must be incomplete, so we would hope that the guiding idea would continue further. The procedures we consider have this "endless" character up to a point, where a result of Kunen sets a delimitation to one kind of *prima facie* natural extension.

We can discern at least four motivating principles behind the large cardinal properties we formulate.

(i) *Generalization.* For instance, it is in many ways quite reasonable to attribute certain properties of ω to uncountable cardinals as well, and these considerations can yield the measurable and strongly compact cardinals. Also, in considerations involving measurable cardinals, natural strengthenings of closure properties on ultrapowers yield the supercompact cardinals (see Sections 1, 2).

(ii) *Reflection.* The ordinary Reflection Principle in set theory invites various generalizations, for instance the Π^n_m-indescribability of cardinals. In one approach more relevant to our context, what is involved is a formulation of various reflection properties Ω, the class of all ordinals, intuitively ought to have (formalized in an extended language), the antithetical realization that Ω ought to be essentially indescribable in set theory, and thus the synthesis in the conclusion that there must already be some cardinal at which these properties obtain. Note that this in itself is a reflection argument. Extendible cardinals especially can be motivated in this way (see Section 5 and Reinhardt [40]).

(iii) *Resemblance.* This is closely related to (ii). Because of reflection considerations and, generally speaking, because the cumulative hierarchy is neutrally defined in terms of just the power set and union operations, it is reasonable to suppose that there are $\langle V_\alpha, \in \rangle$'s which resemble each other. The next conceptual step is to say that there are elementary embeddings $\langle V_\alpha, \in \rangle \to \langle V_\beta, \in \rangle$. Since this argument can just as well be cast in terms of $\langle V_{f(\alpha)}, \in, X(\alpha) \rangle$'s, where $f(\alpha)$ and $X(\alpha)$ are uniformly definable from α, the elementary embeddings may well turn out not to be the identity. Strong axioms like λ-extendibility (see Section 5) or Vopěnka's Principle (see Section 6) can be motivated in this way.

(iv) *Restriction.* Known assertions can be weakened to gain more information and sharpen implications. Ramsey and Rowbottom cardinals can be considered to follow in this way from measurable cardinals, and the axioms of Sections 7–8 can be viewed as introducing a spectrum of perhaps consistent axioms arising from Kunen's inconsistency result (1.12).

In all these approaches, the recurring feature of the various postulations is the notion of *elementary embedding*, and this paper is organized around this main theme.

Let us once again say that this paper is an exposition of mathematics. We consider that many of the methods and technical relationships that we encounter are not without some mathematical elegance. Thus, hopefully our exposition will add an esthetic element to other incentives for considering strong axioms of infinity, and this is by no means a factor to be underrated in the investigation of new mathematical concepts.

Our set theory is ZFC. However, the mathematics in this paper is not strictly formalizable in ZFC, since we discuss elementary embeddings of the whole set theoretical universe V. Kelley–Morse (KM) is adequate, as satisfaction can be expressed there, but Bernays–Gödel (BG) is often sufficient for many purposes. Ultimately, most implications can be formalized in ZFC, since either the elementary embedding involved can be regarded as restricted to some set, or only one formula instance of the elementary schema need be used. For a method of formalizing elementary embeddings of V in ZFC through a system of approximations, see IV of Gaifman [4]; this formalization is adequate to take care of most of the embeddings we consider.

Much of our notation is standard, but we do mention the following: the letters $\alpha, \beta, \gamma, \ldots$ denote ordinals whereas $\kappa, \lambda, \mu, \ldots$ are reserved for cardinals. V_α is the collection of sets of rank $< \alpha$. θ^M denotes the usual relativization of a formula θ to a class M. If x is a set, $|x|$ is its cardinality, $\mathcal{P}x$ is its power set, and $\mathcal{P}_\kappa x = \{y \in \mathcal{P}x \mid |y| < \kappa\}$. If also $x \subseteq \Omega$, $\bar{x}$ denotes its order type in the natural ordering. The identity function with the domain appropriate to the context is denoted by id. Finally, $\square$ signals the end of a proof.

If I is a set, an ultrafilter $\mathcal{U}$ over I is a maximal filter in the Boolean algebra $\mathcal{P}I$. $\mathcal{U}$ is *uniform* iff whenever $X \in \mathcal{U}$, $|X| = |I|$; *non-principal* iff $\bigcap \mathcal{U} = \emptyset$; and *$\kappa$-complete* iff whenever $T \subseteq \mathcal{U}$ and $|T| < \kappa$, $\bigcap T \in \mathcal{U}$. A cardinal $\kappa > \omega$ is *measurable* iff there is a non-principal, κ-complete ultrafilter over κ. In context, "for almost every x" means for x in a set in the ultrafilter involved.

We would like to thank the referee for simplifications in regard to 1.14, 3.2, 3.3, 4.8, and for suggesting the formulation of 2.6 in order to make 2.7 and 5.11 clearer.

1. Elementary embeddings and ultrapowers

Elementary embeddings with domain either the whole set theoretical universe V or just some initial segment V_α play a basic thematic role in this paper. In this initial section we quickly review basic techniques and establish some of our notation and terminology by working through the paradigm case, which can be considered a natural way of motivating measurable cardinals. Also, we consider a result of Kunen which will establish an upper limit to our further efforts.

When we investigate elementary embeddings j of V into some inner model M, it is convenient to have the situation implicit in the notation $j: \mathrm{V} \to M$, which we also take to include the assertion that j is not the identity function.

That j is an *elementary embedding* means that it preserves all relations definable in the language of set theory: if $x_1, \ldots, x_n$ are sets, and $\theta(v_1, \ldots, v_n)$ is a formula and $F(v_1, \ldots, v_n)$ a term in the language of set theory, then

$$\theta(x_1, \ldots, x_n) \quad \text{iff} \quad \theta^M(j(x_1), \ldots, j(x_n))$$

and

$$j(F(x_1, \ldots, x_n)) = F(j(x_1), \ldots, j(x_n)).$$

By an *inner model* we mean a transitive ε-model of ZFC containing all the ordinals. There is a formula $\mathrm{Inn}(M)$ and finitely many axioms $\varphi_0, \ldots, \varphi_n$ of ZFC so that for any class M,

$$\vdash_{\mathrm{ZFC}} (\varphi_0 \,\&\, \cdots \,\&\, \varphi_n)^M \,\&\, \bigcup M \subseteq M \,\&\, \Omega \subseteq M \leftrightarrow \mathrm{Inn}\,(M)$$

and if φ is any theorem of ZFC without free variables,

(*)

$$\vdash_{\mathrm{ZFC}} \mathrm{Inn}\,(M) \to \varphi^M.$$

For example, Inn (M) can assert that M is transitive, contains all the ordinals, and that, in M, the sequence $\langle \mathrm{V}_\alpha^M \mid \alpha \in \Omega \rangle$ is definable. (Here, the V_α^M's satisfy, in M, the usual definition for the V_α's.) Since all instances of ZF axiom schema are needed in the proofs of the theorems (*), this by no means implies that ZFC is finitely axiomatizable. We have:

$$\vdash_{\mathrm{ZFC}} \mathrm{Inn}\,(\mathrm{V}), \qquad \vdash_{\mathrm{ZFC}} \mathrm{Inn}\,(\mathrm{L}), \qquad \vdash_{\mathrm{ZFC}} \mathrm{Inn}\,(\mathrm{HOD}).$$

(HOD is the class of hereditarily ordinal definable sets, cf. Myhill–Scott [36].)

Assume now that $j: \mathrm{V} \to M$. We will frequently use the preservation schema for j without comment, leaving the reader to see that the relations and functions involved are set theoretic. For example, in 1.1 we will use $j(\mathrm{rank}\,(x)) = \mathrm{rank}\, j(x))$, and in 1.2 both

$$j(X_\alpha) = (j(X))_{j(\alpha)} = (j(X))_\alpha \quad \text{when} \quad j(\alpha) = \alpha,$$

and

$$j(\bigcap \{X_\alpha \mid \alpha < \gamma\}) = \bigcap \{(j(X))_\alpha \mid \alpha < j(\gamma)\}.$$

In the latter case, one must of course realize that X and γ are the free variables.

Proposition 1.1. (i) *For every* α, $j(\alpha) \geq \alpha$.

(ii) *j moves some ordinal.*

Proof. (i) By transfinite induction. For (ii), let x be of least rank so that $j(x) \neq x$. Then if $\delta = \operatorname{rank}(x)$, $j(\delta) > \delta$. Otherwise, if $y \in j(x)$, $\operatorname{rank}(y) < \operatorname{rank}(j(x)) = j(\operatorname{rank}(x)) = \delta$, so that $j(y) = y$ and $y \in x$. But also, $y \in x$ implies $j(y) = y \in j(x)$. Hence, $j(x) = x$, which is a contradiction. □

Now let δ be the least ordinal moved by j. We say that δ is the *critical point* of the embedding j. A model theorist might be quick to see

Theorem 1.2. *Let $\mathcal{U}$ be defined by*

$$X \in \mathcal{U} \quad \textit{iff} \quad X \subseteq \delta \;\&\; \delta \in j(X).$$

Then $\mathcal{U}$ is a non-principal δ-complete ultrafilter over δ, and hence, δ is a measurable cardinal.

Proof. $\delta \in \mathcal{U}$, but $\alpha < \delta$ implies $\{\alpha\} \notin \mathcal{U}$ since $j(\{\alpha\}) = \{\alpha\}$. Also, $X \in \mathcal{U}$ iff $\delta \in j(X)$ iff $\delta \notin j(\delta - X)$ iff $\delta - X \notin \mathcal{U}$.
Finally, if $\gamma < \delta$ and $\{X_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{U}$, $\delta \in \bigcap \{j(X_\alpha) \mid \alpha < \gamma\}$. But $j(\gamma) = \gamma$, so $\bigcap \{j(X_\alpha) \mid \alpha < \gamma\} = j(\bigcap \{X_\alpha \mid \alpha < \gamma\})$ i.e., $\bigcap \{X_\alpha \mid \alpha < \gamma\} \in \mathcal{U}$. □

Thus, elementary embeddings already give rise to measurable cardinals, and so these will be the "smallest" of the large cardinals to be considered. It might be worthwhile to note that direct arguments using j show that δ must be large.

(a) Since δ is obviously a limit ordinal, $V_\delta \models Z$.

(b) By the argument of 1.1 (ii), if $x \in V_\delta$, $j(x) = x$. Hence, $y \subseteq V_\delta$ implies $y = j(y) \cap V_\delta \in M$.

(c) If $F: V_\delta \to V_\delta$ and $x \in V_\delta$, then $F''x \in V_\delta$. This is so since $F = j(F) \mid V_\delta$ and $j(F)''j(x) = j(F''x) \subseteq V_\delta \in V_{j(\delta)}$, and thus $F''x \in V_\delta$ by elementarity. Hence, $V_\delta \models \mathrm{ZFC}$ and $V_{\delta+1} \models \mathrm{BG}$.

(d) If $x, y \in V_{\delta+1}$ and $V_{\delta+1} \models \theta(x, y)$ and θ is first-order, then for some $\alpha < \delta$, $V_{\alpha+1} \models \theta(x \cap V_\alpha, y \cap V_\alpha)$: using (b), we have $V_{\delta+1} \subseteq M$, so $V_{\delta+1} \models \theta(x, y)$ in M, so that in M there is an $\alpha < j(\delta)$ with $V_{\alpha+1} \models \theta(j(x) \cap V_\alpha, j(y) \cap V_\alpha)$. Thus, there is an $\alpha < \delta$ with $V_{\alpha+1} \models \theta(x \cap V_\alpha, y \cap V_\alpha)$ by elementarity. It follows that V_δ satisfies Bernays' schema, i.e. δ is second-order indescribable.

Since we will shortly get a converse to 1.2 (that is, if κ is a measurable cardinal, there is a $j: V \to M$ with critical point κ), the foregoing quickly establish the standard facts on the size of measurable cardinals.

To get that converse, we will take an ultrapower of V. So first, let us recall the general process with $\mathcal{D}$ an arbitrary ultrafilter over some index set I. As usual, define for f, g functions with domain I,

$$f \sim_{\mathcal{D}} g \quad \text{iff} \quad \{i \in I \mid f(i) = g(i)\} \in \mathcal{D}.$$

$\sim_{\mathcal{D}}$ is an equivalence relation with each equivalence class a proper class. In order to form the ultrapower, we need to have equivalence types $[f]$ such that $f \sim_{\mathcal{D}} g$ iff

$[f]=[g]$. It is customary to define $[f]$ as the equivalence class of f; in our case these are proper classes and this is inconvenient (since we prefer to stay within the language of ZFC). We may instead use the following device of Scott.

Definition 1.3. (Scott) If f is a function with domain I,

$$S_{\mathcal{D}}(f)=\{g \mid g \sim_{\mathcal{D}} f \,\&\, \forall h(h \sim_{\mathcal{D}} f \rightarrow \operatorname{rank}(g) \leq \operatorname{rank}(h))\}.$$

That is, $S_{\mathcal{D}}(f)$ is *the collection of those* $g \sim_{\mathcal{D}} f$ *of least possible rank.* $S_{\mathcal{D}}(f)$ is a set, so we can define

$$\mathrm{V}^I/\mathcal{D}=\{S_{\mathcal{D}}(f) \mid f: I \rightarrow \mathrm{V}\},$$

and as a membership relation,

$$S_{\mathcal{D}}(f)\, E_{\mathcal{D}}\, S_{\mathcal{D}}(g) \quad \text{iff} \quad \{i \in I \mid f(i) \in g(i)\} \in \mathcal{D}.$$

Thus, $\langle \mathrm{V}^I/\mathcal{D}, E_{\mathcal{D}}\rangle$ is a (class) structure definable within the language of set theory. The following is basic, and proved by induction on length of formulas.

Theorem 1.4. (Łoś) *If $\theta(y_1, \ldots, v_n)$ is a formula of set theory and $f_1, \ldots, f_n$ are functions: $I \rightarrow V$, then*

$$\langle \mathrm{V}^I/\mathcal{D}, E_{\mathcal{D}}\rangle \models \theta(S_{\mathcal{D}}(f_1), \ldots, S_{\mathcal{D}}(f_n)) \quad \text{iff} \quad \{i \in I \mid \theta(f_1(i), \ldots, f_n(i))\} \in \mathcal{D}.$$

If $\mathcal{D}$ is a λ-complete ultrafilter, then this theorem extends to set theoretical formulas in the language $\mathrm{L}_{\lambda\lambda}$. (Recall that $\mathrm{L}_{\kappa\lambda}$ is the infinitary language allowing conjunctions of $<\kappa$ formulas and quantifications over $<\lambda$ variables.) By the theorem, there is a canonical elementary embedding $e_{\mathcal{D}}: \mathrm{V} \rightarrow \mathrm{V}^I/\mathcal{D}$ defined by

$$e_{\mathcal{D}}(x)=S_{\mathcal{D}}(\langle x \mid i \in I\rangle),$$

i.e. $e_{\mathcal{D}}(x)$ is the equivalence class of the constant function on I with value x.

We now assume that $\mathcal{D}$ is both *non-principal* and *ω_1-complete*. The following two propositions are basic tools:

Proposition 1.5. (Mostowski [35]) *Suppose $\langle A, E\rangle$ is a (possibly proper class) structure so that E is a binary relation on A, and*

(i) *E is well-founded,*

(ii) *E is extensional, i.e. if $a, b \in A$ and for any $x \in A$ xEa iff xEb, then $a=b$,*

(iii) *$\{x \mid xEa\}$ is a set for each $a \in A$.*

Then there is a unique isomorphism $h: \langle A, E\rangle \leftrightarrow \langle M, \varepsilon\rangle$ to a transitive ε-structure M, called the transitive collapse.

Proof. Define h by recursion on E by

$$h(x)=\{h(y) \mid yEx\}. \quad \square$$

Proposition 1.6. *$E_{\mathscr{D}}$ on $V^I/\mathscr{D}$ is well-founded.*

Proof. If $\cdots S_{\mathscr{D}}(f_2)\, E_{\mathscr{D}}\, S_{\mathscr{D}}(f_1)\, E_{\mathscr{D}}\, S_{\mathscr{D}}(f_0)$, $x \in \bigcap_{n\in\omega} \{i \in I \mid f_{n+1}(i) \in f_n(i)\}$ implies $\cdots f_2(x) \in f_1(x) \in f_0(x)$, which would be contradictory. □

In view of 1.4, the preceding proposition is essentially the observation that "E is well-founded" is expressible in $L_{\omega_1\omega_1}$. These results show that there is a canonical isomorphism $h_{\mathscr{D}} : \langle V^I/\mathscr{D}, E_{\mathscr{D}}\rangle \leftrightarrow \langle M_{\mathscr{D}}, /\varepsilon\rangle$, where $M_{\mathscr{D}}$ is a transitive class. We have immediately from Łoś' Theorem that $\vdash_{\mathrm{ZFC}} \mathrm{Inn}\,(M_{\mathscr{D}})$. Setting $j_{\mathscr{D}} = h_{\mathscr{D}} \circ e_{\mathscr{D}}$ so that $j_{\mathscr{D}}$ is elementary we will use the typical notation

$$j_{\mathscr{D}} : V \dashrightarrow M_{\mathscr{D}} \approx V^I/\mathscr{D}$$

to depict the situation. Also, for any $f: I \to V$, we set

$$[f]_{\mathscr{D}} = h_{\mathscr{D}}(S_{\mathscr{D}}(f)),$$

i.e. $[f]_{\mathscr{D}}$ is the transitive collapse of the ultrapower equivalence class of f. $\mathscr{D}$ as a subscript will often be dropped in these and similar situations when it is clear from the context. Recalling some comments before 1.3, notice that $f \sim_{\mathscr{D}} g$ iff $[f]_{\mathscr{D}} = [g]_{\mathscr{D}}$. Consequently, $[f]_{\mathscr{D}}$ could initially have played the role of $S_{\mathscr{D}}(f)$ in the construction of well-founded ultrapowers, in which case $V^I/\mathscr{D}$ is immediately identified with $M_{\mathscr{D}}$.

The following proposition is very useful.

Proposition 1.7. *With $\mathscr{D}$ as above, $j = j_{\mathscr{D}}$, $M = M_{\mathscr{D}}$, etc.*

(i) *if $j''x \in M$, and $y \subseteq M$ is such that $|y| \leqslant |x|$, then $y \in M$,*

(ii) *$j''(|I|^+) \notin M$.*

Proof. For (i), let $y = \{[t_a] \mid a \in x\} \subseteq M$, and define $T: j''x \to y$ by $T(j(a)) = [t_a]$. Since T enumerates y, it suffices to show that $T \in M$. So, we need a g so that $[g] = T$, i.e.

(a) domain $([g]) = j''x$.

(b) for all $a \in x$, $[g](j(a)) = [t_a]$.

Let $[f] = j''x$. By Łoś' Theorem (1.4) if for each $i \in I$ we set domain $(g(i)) = f(i)$, and $g(i)(a) = t_a(i)$ for each $a \in$ domain $(g(i))$, then clearly g is as required.

For (ii), assume that $j''(|I|^+) = [f] \in M$. If $A = \{i \in I \mid |f(i)| \leqslant |I|\} \in \mathscr{D}$, since $|I|^+$ is regular, there is an $\alpha \in |I|^+ - \bigcup \{f(i) \mid i \in A\}$. But then $j(\alpha) \notin [f]$. If $B = \{i \in I \mid |f(i)| > |I|\} \in \mathscr{D}$, define h on B by induction on some well-ordering $\leqslant_I$ of I so that

$$h(i) \in f(i) - \{h(j) \mid j <_I i \;\&\; j \in B\}.$$

Then $[h] \in [f]$, yet h is not constant on any set in $\mathscr{D}$, as $\mathscr{D}$ is non-principal. Hence, in either case we get a contradiction from the assumption that $j''(|I|^+) = [f]$. □

If $|x| = \lambda$ in 1.7 (i), the result can simply be stated as: M is *closed under λ-sequences*. In the future, we will also use ${}^{\lambda}M \subseteq M$ to denote this. 1.7(ii) puts an upper limit on the closure of M.

We now consider $\mathcal{U}$ a non-principal κ-complete ultrafilter over κ a measurable cardinal, and $j: \mathrm{V} \to M \approx \mathrm{V}^{\kappa}/\mathcal{U}$. The next theorem deals with this situation, and its (i) completes the converse to 1.2.

Theorem 1.8.
(i) *κ is the critical point of j.*
(ii) ${}^{\kappa}M \subseteq M$ *but* ${}^{\kappa^+}M \nsubseteq M$.
(iii) $\kappa < 2^{\kappa} < j(\kappa) < (2^{\kappa})^+$.
(iv) $\mathcal{U} \notin M$.

Proof. For (i), we first prove by induction on α that if $\alpha < \kappa$, $j(\alpha) = \alpha$. Suppose, towards a contradiction, that $j(\alpha) > \alpha$. Let $[f] = \alpha$. Then $\{\beta \mid f(\beta) < \alpha\} \in \mathcal{U}$, so that by the κ-completeness of $\mathcal{U}$, there is a $\xi < \alpha$ so that $\{\beta \mid f(\beta) = \xi\} \in \mathcal{U}$. But then, $[f] = j(\xi) = \xi$, which is absurd.

Next, if $\mathrm{id}: \kappa \to \kappa$ is the identity function on κ, $\kappa \leqslant [\mathrm{id}] < j(\kappa)$ since for each $\delta < \kappa$, $\{\alpha \mid \delta < \alpha < \kappa\} \in \mathcal{U}$. Thus κ is the critical point of j.

For (ii), use 1.7 and the fact that $j''\kappa = \kappa \in M$.

To show (iii), first note that $j(\kappa) =$ order type of $\{[f]_{\mathcal{U}} \mid f \in {}^{\kappa}\kappa\}$, so that $j(\kappa) < (2^{\kappa})^+$. Also, in M $j(\kappa)$ is measurable, hence strongly inaccessible, so that $(2^{\kappa})^M < j(\kappa)$. But by (ii), $\mathcal{P}(\kappa) = \mathcal{P}^M(\kappa)$ so that $2^{\kappa} \leqslant (2^{\kappa})^M$, since $M \subseteq V$.

For (iv), assume that $\mathcal{U} \in M$. Since ${}^{\kappa}\kappa = ({}^{\kappa}\kappa)^M$, in M one can evaluate $j(\kappa)$, so that $j(\kappa) < ((2^{\kappa})^+)^M$ as in the previous paragraph. But this contradicts the strong inaccessibility of $j(\kappa)$ in M. □

The following corollary has been, of course, much strengthened in recent years by the work of Gaifman, Rowbottom, Silver and others.

Corollary 1.9. (Scott [41]). *If there is a measurable cardinal,* $\mathrm{V} \neq \mathrm{L}$.

Proof. Assume $\mathrm{V} = \mathrm{L}$. Then the M as above is an inner model satisfying the axiom of constructibility. Hence, $M = \mathrm{L}$. But $\mathcal{U} \in \mathrm{V} - M$ by 1.8 (iv), a contradiction. □

This corollary has as an easy consequence the fact that there is no elementary embedding $j: \mathrm{V} \to \mathrm{L}$. The situation with HOD is unclear. But, how about an elementary embedding of V into V itself? Kumen showed that this is not possible in ZFC. His proof uses a simple case of a combinatorial result of Erdös and Hajnal. But first, for the reader's interest a short proof of the general case due to Galvin and Prikry is presented. The result is concerned with the so-called

Jónsson's problem (see Devlin [1] for details), and shows that if we allow an infinitary operation, there are Jónsson algebras of every infinite cardinality.

Definition 1.10. For any set x, a function f is called *ω-Jónsson* over x iff $f: {}^{\omega}x \to x$ and whenever $y \subseteq x$ and $|y| = |x|$, $f''{}^{\omega}y = x$.

Theorem 1.11. (Erdös–Hajnal [2]) *For every infinite cardinal λ, there is an ω-Jónsson function over λ.*

Proof. (Galvin–Prikry [5]) For the special case $\lambda = \omega$, there is a simple inductive argument: Let $\{\langle X_\alpha, n_\alpha\rangle \mid \alpha < 2^\omega\}$ enumerate $[\omega]^\omega \times \omega$. (Recall that $[\omega]^\omega$ is the collection of infinite subsets of ω.) By induction on $\alpha < 2^\omega$ pick $s_\alpha \in {}^{\omega}X_\alpha$ so that $s_\alpha \neq s_\beta$ for $\beta < \alpha$ and set $f(s_\alpha) = n_\alpha$. Then any extension of f to all of ${}^{\omega}\omega$ is ω-Jónsson over ω.

Suppose now that $\lambda > \omega$. Let $\mathcal{S}$ be any maximal collection of subsets of λ so that members of $\mathcal{S}$ have order type ω and are mutually almost disjoint. By the special case above, we can assume that for each $x \in \mathcal{S}$ there is a function f_x ω-Jónsson over x. Define now a function $g: {}^{\omega}\lambda \to \lambda$ by:

$$g(s) = \begin{cases} f_x(s) & \text{if the range of } s \text{ is infinite and } s \in {}^{\omega}x \\ & \text{for some } x \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is well-defined.

It suffices to find an $A \subseteq \lambda$ so that $|A| = \lambda$, yet $B \subseteq A$ and $|B| = \lambda$ implies $g''{}^{\omega}B \supseteq A$, for then an ω-Jónsson function over A can easily be derived from g. If no such A exists, there are sets $\lambda \supseteq A_0 \supseteq A_1 \supseteq A_2 \cdots$ each of cardinality λ, and $a_n \in A_n - A_{n+1}$ so that $a_n \notin g''{}^{\omega}A_{n+1}$. If $y = \{a_n \mid n \in \omega\}$, by maximality of $\mathcal{S}$ there is an $x \in \mathcal{S}$ so that $x \cap y$ is infinite. Let $x \cap y = \{a_{n_0}, a_{n_1}, \ldots\}$, where $n_0 < n_1 < \ldots$. Now $t = \{a_{n_1}, a_{n_2}, \ldots\} \subseteq A_{n_1}$. But by definition of f_x, there is an $s \in {}^{\omega}t$ so that $g(s) = a_{n_0}$. Hence, $a_{n_0} \in g''A_{n_1}$, which is a contradiction. $\square$

Theorem 1.12. (Kunen [15]) *There is no non-trivial elementary embedding of* V *into itself.*

Proof. Argue by contradiction and suppose $j: \mathrm{V} \to \mathrm{V}$ with critical point κ. Set $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$, where $j^0(\kappa) = \kappa$ and $j^{n+1}(\kappa) = j(j^n(\kappa))$. Note that $j(\lambda) = \sup\{j^{n+1}(\kappa) \mid n \in \omega\} = \lambda$.
Now let f be ω-Jónsson over λ; then $j(f)$ is also ω-Jónsson over λ. Consider the set $X = j''\lambda$. Since $|X| = \lambda$, let $x \in {}^{\omega}X$ so that $j(f)(x) = \kappa$. But if $x(n) = j(\alpha_n)$ for $n \in \omega$, $x = j(y)$ where $y \in {}^{\omega}\lambda$ and $y(n) = \alpha_n$. Hence, $\kappa = j(f)(j(y)) = j(f(y))$, contradicting the fact that κ is not in the range of j. $\square$

As Kunen remarks, since the λ in the above proof is a strong limit cardinal of cofinality ω, the argument of the special case of 1.11 suffices to produce an

ω-Jónsson function over λ. (The full result of 1.11 will be used in our forthcoming 3.3 and 3.4) Note also that more generally, if $j:V\to M$ and λ is defined in the same way, then there is actually a subset of λ not in M. This way of formulating Kunen's result is formalizable in ZFC through the strategem of Gaifman [4]; the notion $j:V\to V$ is not. Since AC was used in the proof of 1.11, we may still ask

Open Question 1.13. In ZF (without AC) can there be a non-trivial elementary embedding of the universe into itself?

Kunen's result will limit our efforts in that we cannot embed the universe into too "fat" an inner model. Pending an answer to 1.13, one can perhaps best view this fact as a structural limitation imposed on V by the Axiom of Choice.

Finally, we note the following generalization of 1.8(iv) concerning ultrapowers. This was known before Kunen's result (1.12), and established the special case that no $j:V\to V$ could be the result of taking an ultrapower.

Proposition 1.14. *If $\mathcal{D}$ is a non-principal, ω_1-complete ultrafilter over some cardinal ν, then $\mathcal{D}\notin M_{\mathcal{D}}$.*

Proof. Let $M=M_{\mathcal{D}}$ and $j=j_{\mathcal{D}}$, and assume that $\mathcal{D}\in M$. It follows that $\mathcal{P}\nu\subseteq M$ and ${}^{\nu}\nu\subseteq\mathcal{P}(\nu\times\nu)\subseteq M$. Note that for any ordinal α, $j(\alpha)=$ the order type of $\{[f]_{\mathcal{D}}\mid f\in{}^{\nu}\alpha\}$. Thus, $j''\nu\in M$, since $j''\nu$ is just the collection of such order types for α ranging over ordinals below ν, and can be properly defined in M as $\mathcal{D}\in M$ and ${}^{\nu}\nu=({}^{\nu}\nu)^M$.

By 1.7(i), it follows that M is closed under ν-sequences, and in particular ${}^{\nu}\nu^+=({}^{\nu}\nu^+)^M$. The argument in the previous paragraph can now be used again to show that $j''(\nu^+)\in M$, thereby contradicting 1.7(ii). $\square$

2. Supercompactness

Though there can be no elementary embedding $j:V\to V$, we noted that if $j_{\mathcal{U}}:V\to M_{\mathcal{U}}$ arises from a non-principal κ-complete ultrafilter $\mathcal{U}$ over κ a measurable cardinal, then $M_{\mathcal{U}}$ is not even closed under κ^+-sequences. The following is an intermediary notion, and seems the proper generalization of measurability.

Definition 2.1. If $\kappa\leq\lambda$, κ is *λ-supercompact* iff there is an elementary embedding $j:V\to M$ so that:

(i) j has critical point κ and $j(\kappa)>\lambda$,

(ii) ${}^{\lambda}M\subseteq M$.

κ is *supercompact* iff κ is λ-supercompact for all $\lambda\geq\kappa$.

It follows from (ii) that M contains all sets hereditarily of cardinal $\leq\lambda$. Note that from Section 1, κ is κ-supercompact iff κ is measurable. It will be shown shortly that if κ is 2^κ-supercompact, then κ is actually the κth measurable cardinal (see 3.5). Kunen noticed that (i) in the above definition can be replaced by simply "j has critical point κ," since for some integer n, $j^n(\kappa) > \lambda$ — else we would again get the contradiction of 1.12. A further argument is needed to show that the nth iterate of j embeds V into an inner model closed under λ-sequences (see [15]).

2.2. In the situation of 2.1, since ${}^\lambda M \subseteq M$, the embedding j immediately suggests considering the following ultrafilter:

$$X \in \mathcal{U} \quad \text{iff} \quad j''\lambda \in j(X).$$

Note that since $|j''\lambda| = \lambda < j(\kappa)$ in M, $j''\lambda \in \mathcal{P}_{j(\kappa)}j(\lambda)^M$. Hence $\mathcal{P}_\kappa\lambda \in \mathcal{U}$, and we can consider this set to be the underlying index set for $\mathcal{U}$. $\mathcal{U}$ has the following properties:

(i) $\mathcal{U}$ is a κ-complete ultrafilter,

(ii) $\mathcal{U}$ is non-principal, and for any $a \in \lambda$, $\{x \mid a \in x\} \in \mathcal{U}$,

(iii) If f is a function defined on a set in $\mathcal{U}$ so that $\{x \mid f(x) \in x\} \in \mathcal{U}$, then there is an $a \in \lambda$ so that $\{x \mid f(x) = a\} \in \mathcal{U}$.

By (i) and (ii) if $y \in \mathcal{P}_\kappa\lambda$, $\{x \mid y \subseteq x\} \in \mathcal{U}$. For a proof of (iii), note that if $j(f)(j''\lambda) \in j''\lambda$ then $j(f)(j''\lambda) = j(a)$ for some $a \in \lambda$.

Definition 2.3. If $\kappa \leqslant \lambda$, an ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$ is *normal* iff it satisfies (i), (ii) and (iii) as above. More generally, an ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa I$, where I is a set, is *normal* iff it satisfies (i), (ii) and (iii) with λ replaced by I. Finally, without reference to κ, an ultrafilter $\mathcal{U}$ over $\mathcal{P}I$ (i.e. $\mathcal{U} \subseteq \mathcal{P}\mathcal{P}I$) is *normal* iff it satisfies (ii) and (iii) with λ replaced by I.

If $f: I \to \lambda$ is bijective, then f induces a bijection between normal ultrafilters over $\mathcal{P}_\kappa I$ and those over $\mathcal{P}_\kappa\lambda$, so for most purposes, it suffices to consider ultrafilters over sets of form $\mathcal{P}_\kappa\lambda$.

Note that an ω_1-complete ultrafilter $\mathcal{U}$ over $\mathcal{P}I$ is normal iff $[\mathrm{id}]_\mathcal{U} = j_\mathcal{U}''I$, where $\mathrm{id}: \mathcal{P}I \to \mathcal{P}I$ is the identity map. This easy but central fact will be used repeatedly throughout the rest of this paper.

Just as in Section 1, having produced an ultrafilter from an embedding, one can hope to reverse the process by taking an ultrapower. So, let $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$ be normal and consider the canonical $j: \mathrm{V} \to M \approx \mathrm{V}^{\mathcal{P}_\kappa\lambda}/\mathcal{U}$. Then:

(i) ${}^\lambda M \subseteq M$. Use 1.7(i) and the fact that $[\mathrm{id}] = j''\lambda$.

(ii) κ is the critical point of j and $j(\kappa) > \lambda$. We have $\{x \mid |x| < \kappa\} \in \mathcal{U}$, so $|[\mathrm{id}]| = |j''\lambda| < j(\kappa)$. But $|j''\lambda| = \lambda$ in M, since M is closed under λ-sequences.

We have shown

Theorem 2.4. *If $\kappa \leqslant \lambda$, the following are equivalent:*

(i) *κ is λ-supercompact,*

(ii) *there is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$.*

Note the following reversibility: if $\mathcal{U}$ is normal over $\mathcal{P}_\kappa\lambda$, $j_{\mathcal{U}}:\mathrm{V}\to M_{\mathcal{U}}$, and $\mathcal{U}'$ is defined from $j_{\mathcal{U}}$ as in 2.2, then $\mathcal{U}'=\mathcal{U}$.

Though normal ultrafilters have been defined generally over index sets $\mathcal{P}\lambda$, we have seen how sets of form $\mathcal{P}_\kappa\lambda$ naturally come into play. In fact, if $\mathcal{U}$ is a ω_1-complete normal ultrafilter over $\mathcal{P}\lambda$, then for some $\mu<\lambda$, $\mathcal{P}_\mu\lambda\in\mathcal{U}$:

Let $j:\mathrm{V}\to M$ be the corresponding embedding with critical point κ. Since ${}^\lambda M\subseteq M$ by normality, Kunen's argument shows that for some interger n, we must have $j^n(\kappa)<\lambda\leqslant j^{n+1}(\kappa)$. If $\lambda<j^{n+1}(\kappa)$, set $\mu=j^n(\kappa)$ so that $\mu<\lambda$. If $\lambda=j^{n+1}(\kappa)$, note that a simple argument inducing on the $j^i(\kappa)$'s and using ${}^\lambda M\subseteq M$ shows that λ must be inaccessible in V. Thus, if we set $\mu=(j^n(\kappa))^+$ in this case, we have $\mu<\lambda$. Finally, in either case, not that $\|[\mathrm{id}]\|=|j''\lambda|=\lambda<j(\mu)$, so that $\{x\mid |x|<\mu\}\in\mathcal{U}$, i.e. $\mathcal{P}_\mu\lambda\in\mathcal{U}$.

We now show that if κ is supercompact and $\theta\geqslant\kappa$, then V_θ can be expressed as an ultraproduct of V_γ's with $\gamma<\kappa$.

Theorem 2.5. *Let $\theta\geqslant\kappa$, and assume that κ is $|\mathrm{V}_\theta|$-supercompact. Let $\mathcal{U}$ be a normal ultrafilter over $\mathcal{P}_\kappa\mathrm{V}_\theta$, and $j:\mathrm{V}\to M$ the corresponding embedding. For convenience, set $X=\mathcal{P}_\kappa\mathrm{V}_\theta$. Suppose that in M, $\theta=[\langle\theta_x\mid x\in X\rangle]$. Then*

(i) *for almost every $x\in X$, x is an elementary submodel of V_θ, and the transitive collapse gives an isomorphism $\pi_x:x\cong\mathrm{V}_{\theta_x}$,*

(ii) *set $j_x=\pi_x^{-1}$. Then $j\restriction\mathrm{V}_\theta=[\langle j_x\mid x\in X\rangle]$,*

(iii) *there is an isomorphism*

(*)

$$\mathrm{V}_{\theta+1}\cong\prod_{x\in X}\mathrm{V}_{\theta_x+1}/\mathcal{U},$$

which can be explicated as follows: if $y\in\mathrm{V}_\theta$, $y=[\langle\pi_x(y)\mid x\in X\ \&\ y\in x\rangle]$. If $y\in\mathrm{V}_{\theta+1}$, $y=[\langle\pi_x''(y\cap x)\mid x\in X\rangle]$.

Proof. $j''\mathrm{V}_\theta$ is an elementary submodel of $\mathrm{V}^M_{j(\theta)}$, and the transitization map $\pi:j''\mathrm{V}_\theta\cong\mathrm{V}_\theta$ has inverse $j\restriction\mathrm{V}_\theta$. We also have that $[\mathrm{id}]=j''\mathrm{V}_\theta$ in M, and that $j\restriction\mathrm{V}_\theta$ and π are both in M as they are hereditarily of cardinal $\leqslant|\mathrm{V}_\theta|$. From these facts and Łoś' Theorem (1.4), (i) and (ii) of the theorem now follow.

For (iii), note that the right side of (*) is clearly isomorphic to $\mathrm{V}^M_{\theta+1}$. However, using the fact that M is closed under $|\mathrm{V}_\theta|$-sequences, it is easy to prove by induction on $\alpha\leqslant\theta+1$ that $\mathrm{V}_\alpha=\mathrm{V}^M_\alpha$. This establishes (*).

Now let $y\in\mathrm{V}_\theta$, and suppose $y=[\langle y_x\mid x\in X\rangle]$. Then $y_x\in\mathrm{V}_{\theta_x}$ for almost every $x\in X$. Moreover, $j(y)=[\langle j_x(y_x)\mid x\in X\rangle]$. But by definition, $j(y)=[\langle y\mid x\in X\rangle]$. Hence, for almost every $x\in X$, we have $j_x(y_x)=y$, i.e. $\pi_x(y)=y_x$, which was to be proved.

Finally, let $y\in\mathrm{V}_{\theta+1}$. We wish to show that $y=[\langle\pi_x''(y\cap x)\mid x\in X\rangle]$. For this, it suffices to prove that $y=\pi''(j(y)\cap j''\mathrm{V}_\theta)$. But $j(y)\cap j''\mathrm{V}_\theta=j''y$, and $\pi(j(z))=z$ for $z\in\mathrm{V}_\theta$. So our claim is evident. □

Note that since, in the notation of 2.5, $j(\kappa) > |V_\theta| \geq \theta$, we have that $\theta_x < \kappa$ for almost every x.

Let us say that a property $P(x)$ is *local* iff it has the form $\exists\delta(V_\delta \models \psi(x))$. The ultraproduct representation makes it evident that if $P(x)$ is local and for some γ, $P(\gamma)$ holds, then if κ is supercompact, $P(\gamma)$ holds for some $\gamma < \kappa$. This is worth some elaboration.

Definition 2.6. Recall the Lévy hierarchy of formulas (see Lévy [21], Definition 1). For any transitive M, say Σ_n (resp. Π_n) *relativizes down to* M iff whenever $P(x)$ is Σ_n (resp. Π_n), if $a \in M$ and $P(a)$ holds, then $M \models P(a)$.

Σ_n relativizes down to M iff Π_{n+1} does. According to [21] Theorem 36, if $|V_\theta| = \theta$, then Σ_1 (and hence Π_2) relativizes down to V_θ. Moreover, it is easy to construct a sentence Φ so that $V_\theta \models \Phi$ iff $|V_\theta| = \theta$. With these points in mind, suppose now that $P(x)$ is Σ_2, say $\exists y\, Q(x, y)$ where Q is Π_1. Then,

$$P(x) \quad \text{iff} \quad \exists\beta\,[V_\beta \models (\Phi \,\&\, \exists y\, Q(x, y))].$$

Thus, any Σ_2 property is local (as we have defined this notion just before 2.6). Conversely, it is well known that "x is a V_α" is Π_1, and hence a local property is one given by a Σ_2 formula.

Theorem 2.7. *If κ is supercompact, Σ_2 (and hence Π_3) relativize down to V_κ.*

Proof. Suppose $P(x)$ is $\exists y\, Q(x, y)$ where Q is Π_1. Let $a \in V_\kappa$ so that $P(a)$ holds, and fix b such that $Q(a, b)$ holds. By supercompactness, let $j: V \to M$ with critical point κ so that $b \in M \cap V_{j(\kappa)}$. Note that $j(a) = a$. Thus, $(V_{j(\kappa)} \models P(a))^M$, so $V_\kappa \models P(a)$. $\square$

Observe that 2.6 is optimal, since the Σ_3 sentence "There is a supercompact cardinal" fails in V_κ if κ is the least supercompact cardinal. (Cf. 5.8 below, or note that "x is not supercompact" is Σ_2 and apply 2.6.)

It is pertinent here to discuss the question of cardinal powers in the context of large cardinals. Silver [42] showed that in $L[\mathcal{U}]$, where $\mathcal{U}$ is a normal ultrafilter over a measurable cardinal, the GCH holds, and hence: Con (ZFC & there is a measurable cardinal) implies Con (ZFC & there is a measurable cardinal & GCH). Kunen [18] then showed that Con (ZFC & there is a measurable cardinal κ & $2^\kappa > \kappa^+$) implies $\forall\alpha$ Con (ZFC & there are α measurable cardinals). This surprising result certainly indicated that strong assumptions would be necessary to get a model with a measurable cardinal κ so that $2^\kappa > \kappa^+$.

It was Silver who first found such a model: he showed that if κ is supercompact in the ground model, there is a forcing extension in which $2^\kappa > \kappa^+$ and κ is still measurable. (A more precise formulation of Silver's result is possible in the terminology of Section 5: If κ is $(\eta + \delta + 1)$-extendible in the ground model, there is a forcing extension in which $2^\kappa \geq \aleph_{\kappa+\delta}$ and κ is still η-extendible if $\eta > 0$, or measurable if η was 0.) Combining this result with one further extension using Prikry forcing (to change the cofinality of a measurable cardinal to ω) yielded the first example of a singular strong limit cardinal κ so that $2^\kappa > \kappa^+$. Magidor [27] then proved the following relative consistency result, using in part Silver's result. Given a cardinal with a sufficient degree of supercompactness in the ground model, there is a forcing extension in which $2^{\aleph_\omega} = \aleph_{\omega+2}$ and for every integer n, $2^{\aleph_n} \leq \aleph_{n+2}$. Recent results of Jensen [9] and Mitchell seem to indicate that one cannot expect to weaken the initial large cardinal assumption by very much. See also the end of Section 7 for a result of Magidor with a stronger conclusion, but also starting with the stronger assumption of hugeness.

Silver's method of iterated forcing, first used to get the result mentioned in the previous paragraph, is often called Backward Easton Forcing. It is applicable to a variety of problems in set theory, and in particular can be used to show that the presence of large cardinals has little effect on the cardinal powers of regular cardinals. For example, Menas [30] shows that if κ is supercompact in the ground model, and Φ is a function on regular cardinals so that

(i) $\lambda \leq \mu$ implies $\Phi(\lambda) \leq \Phi(\mu)$,

(ii) $\mathrm{cf}\,(\Phi(\lambda)) > \lambda$, and

(iii) Φ is local (as previously defined, i.e. Σ_2), then there is a Backward Easton forcing extension in which cardinals are preserved, $2^\lambda = \Phi(\lambda)$ for every regular λ, and κ is still supercompact.

When a cardinal κ is supercompact, other than the technical result that there are κ measurable cardinals below κ (see 3.5), little is known about the behavior of the set theoretical universe below κ which does not already follow from the measurability of κ.

Concerning the behavior of the universe above κ, several interesting facts are known. Some of these already follow from the weaker assumption of strong compactness (see Section 4), but one exception is the fact that the second-order Löwenheim–Skolem Theorem holds for structures with underlying set of cardinality at least κ. Magidor [26] shows that, in an appropriate sense, we need the strength of supercompactness in this case.

In Section 4, the main consistency results involving supercompactness and strong compactness are stated. Section 5 contains several results on supercompactness in the context of extendibility, and another characterization (5.7). To conclude this section, we mention that there are combinatorial characterizations of supercompact cardinals (Magidor [23]), in terms of concepts first formulated by Jech [7]. Also, Prikry [37] has recently formulated a concept of *real-supercompactness* in analogy to real-valued measurability, and observed that if it is consistent that a supercompact cardinal exists, then it is consistent that 2^ω is real-supercompact. He also showed that several consequences of supercompactness that we will discuss shortly (see 4.6, 4.7) also follow from real-supercompactness.

3. Normal ultrafilters

With the introduction of normal ultrafilters in the previous section, we now take time to investigate them in some detail. First, some technicalities; recall that if x is a set of ordinals, $\bar{x}$ denotes its order type.

Proposition 3.1. *If $\mathcal{U}$ is normal over $\mathcal{P}_\kappa\lambda$ and $\alpha \leq \lambda$,*
(i) $j''_{\mathcal{U}}\alpha = [\langle x \cap \alpha \mid x \in \mathcal{P}_\kappa\lambda\rangle]_{\mathcal{U}}$,
(ii) $\alpha = [\langle \overline{x \cap \alpha} \mid x \in \mathcal{P}_\kappa\lambda\rangle]_{\mathcal{U}}$.

Proof. $j''\alpha = j''\lambda \cap j(\alpha)$ and $\overline{j''\alpha} = \alpha$. □

Proposition 3.2. *If $\mathcal{U}$ is normal over $\mathcal{P}_\kappa\lambda$, $M_{\mathcal{U}}$ is actually closed under $\lambda^{<\kappa}$-sequences.*

Proof. Let $j: V \to M_{\mathcal{U}} \approx V^{\mathcal{P}_\kappa\lambda}/\mathcal{U}$. By 1.7(i), it suffices to show that $j''(\mathcal{P}_\kappa\lambda) \in M$, since $\lambda^\kappa = |\mathcal{P}_\kappa\lambda|$. However, $j''(\mathcal{P}_\kappa\lambda) = \mathcal{P}_\kappa(j''\lambda) = (\mathcal{P}_\kappa(j''\lambda))^M \in M$. (Here, the first equality holds as $j(x) = j''x$ for any $x \in \mathcal{P}_\kappa\lambda$, and the second, as M is closed under λ-sequences and $j''\lambda \in M$.) □

We next present another structural result about normal ultrafilters, which has already appeared in Solovay [43]. First, a preliminary observation.

Proposition 3.3. *Let $\kappa \leq \lambda$ and G be a ω-Jónsson function over λ. If $\mathcal{U}$ is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$,*

$$\{x \mid G|^{\omega}x \text{ is } \omega\text{-Jónsson over } x\} \in \mathcal{U}.$$

Remarks. G exists by 1.1. Our definition of a function being ω-Jónsson is slightly stronger than the one used in [43], but there is little difference in the manipulations.

Proof. of 3.3. By Łoś' Theorem (1.4), it suffices to show $j(G) \mid {}^{\omega}j''\lambda$ is ω-Jónsson over $j''\lambda$. (Here, we need not distinguish between V and M, as M is closed under λ-sequences.) So, suppose $X \subseteq j''\lambda$ and $|X| = |j''\lambda| = \lambda$. If $Y = j^{-1}(X)$, since G is ω-Jónsson over λ, we have $G''{}^{\omega}Y = \lambda$. So, given any $\alpha < \lambda$, let $s \in {}^{\omega}Y$ such that $G(s) = \alpha$. Then $j(\alpha) = j(G(s)) = j(G)(j(s))$, and $j(s) \in {}^{\omega}X$. Thus, we have shown $j(G)''{}^{\omega}X = j''\lambda$, which was to be proved. □

The following is Theorem 2 of [43].

Theorem 3.4. *Suppose $\kappa \leq \lambda$ are regular cardinals and $\mathcal{U}$ is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$. If $F: \mathcal{P}_\kappa\lambda \to \lambda$ is defined by $F(x) = \sup(x)$, then there is an $X \in \mathcal{U}$ so that $F \mid X$ is one-to-one.*

Proof. Set $X=\{x\in\mathcal{P}_\kappa\lambda \mid \bar{x}$ is inaccessible, x is closed under ω-sequences, and $G\mid{}^\omega x$ is ω-Jónsson over $x\}$. By 3.1, 3.3, and further use of Łoś' Theorem (1.4), $X\in\mathcal{U}$. We show that X has the required property.

Assume that x, $y\in X$, and $\sup(x)=\gamma=\sup(y)$. As $\bar{x}$, $\bar{y}$ are inaccessible and x, y are both closed under ω-sequences, a simple argument shows that $x\cap y$ is cofinal in γ, and hence $|x|=|x\cap y|=|y|$. Thus, by the ω-Jónsson property, $x=G''^\omega(x\cap y)=y$. □

It is not hard to see that if $Y\in\mathcal{U}$, $\bigcup\{\mathcal{P}x \mid x\in Y\}=\mathcal{P}_\kappa\lambda$. Thus, by 3.4 we can conclude that if κ is supercompact and $\kappa\leq\lambda$ is regular, then $\lambda^{<\kappa}=\lambda$. This fact already follows from the strong compactness of κ (Solovay [43]) — see Section 4 for further remarks.

Recall now that the term "normal" is already known in another context: if κ is measurable, a non-principal κ-complete ultrafilter $\mathcal{U}$ over κ is normal iff in $M_\mathcal{U}$, $\kappa=[\mathrm{id}]$. This, of course, just means that whenever $f\in{}^\kappa\kappa$ is such that $\{\alpha<\kappa \mid f(\alpha)<\alpha\}\in\mathcal{U}$, there is a $\gamma<\kappa$ so that $\{\alpha \mid f(\alpha)=\gamma\}\in\mathcal{U}$. Normal ultrafilters turn out to be rather special, but one can always get them from elementary embeddings. In fact, it is easily shown that the ultrafilter of 1.2 is normal.

We now have a notion of normality in two senses, but in fact, there is a one-to-one correspondence between normal ultrafilters over $\mathcal{P}_\kappa\kappa$ and normal ultrafilters over κ. If $\mathcal{V}$ is normal over κ, $\mathcal{U}=\{X\subseteq\mathcal{P}_\kappa\kappa \mid X\cap\kappa\in\mathcal{V}\}$ is normal over $\mathcal{P}_\kappa\kappa$. Conversely, if $\mathcal{U}$ over $\mathcal{P}_\kappa\kappa$ is normal, $\kappa\in\mathcal{U}$. If not, then $\{x \mid x$ is not an ordinal$\}\in\mathcal{U}$. For such x, let $f(x)\in x$ be so that $f(x)$ is the least above some ordinal not in x. By normality, $[f]=j(\gamma)$ for some $\gamma<\kappa$, but this contradicts $\{x \mid \gamma\subseteq x\}\in\mathcal{U}$.

Determining the number of normal ultrafilters possible over a measurable cardinal has turned out ot be an interesting problem. Kunen [17] showed that in $L[\mathcal{U}]$, the universe constructed from a normal ultrafilter $\mathcal{U}$ over κ, $\mathcal{U}\cap L[\mathcal{U}]$ is the only normal ultrafilter over κ. Kunen and Paris [19] showed that if κ is measurable in the ground model, there is a forcing extension in which κ carries the maximal number of normal ultrafilters, i.e. 2^{2^κ}. Then Mitchell [33] more recently showed that if κ is 2^κ-supercompact and τ is $\leq\kappa$ or one of the terms κ^+ or κ^{++}, there is an inner model in which κ is measurable and carries exactly τ normal ultrafilters. It would still be desirable to get Mitchell's relative consistency results starting from just the measurability of κ.

In Mitchell's model with exactly two normal ultrafilters over κ, one contains the set $\{\alpha<\kappa \mid \alpha$ is measurable$\}$ and the other does not. In this regard, consider the following two propositions.

Proposition 3.5. *If κ is 2^κ-supercompact, there is a normal ultrafilter $\mathcal{U}$ over κ so that $\{\alpha<\kappa \mid \alpha$ is measurable$\}\in\mathcal{U}$. Hence, 2^κ-supercompactness is already enough to assure that κ is the* κth *measurable cardinal.*

Proof. Let $j:V \to M$ with critical point κ, so that M is closed under 2^κ-sequences. If $\mathcal{U}$ is defined by

$$X \in \mathcal{U} \quad \text{iff} \quad X \subseteq \kappa \ \& \ \kappa \in j(X),$$

$\mathcal{U}$ is normal over κ, as before. But since M is closed under 2^κ-sequences, it is not hard to see that every ultrafilter over κ is a member of M. Hence, κ is measurable in M, i.e. $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in \mathcal{U}$ by the definition of $\mathcal{U}$. □

Proposition 3.6. *Any measurable cardinal κ carries a normal ultrafilter $\mathcal{U}$ so that $\{\alpha < \kappa \mid \alpha$ is not measurable$\} \in \mathcal{U}$.*

Proof. By induction. Let $\mathcal{V}$ be a normal ultrafilter over κ. Set $T = \{\alpha < \kappa \mid \alpha$ is measurable$\}$. If $T \notin \mathcal{V}$, the theorem certainly holds for κ. So assume $T \in \mathcal{V}$. For each $\alpha \in T$, by induction hypothesis let $\mathcal{U}_\alpha$ be normal over α so that $\{\beta < \alpha \mid \beta$ is not measurable$\} \in \mathcal{U}_\alpha$. Define $\mathcal{U}$ over κ by

$$X \in \mathcal{U} \quad \text{iff} \quad \{\alpha < \kappa \mid X \cap \alpha \in \mathcal{U}_\alpha\} \in \mathcal{V}.$$

It is not hard to check that $\mathcal{U}$ is a normal ultrafilter over κ so that $\{\alpha < \kappa \mid \alpha$ is not measurable$\} \in \mathcal{U}$. □

These two propositions show that if κ is 2^κ-supercompact, there are at least two normal ultrafilters over κ. In fact, there are 2^{2^κ} normal ultrafilters over κ, and this result is a special case of a general result on the number of normal ultrafilters over $\mathcal{P}_\kappa\lambda$. To prove the main theorem (3.8) from which this will follow, we develop some technical machinery of independent interest.

Consider the following situation. $\kappa \leq \lambda < \mu$ and there is a normal ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa\mu$. For $X \subseteq \mathcal{P}_\kappa\mu$ if we let $X \mid \lambda = \{x \cap \lambda \mid x \in X\}$ and set

$$\mathcal{U} \mid \lambda = \{X \mid \lambda \mid X \in \mathcal{U}\},$$

it is not hard to see that $\mathcal{U} \mid \lambda$ is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$. In fact,

$$X \in \mathcal{U} \mid \lambda \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \ \& \ j_{\mathcal{U}}''\lambda \in j_{\mathcal{U}}(X).$$

Consider the following diagram:

$$\begin{tikzcd} V \arrow[r, "j_0"] \arrow[dr, "j_1"'] & M_0 \approx V^{\mathcal{P}_\kappa\lambda}/\mathcal{U}\mid\lambda \arrow[d, "k"] \\ & M_1 \approx V^{\mathcal{P}_\kappa\mu}/\mathcal{U} \end{tikzcd}$$

where k is defined by

$$k([\langle f(x) \mid x \in \mathcal{P}_\kappa\lambda\rangle]_{\mathcal{U}\mid\lambda}) = [\langle f(x \cap \lambda) \mid x \in \mathcal{P}_\kappa\mu\rangle]_{\mathcal{U}}.$$

Proposition 3.7. *k is an elementary embedding and the diagram commutes.*

Proof. Straightforward.

For those familiar with the Rudin–Keisler ordering on ultrafilters, note that $\mathcal{U} | \lambda$ is just $h_*(\mathcal{U})$, where $h: \mathcal{P}_\kappa\mu \to \mathcal{P}_\kappa\lambda$ is defined by $h(x) = x \cap \lambda$, and k is the associated elementary embedding.

Concerning the action of k, first note that if $\alpha \leq \lambda$, $k(\alpha) = \alpha$ since

$$k(\alpha) = k([\langle \overline{x \cap \alpha} \mid x \in \mathcal{P}_\kappa\lambda \rangle]) = [\langle \overline{x \cap \lambda \cap \alpha} \mid x \in \mathcal{P}_\kappa\mu \rangle] = \alpha.$$

Next, we show $k(x) = x$ for $x \in \mathcal{P}_\kappa\lambda$. Let $\gamma < \kappa$, and suppose $h: \gamma \to x$ is surjective. Then $h \in M_0$. Since $h(\gamma) = \gamma$, $k(h)$ is a function with domain γ. For $\xi < \gamma$, $k(h)(\xi) = k(h(\xi)) = h(\xi)$ by the preceding paragraph. So $k(h) = h$, whence $k(x) = \text{range}\,(k(h)) = \text{range}\,(h) = x$.

Next, since $k(\mathcal{P}_\kappa\lambda) = \mathcal{P}_\kappa\lambda^{M_1} = \mathcal{P}_\kappa\lambda$, an argument using elementarity and the preceding paragraph shows that $k(X) = X$ for $X \in \mathcal{P}\mathcal{P}_\kappa\lambda$.

Finally, if $X \in \mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_0$, then $k(X) = X$, since $\mathcal{P}\mathcal{P}_\kappa\lambda \subseteq M_0$ by 3.2. Using these results, a straightforward argument (Menas [32], 2.6 and preceding) shows that if $\mu \geq 2^{\lambda^{<\kappa}}$, the least ordinal moved by k is $(2^{\lambda^{<\kappa}})^{+M_0}$. However, this particular fact will not be used in this paper.

We are now is a position to prove the main result of this section

Theorem 3.8. *If κ is $2^{\lambda^{<\kappa}}$-supercompact, $X \in \mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda$ implies that there is a $\mathcal{V}$ normal over $\mathcal{P}_\kappa\lambda$ so that $X \in M_{\mathcal{V}}$.*

Proof. Let $\mathcal{U}$ be normal over $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ and consider the diagram:

$$\begin{tikzcd} V \arrow[r, "j_0"] \arrow[dr, "j_1"'] & M_0 \approx V^{\mathcal{P}_\kappa\lambda}/\mathcal{U}|\lambda \arrow[d, "k"] \\ & M_1 \approx V^{\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})}/\mathcal{U} \end{tikzcd}$$

Argue by contradiction. Let $\varphi(X, \kappa, \lambda)$ iff $X \in \mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda$ and for any $\mathcal{V}$ normal over $\mathcal{P}_\kappa\lambda$, $X \notin (\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_{\mathcal{V}})$. Suppose $\exists X\, \varphi(X, \kappa, \lambda)$. Then $M_1 \vDash \exists X\, \varphi(X, \kappa, \lambda)$. This is so since every normal $\mathcal{V}$ over $\mathcal{P}_\kappa\lambda$ is in M_1, and $\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_{\mathcal{V}}$ is correctly "computed" in M_1: every function $\mathcal{P}_\kappa\lambda \to V_\kappa$ is in M_1 and $\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_{\mathcal{V}} \subseteq (V_{\lambda+4})^{M_{\mathcal{V}}} \subseteq (V_{j_{\mathcal{V}}(\kappa)})^{M_{\mathcal{V}}} = j_{\mathcal{V}}(V_\kappa)$.

Recall now the properties of k discussed just before the theorem. Since $k(\kappa) = \kappa$ and $k(\lambda) = \lambda$, $M_0 \vDash \exists X\, \varphi(X, \kappa, \lambda)$. Let $X_0 \in M_0$ so that $M_0 \vDash \varphi(X_0, \kappa, \lambda)$. $X_0 \in \mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda$ so $k(X_0) = X_0$. Hence, $M_1 \vDash \varphi(X_0, \kappa, \lambda)$. But this contradicts $X_0 \in M_0 = M_{\mathcal{U}|\lambda}$ and the definition of φ. $\square$

The following improves the result of Magidor [25].

Corollary 3.9. *If κ is $2^{\lambda^{<\kappa}}$-supercompact, there are $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters over $\mathcal{P}_\kappa\lambda$. Hence, if κ is 2^κ-supercompact, there are 2^{2^κ} normal ultrafilters over κ.*

Proof. If $\mathcal{V}$ is normal over $\mathcal{P}_\kappa\lambda$, note first that $|\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_\mathcal{V}| < (2^{\lambda^{<\kappa}})^+$. This is so because

$$(2^{2^{\lambda^{<\kappa}}})^{M_\mathcal{V}} < j_\mathcal{V}(\kappa) < (2^{\lambda^{<\kappa}})^+,$$

the first inequality because $j_\mathcal{V}(\kappa)$ is inaccessible in $M_\mathcal{V}$ and the second, a trival upper bound (see the analogous 1.8 (iii)). Hence, since by 3.8,

$$\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda = \bigcup \{\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda \cap M_\mathcal{V} \mid \mathcal{V} \quad \text{normal over } \mathcal{P}_\kappa\lambda\},$$

there must be

$$|\mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda| = 2^{2^{\lambda^{<\kappa}}}$$

normal ultrafilters over $\mathcal{P}_\kappa\lambda$. □

By examining the proofs of 3.8 and 3.9, one can check that if κ is 2^κ-supercompact there are 2^{2^κ} normal ultrafilters over κ containing the set $\{\alpha < \kappa \mid \alpha$ is measurable$\}$. But paradoxically, the following question is still open.

Open Question 3.10. If κ is 2^κ-supercompact, is it provable that there is more than one normal ultrafilter over κ containing the set $\{\alpha < \kappa \mid \alpha$ is not measurable$\}$?

We can also prove something about the following ordering defined on ultrafilters.

Definition 3.11. If $\mathcal{U}$ and $\mathcal{V}$ are ω_1-complete ultrafilters, $\mathcal{U} \lhd \mathcal{V}$ iff $\mathcal{U} \in M_\mathcal{V}$.

$\lhd$ on normal ultrafilters over fixed $\mathcal{P}_\kappa\lambda$ is a well-founded partial ordering (by a generalization of 1.1 of Mitchell [33]). Notice that if $\mathcal{V}$ is normal over $\mathcal{P}_\kappa\lambda$, there are at most $2^{\lambda^{<\kappa}}$ normal ultrafilters over $\mathcal{P}_\kappa\lambda$ which are $\lhd$ predecessors of $\mathcal{V}$, by the first fact used in the proof of 3.9.

Corollary 3.12 *If κ is $2^{\lambda^{<\kappa}}$-supercompact, there is a $\lhd$ chain of normal ultrafilters over $\mathcal{P}_\kappa\lambda$ of length $(2^{\lambda^{<\kappa}})^+$. Hence, if κ is 2^κ-supercompact there is a $\lhd$ chain of normal ultrafilters over κ of length $(2^\kappa)^+$.*

Proof. Given at most $2^{\lambda^{<\kappa}}$ normal ultrafilters, one can code them as some $X \in \mathcal{P}\mathcal{P}\mathcal{P}_\kappa\lambda$. Thus, 3.8 can be used to find some normal $\mathcal{V}$ over $\mathcal{P}_\kappa\lambda$ so that $X \in M_\mathcal{V}$. □

For an application of the order $\lhd$, see Magidor [22]. He shows that if $\mu < \kappa$ are regular, a $\lhd$ chain of order type μ of normal ultrafilters over κ can be used to define a partial ordering such that forcing with it preserves all cardinals but changes the cofinality of κ to μ. This, of course, complements Prikry forcing.

To conclude this section, we mention that, analogous to Rowbottom's partition result for normal ultrafilters over a measurable cardinal, there is the following partition property: say that a normal ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$ has the *partition property* iff whenever $f:[\mathcal{P}_\kappa\lambda]^2\to 2$, there is an $X\in\mathcal{U}$ and an $i<2$ so that $x, y\in X$, $x\subseteq y$, and $x\neq y$ imply that $f(\{x, y\})=i$. Menas established a characterization of this property and showed that 3.8 holds with the additional requirement that the normal ultrafilters over $\mathcal{P}_\kappa\lambda$ each satisfy the partition property. However, Solovay has shown that if $\kappa<\lambda$, κ is λ-supercompact and λ is measurable, then there is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$ without the partition property. Also, Kunen proved that the least $\mu>\kappa$ so that there is a normal ultrafilter over $\mathcal{P}_\kappa\mu$ without the partition property is Π_1^2-indescribable and strictly less than the least ineffable cardinal greater than κ. See Menas [31] for the details.

4. Strong compactness

The concept of strong compactness is discussed in Keisler–Tarski [12] and historically was motivated by efforts to generalize the Compactness Theorem of lower predicate calculus to infinitary languages $\mathcal{L}_{\kappa\kappa}$. Supercompactness was conceived partly in order to ostensibly strengthen the definition of strong compactness in a desirable manner, but Solovay conjectured that the two concepts coincide. Though this conjecture stood for some time, it is now known to be false (see 4.4 and after).

In view of our thematic approach, in this section we consider strong compactness as formulated in terms of elementary embeddings and ultrafilters. The connection with $\mathcal{L}_{\kappa\kappa}$ and other equivalent formulations are given in a variety of sources.

Definition 4.1. If $\kappa\leqslant\lambda$ and $\mathcal{U}$ is an ultrafilter over $\mathcal{P}_\kappa\lambda$, then $\mathcal{U}$ is *fine* iff $\mathcal{U}$ is κ-complete and for each $\alpha<\lambda$, $\{x \mid \alpha\in x\}\in\mathcal{U}$.

Thus, the definition leaves out clause (iii) of the definition of normality (2.3).

Definition 4.2. If $\kappa\leqslant\lambda$, κ is *λ-compact* iff there is a fine ultrafilter over $\mathcal{P}_\kappa\lambda$. κ is *strongly compact* iff κ is λ-compact for all $\lambda\geqslant\kappa$.

Trivially, if κ is λ-supercompact, κ is λ-compact, and κ is measurable iff κ is κ-compact. If $\kappa\leqslant\lambda<\mu$, κ is μ-compact, and $\mathcal{U}$ is a fine ultrafilter over $\mathcal{P}_\kappa\mu$, then $\mathcal{U}\mid\lambda$ is a fine ultrafilter over $\mathcal{P}_\kappa\lambda$, so that $\mathcal{U}$ is λ-compact. The reader is cautioned that there is a different, equally natural notion of λ-compactness often seen in the literature.

We proceed immediately to the characterization

Theorem 4.3. *If $\kappa \leq \lambda$, the following are equivalent:*

(i) *κ is λ-compact,*

(ii) *there is a $j: V \to M$ with critical point κ so that: $X \subseteq M$ and $|X| \leq \lambda$ implies there is a $Y \in M$ so that $X \subseteq Y$, and $M \models |Y| < j(\kappa)$.*

(iii) *if $\mathcal{F}$ is any κ-complete filter over an index set I so that $\mathcal{F}$ is generated by $\leq \lambda$ sets, then $\mathcal{F}$ can be extended to a κ-complete ultrafilter over I.*

Proof. (i)$\to$(ii). Let $\mathcal{U}$ be fine over $\mathcal{P}_\kappa\lambda$, and consider $j: V \to M \approx V^{\mathcal{P}_\kappa\lambda}/\mathcal{U}$. If $X = \{[f_\alpha] \mid \alpha < \lambda\} \subseteq M$. set $G(x) = \{f_\alpha(x) \mid \alpha \in x\}$ amd $Y = [G]$.

(ii)$\to$(iii). Suppose $\mathcal{F}$ is as hypothesized, and generated by elements of $T \subseteq \mathcal{P}(I)$, where $|T| \leq \lambda$. By (ii) let $Y \supseteq j''T$ so that $Y \in M$ and $M \models |Y| < j(\kappa)$. In M, $j(\mathcal{F})$ is a $j(\kappa)$-complete filter and $j(\mathcal{F}) \cap Y$ is a subset of cardinality $< j(\kappa)$. Hence, there is a $c \in M$ so that $c \in \bigcap (j(\mathcal{F}) \cap Y)$. Set $X \in \mathcal{U}$ iff $X \subseteq I$ & $c \in j(X)$. Then $\mathcal{U}$ is a κ-complete ultrafilter which extends $\mathcal{F}$.

(iii)$\to$(i). Extend the κ-complete filter over $\mathcal{P}_\kappa\lambda$ generated by the sets $\{x \mid \alpha \in x\}$ for $\alpha < \lambda$ to a κ-complete ultrafilter. □

4.3 (ii) thus shows the weakness of λ-compactness; with λ-supercompactness one can always assert that $X = Y$. We mention here that Ketonen [13] has another characterization: if $\kappa \leq \lambda$ are regular, κ is λ-compact iff for every regular μ so that $\kappa \leq \mu \leq \lambda$ there is a uniform κ-complete ultrafilter over μ. For a further discussion of fine ultrafilters, normal ultrafilters, and connections involving the Rudin–Keisler ordering, consult Sections 2.1–2.3 of Menas [32].

The following result indicates that strong compactness and supercompactness are not the same concept.

Proposition 4.4. (Menas [32] (i) *If κ is measurable and a limit of strongly compact cardinals, then κ is strongly compact.*

(ii) *If κ is the least cardinal as in* (i), *then κ is not 2^κ-supercompact.*

Proof. For (i), let $\mathcal{U}$ be a non-principal κ-complete ultrafilter over κ so that $A = \{\alpha < \kappa \mid \alpha$ is strongly compact$\} \in \mathcal{U}$. If $\lambda \geq \kappa$, for $\alpha \in A$ let $\mathcal{U}_\alpha$ be fine over $\mathcal{P}_\alpha\lambda$. Define $\mathcal{V}_\lambda$ by

$$x \in \mathcal{V}_\lambda \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \;\&\; \{\alpha \mid X \cap \mathcal{P}_\alpha\lambda \in \mathcal{U}_\alpha\} \in \mathcal{U}.$$

Then $\mathcal{V}_\lambda$ is fine over $\mathcal{P}_\kappa\lambda$.

For (ii), argue by contradiction, and suppose κ were 2^κ-supercompact. Let $j: V \to M$ with critical point κ so that M is closed under 2^κ-sequences. By definition of κ and elementarity, we have in M that $j(\kappa)$ is the *least* measurable cardinal which is a limit of strongly compact cardinals. But M is closed under 2^κ-sequences so that κ is measurable in M, and also, if $\alpha < \kappa$ is strongly compact, $j(\alpha) = \alpha$ is strongly compact in the sense of M. Hence, in M, κ is also measurable and a limit of strongly compact cardinals, which is a contradiction. □

It is a consequence of the existence of an extendible cardinal (by an easy strengthening of 5.9 of the next section) that there are many cardinals as in 4.4 (i). Menas was able to establish the following result.

(a) (Menas [32]) Con (ZFC & there is a measurable cardinal that is the limit of strongly compact cardinals) implies Con (ZFC & there is exactly one strongly compact cardinal κ, and κ is not even κ^+-supercompact).

The following are results of Magidor.

(b) (Magidor [24]) Con (ZFC & there is a supercompact cardinal) implies Con (ZFC & there is a supercompact cardinal with no strongly compact cardinals below it).

(c) (Magidor [24]) Con (ZFC & there is a strongly compact cardinal) implies Con (ZFC & there is a strongly compact cardinal with no measurable cardinals below it).

Though Kunen [17] has shown that the existence of a strongly compact cardinal implies the existence of inner models with any specified number of measurable cardinals, the results (a), (b) and (c) indicate that strong compactness is a rather pathological concept in the hierarchy of large cardinals. Perhaps it should ultimately be regarded as a generalization of weak compactness in the same spirit that supercompactness is a generalization of measurability.

In connection with these considerations, recall that in the previous section we showed that if κ is supercompact there are many normal ultrafilters over κ. However, the following is still open.

Open Question 4.5. If κ is strongly compact, can it be proved that there is more than one normal ultrafilter over κ?

4.6. Concerning the effect of the existence of a strongly compact cardinal κ on the behavior of the set theoretical universe, Solovay [43] has proved that if $\lambda \geq \kappa$ and λ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$. As noted by Paris, Prikry and probably others, this result now follows from the easier result of [43] that if $\lambda \geq \lambda$ and λ is regular, then $\lambda^{<\kappa} = \lambda$, and Silver's recent solution to most cases of the singular cardinals problem:

Assume $\lambda \geq \kappa$ and λ is a singular strong limit cardinal. If $\mathrm{cf}(\lambda) < \kappa$, it is immediate that $2^\lambda = \lambda^{\mathrm{cf}(\lambda)} \leq (\lambda^+)^{\mathrm{cf}(\lambda)} = \lambda^+$. But if $\mathrm{cf}(\lambda) \geq \kappa$, $S = \{\alpha < \lambda \mid \alpha$ is a singular strong limit cardinal of cofinality $< \kappa\}$ is a stationary subset of λ and $\alpha \in S$ implies $2^\alpha = \alpha^+$ by the previous sentence. Silver's result states that if μ is a singular cardinal of uncountable cofinality so that those $\alpha < \mu$ with $2^\alpha = \alpha^+$ forms a stationary subset of μ, then $2^\mu = \mu^+$. Thus, we can conclude $2^\lambda = \lambda^+$.

However, we note that the further results in [43] on powers of cardinals cannot ostensibly be simplified in this way.

It is also proved in [43] that if $\kappa \leq \lambda$ and κ is λ^+-supercompact, then Jensen's combinatorial principle $\Box_\lambda$ fails (see Jensen [8] for the result that if $V = L$, $\Box_\mu$ holds for every infinite cardinal μ). Since then, Gregory proved that the failure of

$\square_\lambda$ already follows from the λ^+-compactness of κ. His proof used the notion of a λ-free Abelian group, but a direct combinatorial proof is possible. Following a comment of Kunen, we present the ideas involved in general context.

For regular $\lambda > \omega$, let us call the following principle E_λ: there is a set $S \subseteq \{\alpha < \mid \mathrm{cf}(\alpha) = \omega\}$ stationary in λ so that for all limit ordinals $\xi < \lambda$, $S \cap \xi$ is not stationary in ξ. Jensen [8] proved that if $V = L$, then E_λ fails just in case λ is a weekly compact cardinal. The following proposition is well-known.

Proposition 4.7. *Suppose $\lambda > \omega$ and $\square_\lambda$ holds. Then E_{λ^+} holds, and in fact, whenever T is a stationary subset of λ^+, there is a stationary $S \subseteq T$ so that for any limit ordinal $\xi < \lambda^+$, $S \cap \xi$ is not stationary ξ.*

Proof. Let us first recall the principle $\square_\lambda$: there is a sequence $\langle C_\xi \mid \xi < \lambda^+ \rangle$ so that for any $\xi < \lambda^+$, we have

(i) $C_\xi \subseteq \xi$, and if ξ in a limit ordinal, C_ξ is closed and unbounded in ξ,

(ii) The order type of C_ξ is $\leq \lambda$,

(iii) If γ is a limit point of C_ξ, then $C_\gamma = C_\xi \cap \gamma$.

Suppose now that T is a stationary subset of λ^+. Without loss of generality, we can assume that T consists of limit ordinals. For $\alpha \leq \lambda$, set $S_\alpha = \{\xi \in T \mid C_\xi$ has order type $\alpha\}$. Then, as the S_α's partition T into λ parts by property (ii), there is some $\alpha_0 \leq \lambda$ so that $S = S_{\alpha_0}$ is still stationary in λ^+. We now claim that for any limit ordinal $\xi < \lambda^+$, $S \cap \xi$ is not stationary in ξ. There are three cases.

(a) $\mathrm{cf}(\xi) = \omega$. Then as S consists of limit ordinals, S is disjoint from any sequence of type ω of successor ordinals, cofinal in ξ.

(b) $\mathrm{cf}(\xi) > \omega$ and C_ξ has order type $\leq \alpha_0$. Let $\bar{C}_\xi$ consist of the limit points of C_ξ. Then as $\mathrm{cf}(\xi) > \omega$, $\bar{C}_\xi$ is closed and unbounded in ξ, and $(S \cap \xi) \cap \bar{C}_\xi = \emptyset$ by property (iii).

(c) $\mathrm{cf}(\xi) > \omega$ and C_ξ has order type $> \alpha_0$. Let $\gamma \in C_\xi$ so that $C_\xi \cap \gamma$ is of type α_0. Then if $\bar{C}_\xi$ is defined as in (b), we have by property (iii) that $(S \cap \xi) \cap (\bar{C}_\xi - (\gamma + 1)) = \emptyset$

Thus, the claim is proved, and the proposition follows. $\square$

The following theorem establishes the connection to large cardinals.

Theorem 4.8. *If λ is regular and there is a uniform, ω_1-complete ultrafilter over λ, then E_λ fails.*

Proof. Let $\mathcal{U}$ be uniform, ω_1-complete over λ. Let $f: \lambda \to \lambda$ so that $\sup\{j_\mu(\alpha) \mid \alpha < \lambda\} = [f]_{\mathcal{U}}$. Such an f exists since the supremum in question is $\leq [\mathrm{id}]$. Set

$$\mathcal{V} = \{X \subseteq \lambda \mid f^{-1}(X) \in \mathcal{U}\}.$$

It is not hard to see that $\mathscr{V}$ is a uniform, ω_1-complete ultrafilter over λ, which is *weakly normal* in the following sense: if $g:\lambda\to\lambda$ so that $\{\alpha \mid g(\alpha)<\alpha\}\in\mathscr{V}$, then for some $\delta<\lambda$, $\{\alpha \mid g(\alpha)<\delta\}\in\mathscr{V}$.

Let us first prove a preliminary

Fact. $\{\alpha \mid \mathrm{cf}(\alpha)>\omega\}\in\mathscr{V}$.

Otherwise, for almost every α, let $\langle\alpha_n \mid n\in\omega\rangle$ be cofinal in α. By the weak normality of $\mathscr{V}$, for every $n\in\omega$ there is a $\delta_n<\lambda$ so that $X_n=\{\alpha \mid \alpha_n<\delta_n\}\in\mathscr{V}$. Then $Y=\bigcap X_n\in\mathscr{V}$, but if $\delta=\sup\delta_n<\lambda$, notice that $\alpha\in Y$ implies $\alpha\leqslant\delta$, contradicting the fact that $\mathscr{V}$ is uniform. The Fact is thus proved.

We now proceed with the main proof. Let S be a stationary subset of λ so that $S\subseteq\{\alpha \mid \mathrm{cf}(\alpha)=\omega\}$. To show that E_λ fails, we establish in fact that

(*)
$$\{\xi \mid S\cap\xi \text{ is stationary in } \xi\}\in\mathscr{V}.$$

Let us suppose to the contrary, and derive a contradiction. Then for almost every ξ, there is a closed and unbounded $C_\xi\subseteq\xi$ so that $S\cap C_\xi=\emptyset$. Define

$$C=\{\alpha \mid \{\xi \mid \alpha\in C_\xi\}\in\mathscr{V}\}.$$

Because $\mathscr{V}$ is ω_1-complete, and as almost every ξ has cofinality $>\omega$ by the Fact, C is ω-closed: given $\alpha_n\in C$ for $n\in\omega$, $\sup\alpha_n\in C$. Also, we claim that C is an unbounded subset of λ. To show this, let $\beta_0<\lambda$ be arbitrary. Using the weak normality of $\mathscr{V}$, for each integer n we can define functions $f_n:\lambda\to\lambda$ and ordinals β_n by induction so that

(a) $A_n=\{\xi \mid f_n(\xi)\in C_\xi \;\&\; f_n(\xi)>\beta_n\}\in\mathscr{V}$,

and

(b) $B_n=\{\xi \mid f_n(\xi)<\beta_{n+1}\}\in\mathscr{V}$.

Since $\mathscr{V}$ is ω_1-complete, $X=\bigcap(A_n\cap B_n)\in\mathscr{V}$. But then if $\beta=\sup\beta_n$, it is not hard to show using the set X (and the Fact proved earlier) that for almost every ξ, $\beta\in C_\xi$.

We have just established that C is ω-closed and unbounded in λ. Since S is a stationary subset of $\{\alpha<\lambda \mid \mathrm{cf}(\alpha)=\omega\}$, we must have $C\cap S\neq\emptyset$. Having arrived at this contradiction, we have thus established (*) and the theorem. $\square$

The following is now immediate from 4.7 and 4.8.

Corollary 4.9. (Gregory) *If κ is λ^+-compact, then $\square_\lambda$ fails.*

We remark that the main idea in the proof of 4.8 is due to Jensen, Prikry and Silver, and is stated (somewhat obscurely) in Theorem 20 of Prikry [39]. Let E_λ^κ

for regular $\kappa<\lambda$ be like E_λ but with the required stationary $S\subseteq\{\alpha \mid \mathrm{cf}(\alpha)=\kappa\}$. The method of proof actually shows (see [39]) that if there is a uniform ultrafilter $\mathcal{U}$ over λ which is κ-descendingly complete, then E_λ^κ fails. Prikry had to assume some form of the GCH to get a *weakly normal* ultrafilter related to such a $\mathcal{U}$, but it is now known that no such hypothesis is needed (see Theorem 2.5 of [10]).

The referee has outlined a proof of 4.8 (which similarly generalizes for E_λ^κ) that does not use weak normality. Suppose that $S\subseteq\lambda$ witnesses E_λ. For $\alpha\in S$, let $\langle\gamma_\alpha^n \mid n\in\omega\rangle$ be cofinal in α. Call a function $f:S\to\omega$ a *disjointer* iff whenever $\alpha\neq\beta\in S$, $m>f(\alpha)$ and $n>f(\beta)$ implies $\gamma_\alpha^m\neq\gamma_\beta^n$. By the regressive function lemma, there is no disjointer for S. However, one can show by induction on $\xi<\lambda$ that there are disjointers f_ξ for $S\cap\xi$, using the fact that $S\cap\xi$ is not stationary in ξ. If $\mathcal{U}$ were uniform, ω_1-complete over λ, one can obtain a disjointer for S (and hence a contradiction) by taking the ultraproduct of the f_ξ.

5. Extendible cardinals

We now consider an axiom which implies the existence of many supercompact cardinals. The notion of *extendibility* is motivated in Reinhardt [40] by considerations involving strong principles of reflection and resemblance formalized in an extended theory which allows transfinite levels of higher type objects over the set theoretical universe V. Essentially, Cantor's Ω, the class of all ordinals, is hypothesized to be extendible in this context. With the natural reflection down into the realm of sets, we have the concept of an extendible cardinal. (As Reinhardt [40] points out, however, this sort of internal formalization within V rather begs the question if we want to discuss fundamental issues about the nature of V and Ω.)

More simplistically, recall that Kunen's Theorem (1.12) showed that one cannot embed V into too fat an inner model. As a natural weakening one can instead consider embedding initial segments of the universe into larger initial segments, $j:\mathrm{V}_\alpha\to\mathrm{V}_\beta$ where $\alpha\leq\beta$. (As before, implicit in this notation is the assertion that j is not the identity.) This approach may be conceptually helpful, but the exact form of the following definition owes its origin to the considerations of [40].

Definition 5.1. If $\eta>0$, κ is *η-extendible* iff there is a ζ and a $j:\mathrm{V}_{\kappa+\eta}\to\mathrm{V}_\zeta$ with critical point κ, where $\kappa+\eta<j(\kappa)<\zeta$. κ is *extendible* iff κ is η-extendible for every $\eta>0$.

Remarks. (i) Since $\eta\geq\kappa\cdot\kappa$ implies $\kappa+\eta=\eta$, the exact form of the above definition is distinctive only for small η.

(ii) When $\eta<\kappa$, it is not hard to see that $\zeta=j(\kappa)+\eta$. This fact will be used without further reference. Note that in such cases and especially for η an integer, it is clear that η-extendibility is just a postulate of resemblance: With $j: V_{\kappa+\eta} \to V_{j(\kappa)+\eta}$, V_κ and $V_{j(\kappa)}$ are indistinguishable as far as $(\eta+1)$th order properties are concerned.

(iii) If κ is η-extendible and $0<\delta<\eta$, then κ is δ-extendible. Since the term V_α is definable from α, if ζ and j are as in 5.1 for the η-extendibility of κ, then $j \mid V_{\kappa+\delta}: V_{\kappa+\delta} \to V_{j(\kappa+\delta)}$ is also an elementary embedding, as

$$\begin{aligned} V_{\kappa+\delta} \Vdash \varphi(x) &\leftrightarrow V_{\kappa+\eta} \Vdash (V_{\kappa+\delta} \Vdash \varphi(x)) \\ &\leftrightarrow V_\zeta \Vdash (V_{j(\kappa+\delta)} \Vdash \varphi(j(x))) \\ &\leftrightarrow V_{j(\kappa+\delta)} \Vdash \varphi(j(x)). \end{aligned}$$

That restrictions of elementary embeddings in this way are also elementary will be assumed in what follows.

(iv) For the related concept of *complete* η-extendibility and some results concerning it, see Gaifman [4].

(v) The requirement $j(k)>\kappa+\eta$ in 5.1 can be regarded as a natural one, reminiscent of the definition 2.1 of λ-supercompactness. It is a useful, but not stringent, condition; in fact, if it were ever the case that $j(\kappa)\leq\kappa+\eta$, we would have a much stronger axiom ($A_2(\kappa)$ of Section 8). In this connection, we remark that the definition of extendibility given in [40] contains an equivocation (pointed out by Wang) between statements E_0 and E in 6.2c, which leaves Axiom 6.3 unclear. We take this opportunity to resolve this ambiguity: what was intended was E, rather than E_0 (this corresponds to including the condition $j(\kappa)>\kappa+\eta$ in our Definition 5.1). This should not affect the ensuing discussion in [40], since it is argued there that the additional condition is natural, though not forced by the guiding idea. (In 6.3 of [40], it is suggested that the critical step in arriving at Kunen's contradiction is the treatment of $V_{\kappa+\eta}$ as a universe in itself, which moreover is absolute in a strong sense (in our formulation, this amounts to setting $\kappa+\eta=\zeta$), rather than anything in the guiding idea behind extendibility as expressed in E or E_0.) The following result shows that (full) extendibility as a concept is not affected, in any case.

Proposition 5.2. *κ is extendible iff for every $\delta>\kappa$, there is a ζ and a $j: V_\delta \to V_\zeta$ with critical point κ.*

Proof. The forward direction is immediate. For the converse, it suffices to show from the hypothesis that if $\eta>\kappa\cdot\kappa$, then κ is η-extendible.

Given such an η, first get an auxiliary ordinal $\gamma>\eta$ so that

(a) $\mathrm{cf}(\gamma)=\omega_1$,

(b) whenever there is a $k: V_\eta \to V_\zeta$ with critical point κ so that $k(\kappa)<\gamma$, there is such a k (with the same value for $k(\kappa)$) for some ζ so that $\zeta<\gamma$.

We can use a reflection argument to get (b), and (a) can easily be arranged.

By hypothesis, let $j: \mathrm{V}_\gamma \to \mathrm{V}_\rho$ with critical point κ. Set $\kappa_0 = \kappa$, and for $n \in \omega$, $\kappa_{n+1} = j(\kappa_n)$ whenever possible (i.e. whenever $\kappa_n < \gamma$). If κ_n is defined for each integer n, then $\sup\{\kappa_n \mid n \in \omega\} < \gamma$ since $\mathrm{cf}(\gamma) = \omega_1$. But then, we can now get a contradiction using Kunen's argument (1.12). Thus, it follows that there is an n so that $\kappa_n < \gamma \leq \kappa_{n+1}$.

To conclude the proof, it suffices to establish $P(m)$ for every $m < n+1$ by induction on m, where $P(m)$ states: there is a ζ and an $i: \mathrm{V}_\eta \to \mathrm{V}_\zeta$ with critical point κ so that $i(\kappa) = \kappa_{m+1}$. This is so, since η-extendibility would follow from $P(n)$ and [illegible] $\gamma \leq \kappa_{n+1}$.

Define $\bar{j} = j \mid \mathrm{V}_\eta$. Then $\bar{j}: \mathrm{V}_\eta \to \mathrm{V}_{j(\eta)}$ with critical point κ so that $\bar{j}(\kappa) = \kappa_1$. Hence, we have $P(0)$. Now assume $P(m)$, where $m < n$. Then, because $\kappa_{m+1} < \gamma$, by the property (b) of γ, there is an $i: \mathrm{V}_\eta \to \mathrm{V}_\zeta$ for some $\zeta < \gamma$, with critical point κ so that $i(\kappa) = \kappa_{m+1}$. Thus, by the elementarity of j, we have that in V_ρ (and hence in V), there is an $\bar{i}: \mathrm{V}_{j(\eta)} \to \mathrm{V}_{j(\zeta)}$ with critical point $j(\kappa)$ so that $\bar{i}(j(\kappa)) = j(\kappa_{m+1}) = \kappa_{m+2}$. Recalling the definition of $\bar{j}$ above, we can now conclude that $\bar{i} \cdot \bar{j}: \mathrm{V}_\eta \to \mathrm{V}_{j(\zeta)}$ with critical point κ so that $\bar{i} \cdot \bar{j}(\kappa) = \kappa_{m+2}$. Hence, $P(m+1)$ holds, and the proof is complete. □

If κ is supercompact, it is consistent that there is no strongly inaccessible cardinal $>\kappa$, since if there were one, we can cut off the universe at the least one and still have a model of set theory in which κ is supercompact. However, suppose that κ is even 1-extendible, with $j: \mathrm{V}_{\kappa+1} \to \mathrm{V}_{j(\kappa)+1}$. Then by elementarity $j(\kappa)$ is inaccessible in $\mathrm{V}_{j(\kappa)+1}$, and hence in V. Similarly, if κ is 2-extendible with $j: \mathrm{V}_{\kappa+2} \to \mathrm{V}_{j(\kappa)+2}$, by elementarity $j(\kappa)$ is measurable in $\mathrm{V}_{j(\kappa)+2}$, and hence in V. Thus, the extendibility of a cardinal κ implies the existence of large cardinals $>\kappa$.

These considerations begin to show how strongly the existence of an extendible cardinal affects the higher levels of the cumulative hierarchy, and why λ-extendibility cannot be formulated, as λ-supercompactness can, merely in terms of the existence of certain ultrafilters. They also point to the close relationship between extendibility and principles of reflection and resemblance. See the end of this section for an elaboration in terms of the Lévy hierarchy of formulas.

We now proceed to investigate extendibility, particularly in connection with supercompactness. Ultimately, we will establish that any extendible cardinal κ is supercompact and is the κth supercompact cardinal, and that the least supercompact cardinal is not even 1-extendible.

Note first that 1-extendibility is already quite strong.

Proposition 5.3. *If κ is 1-extendible, then κ is measurable and there is a normal ultrafilter $\mathcal{U}$ over κ so that $\{\alpha < \kappa \mid \alpha$ is measurable$\} \in \mathcal{U}$.*

Proof. Let $j: \mathrm{V}_{\kappa+1} \to \mathrm{V}_{j(\kappa)+1}$ with critical point κ. Then $\mathcal{U}$ defined by $X \in \mathcal{U}$ iff $X \subseteq \kappa$ & $\kappa \in j(X)$ is normal over κ. Certainly $\mathcal{U} \in \mathrm{V}_{j(\kappa)+1}$, so $\mathrm{V}_{j(\kappa)+1} \vDash \kappa$ is measurable, i.e. $\{\alpha < \kappa \mid \alpha$ is measurable$\} \in \mathcal{U}$. □

We now proceed to establish some connections between degrees of supercompactness and degrees of extendibility.

Proposition 5.4. *If κ is $|V_{\kappa+\eta}|$-supercompact and $\eta<\kappa$, there is a normal ultrafilter $\mathcal{U}$ over κ so that $\{\alpha<\kappa \mid \alpha$ is η-extendible$\}\in\mathcal{U}$.*

Proof. Let $j: V\to M$ be as in $|V_{\kappa+\eta}|$-supercompactness. Then $V_{\kappa+\eta}\in M$, since $V_{\kappa+\eta}$ is hereditarily of cardinal $\leq|V_{\kappa+\eta}|$. Similarly, if we set $e=j \restriction V_{\kappa+\eta}$, $e\in M$. Now in M, $e: V_{\kappa+\eta}\to j(V_{\kappa+\eta})$ is an elementary embedding with critical point κ and $e(\kappa)>\kappa+\eta$, since this is true in V. Thus, κ is η-extendible in M. If we let $\mathcal{U}$ be the usual normal ultrafilter over κ corresponding to j, then it follows that $\{\alpha<\kappa \mid \alpha$ is η-extendible$\}\in\mathcal{U}$. □

Thus, supercompactness implies the existence of many cardinals with some degree of extendibility. One conjecture which expresses confidence in even 2-extendibility is the following.

Open Question 5.5. Does Con (ZFC & there is a 2-extendible cardinal) imply Con (ZFC & there is a strongly compact cardinal)?

Of course, by 5.4 an affirmative answer to this question would imply that the consistency strength of strong compactness is very weak compared to that of supercompactness.

The following proposition reverses the process of 5.4.

Proposition 5.6. *If κ is η*-extendible and $\delta+1<\eta$, *then κ is $|V_{\kappa+\delta}|$-supercompact. Hence, if κ is extendible, then it is supercompact.*

Proof. Suppose $j: V_{\kappa+\eta}\to V_{\zeta}$ is as in η-extendibility. Since $j(\kappa)$ is really inaccessible and $j(\kappa)>\kappa+\delta$, $j(\kappa)>|V_{\kappa+\delta}|$. Hence, since $\delta+1<\eta$ so that $\mathcal{P}\mathcal{P}V_{\kappa+\delta}\subseteq V_{\kappa+\eta}$, we can define a normal ultrafilter over $\mathcal{P}V_{\kappa+\delta}$ as usual:

$$X\in\mathcal{U} \quad\text{iff}\quad j''V_{\kappa+\delta}\in j(X). \quad \square$$

Incidentally, the methods of 5.4 and 5.6 yield another characterization of supercompactness, which was also noticed by Magidor [26].

Theorem 5.7. *κ is supercompact iff for every $\eta>\kappa$ there is an $\alpha<\kappa$ and a $j: V_\alpha\to V_\eta$ with critical point γ so that $j(\gamma)=\kappa$.*

Proof. For the forward direction, fix $\eta>\kappa$ and let $j: V\to M$ be as in the $|V_\eta|$-supercompactness of κ. Then just as in 5.4, $j \restriction V_\eta: V_\eta\to V_{j(\eta)}^M$ is an elementary embedding which is in M. Thus, M models the following: "there is an

$\alpha < j(\kappa)$ and an elementary embedding $e: \mathrm{V}_\alpha \to \mathrm{V}_{j(\eta)}$ with a critical point γ such that $e(\gamma) = j(\kappa)$." The result now follows from the elementarity of j.

For the converse, fix $\eta > \kappa$ and let $j: \mathrm{V}_{\alpha+\omega} \to \mathrm{V}_{\eta+\omega}$ for some $\alpha < \kappa$, with critical point γ so that $j(\gamma) = \kappa$. As in 5.6, since $\mathscr{P}\mathscr{P}_\gamma\alpha \subseteq \mathrm{V}_{\alpha+\omega}$, j determines a normal ultrafilter $\mathscr{U}$ over $\mathscr{P}_\gamma\alpha$. But $\mathscr{U} \in \mathrm{V}_{\alpha+\omega}$ and so $j(\mathscr{U})$ is a normal ultrafilter over $\mathscr{P}_{j(\gamma)}j(\alpha) = \mathscr{P}_\kappa\eta$. $\square$

The following proposition on supercompactness will be used in the next theorem, but is interesting in its own right.

Proposition 5.8. *If κ is α-supercompact for every $\alpha < \lambda$ and λ is supercompact, then κ is supercompact.*

Proof. Let $\mu \geq \lambda$. We must get a normal ultrafilter over $\mathscr{P}_\kappa\mu$. For each $x \in \mathscr{P}_\lambda\mu$, so that $|x| \geq \kappa$, let $\mathscr{U}_x$ be normal over $\mathscr{P}_\kappa x$. Such $\mathscr{U}_x$ exist since $|x| < \lambda$. Let $\mathscr{V}$ be normal over $\mathscr{P}_\lambda\mu$, and define $\mathscr{U}$ over $\mathscr{P}_\kappa\mu$ by

$$X \in \mathscr{U} \quad \text{iff} \quad \{x \in \mathscr{P}_\lambda\mu \mid \mathscr{P}_\kappa x \cap X \in \mathscr{U}_x\} \in \mathscr{V}.$$

Since $j_{\mathscr{U}_x}{}''x = [\mathrm{id}]_{\mathscr{U}_x}$, it is not hard to see that $\mathscr{U}$ is a normal ultrafilter over $\mathscr{P}_\kappa\mu$. $\square$

Theorem 5.9. *If κ is supercompact and 1-extendible, then there is a normal ultrafilter $\mathscr{U}$ over κ so that $\{\alpha < \kappa \mid \alpha$ is supercompact$\} \in \mathscr{U}$. Hence, the least supercompact cardinal is not 1-extendible.*

Proof. Let $j: \mathrm{V}_{\kappa+1} \to \mathrm{V}_{j(\kappa)+1}$ be as in 1-extendibility, and let $\mathscr{U}$ be the usual normal ultrafilter over κ corresponding to j, as in 5.3. Now $\mathrm{V}_{j(\kappa)+1} \models \kappa$ is δ-supercompact for all $\delta < j(\kappa)$, since $j(\kappa)$ is inaccessible. Hence, $A = \{\alpha < \kappa \mid \alpha$ is δ-supercompact for all $\delta < \kappa\} \in \mathscr{U}$. But by the previous proposition, $\alpha \in A$ implies α is supercompact. $\square$

Note that if κ is extendible, $\gamma_0 = \kappa + 1$, and $\gamma_{n+1} =$ least ordinal ζ so that there is a $j: \mathrm{V}_{\gamma_n} \to \mathrm{V}_\zeta$ as in γ_n-extendibility, then if $\gamma = \sup \gamma_n$, we have $\mathrm{V}_\gamma \models \kappa$ is extendible. However, V_γ may not model ZFC. In contrast, consider the following.

Proposition 5.10. *If $\kappa < \lambda$, κ is extendible and λ supercompact, then $\mathrm{V}_\lambda \models \kappa$ is extendible.*

Proof. Suppose that $\kappa \cdot \kappa < \alpha < \lambda$. We must show that $\mathrm{V}_\lambda \models \kappa$ is α-extendible. Since κ is α-extendible in V, there is a $j: \mathrm{V}_\alpha \to \mathrm{V}_\beta$ with critical point κ, so that $j(\kappa) > \alpha$. If $\beta < \lambda$, we are done, so assume $\beta \geq \lambda$.

Let $k: \mathrm{V} \to M$ be as in the $|\mathrm{V}_\beta|$-supercompactness of λ. Then $M \models j: \mathrm{V}_\alpha \to \mathrm{V}_\beta$ with critical point κ, and $j(\kappa) > \alpha$ and $\beta < k(\lambda)$. So, by elementarity, in V there is a $\delta < \lambda$, and $\bar{j}: \mathrm{V}_\alpha \to \mathrm{V}_\delta$ with critical point κ, and $\bar{j}(\kappa) > \alpha$. Thus, $\mathrm{V}_\lambda \models \kappa$ is α-extendible. $\square$

One can also prove 5.10 using the fact that the property "κ is extendible" is Π_3 in the Lévy hierarchy and so reflects down to V_λ for λ supercompact, by 2.7.

We thus see that the extendibility of a cardinal κ can already be comprehended in V_λ where λ is a supercompact cardinal $> \kappa$. This shows in particular that it is consistent to assume there are no supercompact cardinals above an extendible cardinal. Perhaps, 5.10 may serve to allay suspicions about extendibility, which might arise from the fact that it has as a consequence the existence of proper classes of various large cardinals.

In terms of the Lévy hierarchy, this last fact about extendibility can be expressed as follows. Any local property (i.e. Σ_2, see Section 2) of an extendible cardinal κ holds for a proper class of cardinals. The example of supercompactness shows that one cannot prove in ZFC that every Π_2 property of an extendible cardinal κ holds for some $\lambda > \kappa$. Finally, it follows readily from 5.10 that "κ is the least extendible cardinal" is Π^1_3 (being equivalent to "κ is extendible & $\forall \mu < \kappa\ (V_\kappa \models \mu$ is not extendible)") — and certainly holds for no $\lambda > \kappa$.

Theorem 5.11. *If κ is extendible, then Σ_3 (and hence Π_4) relativize down to V_κ.* (Recall 2.6 for this notion.)

Proof. Actually, we only use the fact that there are arbitrarily large inaccessibles $\lambda > \kappa$ with $V_\kappa \prec V_\lambda$. Suppose $P(x)$ is $\exists y\, Q(x, y)$ where Q is Π_2. Let $a \in V_\kappa$ so that $P(a)$ holds, and fix b such that $Q(a, b)$. Let $\lambda > \kappa$ be inaccessible so that $b \in V_\lambda$ and $V_\kappa \prec V_\lambda$. Since, by a remark just after 2.6, Π_2 relativizes down to V_λ, we have $V_\lambda \models Q(a, b)$. Thus, $V_\lambda \models P(a)$ and so $V_\kappa \models P(a)$. $\square$

Note that the Σ_4 sentence "There is an extendible cardinal" is false in V_λ if λ is the least extendible cardinal (by 5.6 and 5.10). Thus, Π_4 in 5.11 is optimal.

6. Vopěnka's principle

We next consider an axiom of a different character both from supercompactness and from extendibility. Bearing in mind our theme of elementary embedding and considerations of resemblance, the motivation behind the following statement is evident, especially in the context of model theory.

Vopěnka's principle. Given a proper class of (set) structures of the same type, there exists one that can be elementarily embedded in another.

This concept was also considered independently by Keisler. It may not be immediately clear that Vopěnka's Principle is a very strong axiom of infinity at all, but we shall prove that the principle implies the existence of many extendible cardinals in a strong sense. In Section 8, it will be shown that the principle actually has a natural place in a hierarchy of strong axioms of infinity.

Vopěnka's Principle definitely cannot be formulated in ZFC, and in this section, we will freely use quantification and comprehension over classes. However, all the manipulations can be carried out in ZFC within some V_κ, where κ is inaccessible (indeed, this is how A_6 of Section 8 is stated). Formulated in this way, note that Vopěnka's Principle is a second order statement about V_κ, whereas even measurability is third order. Indeed, one significant way in which Vopěnka's Principle differs from our previous axioms is that it does not merely assert the existence of a large cardinal with higher order properties, but provides a framework in which many such cardinals can be shown to exist.

We now assume Vopěnka's Principle throughout this section. The approach here is reminiscent of Lévy [20] in that a natural filter is developed and used as a tool. Recall that Ω is the class of all ordinals. Call a sequence of structures $\langle \mathcal{M}_\alpha \mid \alpha \in \Omega \rangle$ a *natural sequence* iff each $\mathcal{M}_\alpha$ is of the same fixed type and specifically of form $\langle V_{\xi_\alpha}; \in, \{\alpha\}, A_\alpha \rangle$, where A_α codes a finite number of relations and $\alpha < \beta$ implies $\alpha < \xi_\alpha \leq \xi_\beta$. The specification of $\{\alpha\}$ in $\mathcal{M}_\alpha$ insures that whenever $\alpha < \beta$ and $j: \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is elementary, j moves some ordinal, since $j(\alpha) = \beta$.

Definition 6.1. If $X \subseteq \Omega$ is a class, X is *enforced* by a natural sequence $\langle \mathcal{M}_\alpha \mid \alpha \in \Omega \rangle$ iff whenever $\alpha < \beta$ and $j: \mathcal{M}_\alpha \to \mathcal{M}_\beta$, the critical point of j is a member of X. $X \subseteq \Omega$ is *enforceable* iff X is enforced by some natural sequence.

Proposition 6.2. *The enforceable classes form a proper filter over* Ω.

Proof. Clearly, if X is enforceable and $X \subseteq Y \subseteq \Omega$, then Y is enforceable. $\emptyset$ is not enforceable since we are assuming Vopěnka's Principle.

Suppose now that X and Y are enforced by $\langle \mathcal{M}_\alpha \mid \alpha \in \Omega \rangle$ and $\langle \mathcal{N}_\alpha \mid \alpha \in \Omega \rangle$ respectively. Set $\mathcal{A}_\alpha = \langle V_{\xi_\alpha}; \in, \{\alpha\}, \langle \mathcal{M}_\alpha, \mathcal{N}_\alpha \rangle \rangle$ where V_{ξ_α} is the union of the underlying sets of $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$. Then $X \cap Y$ is enforced by $\langle \mathcal{A}_\alpha \mid \alpha \in \Omega \rangle$: If $j: \mathcal{A}_\alpha \to \mathcal{A}_\beta$ with critical point κ, $j \mid \mathcal{M}_\alpha$ and $j \mid \mathcal{N}_\alpha$ both have critical point κ, i.e. $\kappa \in X \cap Y$. $\square$

Proposition 6.3. *Every closed unbounded subclass of* Ω *is enforceable.*

Proof. Suppose $C \subseteq \Omega$ is closed unbounded. For each ordinal α let γ_α be the least limit point of C greater than α, and set

$$\mathcal{M}_\alpha = \langle V_{\gamma_\alpha}; \in, \{\alpha\}, C \cap \gamma_\alpha \rangle.$$

We show that C is enforced by $\langle \mathcal{M}_\alpha \mid \alpha \in \Omega \rangle$. Suppose $j: \mathcal{M}_\alpha \to \mathcal{M}_\beta$ with critical point κ, and assume $\kappa \notin C$. Then $\rho = \sup(C \cap \kappa) < \kappa$ and if η is the least element of C greater than ρ, $\kappa < \eta < \gamma_\alpha$ since γ_α is a limit point of C. As η is definable from ρ in $\mathcal{M}_\alpha$ and $j(\rho) = \rho$, $j(\eta) = \eta$. But, as usual, $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$ is the least ordinal greater than κ fixed by j, so that $\lambda \leq \eta$. We can now derive a contradiction by Kunen's argument (1.12), since γ_α is a limit ordinal $> \eta$. Thus, $\kappa \in C$. $\square$

In fact, the enforceable classes form a normal, Ω-complete filter over Ω, see [11]. This paper discusses strong versions of Vopěnka's Principle related to n-hugeness (see Section 7 for this concept), which are analogous to the n-subtle and n-ineffable cardinals studied by Baumgartner.

Proposition 6.4. *$\{\alpha \in \Omega \mid \alpha$ is extendible$\}$ is enforceable.*

Proof. Define $F:\Omega \to \Omega$ by

$$F(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ is extendible,} \\ \alpha + \beta & \text{where } \beta \text{ is the least so that } \alpha \text{ is not} \\ & \beta\text{-extendible otherwise.} \end{cases}$$

If $C = \{\delta \mid F:\delta \to \delta\}$, C is closed unbounded. Define $\langle \mathcal{M}_\alpha \mid \alpha \in \Omega\rangle$ for this C, just as in the previous proposition.

It suffices to show that if $j:\mathcal{M}_\alpha \to \mathcal{M}_\beta$ with critical point κ, then κ is extendible. If not, let $F(\kappa) = \rho > \kappa$. Since $\kappa < \gamma_\alpha$ and $\gamma_\alpha \in C$, we have $\rho < \gamma_\alpha$ by definition of C. So, $j \mid V_\rho : V_\rho \to V_{j(\rho)}$ is elementary with critical point κ. Finally, by the proof of the previous proposition, $\kappa \in C$ and so $j(\kappa) \in C$ (recall that $C \cap \gamma_\alpha$ is a specified relation in $\mathcal{M}_\alpha$). Hence, $\kappa < j(\kappa)$ implies $\rho < j(\kappa)$. This also implies $\rho < j(\rho)$. If $\rho = \kappa + \beta$, all these facts show that κ was β-extendible after all, which contradicts the definition of F. $\square$

Proposition 6.5. *If X is enforceable and $Y = \{\alpha \mid \alpha$ is measurable and for some normal ultrafilter $\mathcal{U}$ over α, $X \cap \alpha \in \mathcal{U}\}$, then Y is also enforceable.*

Proof. For each ordinal α, set

$$\mathcal{M}_\alpha = \langle V_{\alpha+\omega}; \in, \{\alpha\}, \{X \cap \alpha\}\rangle.$$

It suffices to show that if $j:\mathcal{M}_\alpha \to \mathcal{M}_\beta$ with critical point κ and $\kappa \in X$, then $\kappa \in Y$. Let $\mathcal{U}$ be the normal ultrafilter over κ corresponding to j. We have

$$j(X \cap \kappa) = j(X \cap \alpha \cap \kappa) = j(X \cap \alpha) \cap j(\kappa) = X \cap \beta \cap j(\kappa) = X \cap j(\kappa).$$

But $\kappa \in X \cap j(\kappa)$, so $X \cap \kappa \in \mathcal{U}$. $\square$

Theorem 6.6. *Assuming Vopěnka's Principle, the class of extendible cardinals κ which carry a normal ultrafilter containing $\{\alpha < \kappa \mid \alpha$ is extendible$\}$ is stationary in Ω.*

Proof. See 6.3, 6.4, and 6.5. $\square$

All in all, Vopěnka's Principle seems to have an unbridled strength, and the relative ease with which strong consequences can be derived from it may lead one to be rather suspicious of the principle. However, it is the weakest of the hierarchical list of axioms we will consider in Sections 7–8.

In the remainder of this section, we present two alternative characterizations of Vopěnka's Principle. In addition to their intrinsic interest, these results will be useful in the discussion of Backward Easton forcing in a projected sequel to this paper.

Let A be a class. We are going to relativize the notions of supercompactness and extendibility to the class A.

Definition 6.7. A cardinal κ is A-*extendible* iff for every $\alpha > \kappa$, there is an elementary embedding

$$j : \langle V_\alpha ; \in, A \cap V_\alpha \rangle \to \langle V_\beta ; \in, A \cap V_\beta \rangle,$$

with κ the critical point of j and $\alpha < j(\kappa) < \beta$.

A cardinal κ is A-*supercompact* iff for every $\eta > \kappa$ there is an $\alpha < \kappa$, and an elementary embedding

$$j : \langle V_\alpha ; \in, A \cap V_\alpha \rangle \to \langle V_\eta ; \in, A \cap V_\eta \rangle$$

with critical point γ such that $j(\gamma) = \kappa$ (cf. 5.7).

We can also give a characterization in terms of normal ultrafilters. Let $\mathcal{U}$ be a normal ultrafilter over $\mathcal{P}_\kappa V_\eta$. Recall (2.5) that if $\eta = [\langle \eta_x \mid x \in \mathcal{P}_\kappa V_\eta \rangle]$, for almost all $x \in \mathcal{P}_\kappa V_\eta$, there is an elementary embedding $j_x : V_{\eta_x} \to V_\eta$ with $x = j_x'' V_{\eta_x}$.

We say that $\mathcal{U}$ is A-*normal* iff for almost all x,

$$j_x : \langle V_{\eta_x} ; \in, A \cap V_{\eta_x} \rangle \to \langle V_\eta ; \in, A \cap V_\eta \rangle$$

is elementary. Evidently $\mathcal{U}$ is A-normal iff $\mathcal{U}$ is $(A \cap V_\eta)$-normal.

Proposition 6.8. *κ is A-supercompact iff for every $\eta \geq \kappa$, there is an A-normal ultrafilter over $\mathcal{P}_\kappa V_\eta$.*

Proof. We show that if κ is A-supercompact and $\eta \geq \kappa$, there is an A-normal ultrafilter over $\mathcal{P}_\kappa V_\eta$. The other implication is left to the reader. (Cf. the proof of 5.7.)

By the A-supercompactness of κ, let $\alpha < \kappa$ and

$$j : \langle V_{\alpha+\omega} ; \in, A \cap V_{\alpha+\omega} \rangle \to \langle V_{\eta+\omega} ; \in, A \cap V_{\eta+\omega} \rangle$$

with critical point γ such that $j(\gamma) = \kappa$. Note that $j(A \cap V_\alpha) = A \cap V_\eta$.

Define $\mathcal{U}$ over $\mathcal{P}_\gamma V_\alpha$ by $X \in \mathcal{U}$ iff $j'' V_\alpha \in j(X)$. Then, as usual, $\mathcal{U}$ is a normal ultrafilter over $\mathcal{P}_\gamma V_\alpha$. Let $\alpha = [\langle \alpha_x \mid x \in \mathcal{P}_\gamma V_\alpha \rangle]_{\mathcal{U}}$, and recall 2.5(i) in what follows. For almost every x, $\langle x ; \in, A \cap x \rangle$ is an elementary submodel of $\langle V_\alpha ; \in, A \cap V_\alpha \rangle$.

Thus, $\mathcal{U}$ is A-normal iff the transitive collapse $\pi_x : x \cong V_{\alpha_x}$ "preserves A" for almost every x iff the transitive collapse $\pi : j''V_\alpha \cong V_\alpha$ "preserves A". But this last formulation is evident since $\pi(j(x)) = x$ for $x \in V_\alpha$, and j "preserves A".

Thus, $\mathcal{U}$ is an $(A \cap V_\alpha)$-normal ultrafilter over $\mathcal{P}_\gamma V_\alpha$. But then $j(\mathcal{U})$ is an $j(A \cap V_\alpha)$-normal ultrafilter over $\mathcal{P}_\kappa V_\eta$. Since $j(A \cap V_\alpha) = A \cap V_\eta$, $j(\mathcal{U})$ is the desired A-normal ultrafilter on $\mathcal{P}_\kappa V_\eta$. $\square$

Note that if $A = \{\gamma\}$, then γ is not A-supercompact or A-extendible. However,

Theorem 6.9. *The following are equivalent:*

(1) *Vopěnka's Principle,*

(2) *For every class A, there is an A-extendible cardinal,*

(3) *For every class A, there is an A-supercompact cardinal.*

We remark that our proof will show in fact that if (1) holds, the classes of A-extendible cardinals and A-supercompact cardinals are enforceable.

Proof. The proof of 6.4 adapts to show that $(1) \rightarrow (2)$. Also, the proof of 6.8 can be used to show that every A-extendible cardinal is A-supercompact. Thus $(2) \rightarrow (3)$.

Finally, let $\langle \mathcal{M}_\xi \mid \xi \in \Omega \rangle$ be as in the statement of Vopěnka's Principle. Let $A = \{\langle \xi, \mathcal{M}_\xi \rangle \mid \xi \in \Omega\}$. Let κ be A-supercompact. Let $\theta > \kappa$ be such that:

(i) $\xi < \theta \rightarrow \mathcal{M}_\xi \in V_\theta$, and

(ii) $|V_\theta| = \theta$.

Let $\alpha < \kappa$, and

$$j : \langle V_\alpha ; \in, A \cap V_\alpha \rangle \rightarrow \langle V_\theta ; \in, A \cap V_\theta \rangle$$

be an elementary embedding that maps its critical point γ onto κ. Then j induces an elementary embedding of $\mathcal{M}_\gamma$ into $\mathcal{M}_\kappa$. $\square$

7. On the verge of inconsistency

Having examined several axioms increasing in strength and motivated with different but definite plausibility arguments in mind, we now take a more pragmatic approach. Kunen's result (1.12) sets an upper bound to our efforts in an essential way, but it is still of interest to see what weaker principles can possibly be retained without inconsistency in ZFC. In this and the next section we work downward through weaker and weaker axioms that suggest themselves, are at least as strong as Vopěnka's Principle, but are not directly ruled out by Kunen's argument.

Tacit in this section is the assumption that if j is some elementary embedding with critical point κ, then $\kappa_0 = \kappa$ and for each integer n, $\kappa_{n+1} = j(\kappa_n)$ if κ_n is still in the domain of j, and $\kappa_\omega = \sup\{\kappa_n \mid n \in \omega\}$, again, if definable at all. (Of course, the κ_n's depend on j, but the j being discussed should be clear from the context.)

If there were a $j: V_\alpha \to V_\beta$ with $\kappa_\omega \leq \alpha$, note that we must have $\alpha < \kappa_\omega + 2$. This is so since what is needed to get Kunen's contradiction is that a function ω-Jónsson over κ_ω, i.e. a certain function: ${}^\omega\kappa_\omega \to \kappa_\omega$, be in the domain of j — but all such functions are in $V_{\kappa_\omega+2}$. When a function ω-Jónsson over κ_ω is in the domain of an elementary embedding as in, say, $j: V \to M$, then by Kunen's argument we must have $V_{\kappa_\omega+1} \not\subseteq M$. However, we can still consider the following statements.

I1. There is a $j: V_{\kappa_\omega+1} \to V_{\kappa_\omega+1}$.

I2. There is a $j: V \to M$ with $V_{\kappa_\omega} \subseteq M$.

I3. There is a $j: V_{\kappa_\omega} \to V_{\kappa_\omega}$.

Notice that in I1 we have specified the range of j to be included in $V_{\kappa_\omega+1}$, but this is true since $j(\kappa_\omega) = \kappa_\omega$; similarly for I3. In fact, I1 and I3 are the only possible forms that an axiom of the type "there is a non-trivial elementary embedding of some V_α into itself" can take.

The proof of the next proposition uses iteration and limit ultrapower techniques.

Proposition 7.1. (Gaifman). *I1 implies I2.*

Proof. See IV. 8 of Gaifman [4]. □

Proposition 7.2. *I2 implies I3.*

Proof. If j is as in I2, then $j(V_{\kappa_\omega}) = V^M_{\kappa_\omega} = V_{\kappa_\omega}$, so that

$$j \mid V_{\kappa_\omega}: V_{\kappa_\omega} \to V_{\kappa_\omega}. \quad \square$$

Next, it is natural to consider postulations with weaker closure requirements on the range of the embedding.

Definition 7.3. If n is an integer, κ is *n-huge* iff there is a $j: V \to M$ with critical point κ so that ${}^{\kappa_n}M \subseteq M$. κ is *huge* (Kunen) iff κ is 1-huge.

Note that κ is 0-huge iff κ is measurable, and κ is n-huge implies κ_n is inaccessible in V itself. It is interesting to note that, reminiscent of λ-supercompactness, a characterization exists in terms of the existence of certain ultrafilters.

Theorem 7.4. *κ is n-huge iff there is a κ-complete normal ultrafilter $\mathcal{U}$ over some $\mathcal{P}\lambda$, and cardinals $\kappa=\lambda_0<\lambda_1<\cdots<\lambda_n=\lambda$ so that for each $i<n$, $\{x \mid \overline{x\cap\lambda_{i+1}}=\lambda_i\}\in\mathcal{U}$.*

Proof. If $j:V\to M$ as in n-hugeness, define $\mathcal{U}$ over $\mathcal{P}\kappa_n$ by

$$X\in\mathcal{U} \quad \text{iff} \quad j''\kappa_n\in j(X).$$

Then as in 2.2, we can show that $\mathcal{U}$ is normal and κ-complete. Also, note that

$$\overline{j''\kappa_n\cap j(\kappa_i)}=\overline{j''\kappa_i}=\kappa_i \quad \text{for} \quad 0\leqslant i\leqslant n,$$

so that we can set $\kappa_i=\lambda_i$.

Conversely, take j, M the ultrapower as usual. Then, as in 2.4, $[\mathrm{id}]=j''\lambda$ and M is closed under λ-sequences. Also, as in 3.1, we have that for $0\leqslant i<n$,

$$j(\lambda_i)=[\langle\overline{x\cap\lambda_{i+1}} \mid x\in\mathcal{P}\lambda\rangle]=\lambda_{i+1}. \;\square$$

Theorem 7.5. *If I3 or κ is $n+1$-huge, then there is a normal ultrafilter $\mathcal{U}$ over κ so that $\{\alpha<\kappa \mid \alpha \text{ is } n\text{-huge}\}\in\mathcal{U}$.*

Proof. Suppose, for example, that $j:V\to M$ as in the $n+1$-hugeness of κ. Since ${}^{\kappa_{n+1}}M\subseteq M$, M certainly contains the ultrafilter described in 7.4 for the n-hugeness of κ, arising from j. Hence, $M\models\kappa$ is n-huge, and we can take $\mathcal{U}$ to be the (usual) normal ultrafilter over κ corresponding to j. $\square$

It seems likely that I1, I2 and I3 are all inconsistent since they appear to differ from the proposition proved inconsistent by Kunen only in an inessential technical way. The axioms asserting the existence of n-huge cardinals, for $n>1$, seem (to our unpracticed eyes) essentially equivalent in plausibility: far more plausible that I3, but far less plausible than say extendibility.

Kunen's work [16] relates 1-hugeness (i.e. hugeness) to the theory of saturated ideals. He shows that Con (ZFC & there is a huge cardinal) implies Con (ZFC & there is a countably complete, ω_2-saturated ideal on ω_1 containing all the singletons). Kunen also indicates a heuristic argument suggesting that the consistency of something slightly weaker than hugeness ($A_3(\kappa)$ of the next section) should follow from the consistency of the existence of such a non-trivial ω_2-saturated ideal on ω_1.

More recently, Laver has announced a refinement of Kunen's argument to get an ideal over ω_1 with an even stronger property, which has as a consequence the following polarized partition relation:

$$\begin{pmatrix}\aleph_2\\ \aleph_1\end{pmatrix}\to\begin{pmatrix}\aleph_1\\ \aleph_1\end{pmatrix}_{\aleph_0}$$

Since the GCH can also be arranged in this particular case, this answers problem 27 of Erdös–Hajnal [3] in the negative (at least if we assume the consistency of the existence of a huge cardinal!). Prikry [38] had previously shown that there is a combinatorial principle which implies, in a strong sense, that the partition relation does *not* hold, and that this principle can be made to hold by forcing. Jensen then showed that this principle holds in L.

An ultrafilter $\mathcal{U}$ is called (κ, λ)-regular iff there are λ sets in $\mathcal{U}$ any κ of them having empty intersection. This concept was formulated by Keisler in the context of model theory some time ago. Recently, it has been shown that the existence of, for instance, a uniform ultrafilter over ω_1 which is not (ω, ω_1)-regular leads to consequences of large cardinal character (see Kanamori [10] and Ketonen [14]). In this context, we state the following result of Magidor [28], proved by using a variation of the Kunen [16] argument: Con (ZFC & there is a huge cardinal) implies Con (ZFC & there is a uniform ultrafilter over ω_2 not (ω_0, ω_2)-regular). At present, there is no other known way to get uniform ultrafilters with any degree of irregularity over any ω_n.

Finally, the most recent relative consistency result involving hugeness is the following, due once again to Magidor [27]: Con (ZFC & there is a huge cardinal with a supercompact cardinal below it) implies Con (ZFC & $2^{\aleph_\omega} = \aleph_{\omega+2}$, yet for every integer n, $2^{\aleph_n} = \aleph_{n+1}$). This, of course, solves the so-called singular cardinals problem at $\aleph_\omega$. Previously, we had remarked (end of Section 2) that Magidor had found a model in which $2^{\aleph_\omega} = \aleph_{\omega+2}$, and $\aleph_\omega$ is a strong limit cardinal, assuming the consistency strength of the existence of a cardinal with sufficient degree of supercompactness. To get the exactitude of the GCH actually holding below $\aleph_\omega$, Magidor found it necessary to start from his stronger assumption.

It is tempting to speculate on the further relevance of huge cardinals in considerations involving the lower orders of the cumulative hierarchy. After all, it is such empirical evidence which gained for measurability a certain respectability, if not acceptance

8. Below huge

This section contains the rest of the new axioms to be considered in this paper. They are intended to fill in the gap between the concept of hugeness and the relatively weak one of extendibility with a spectrum of statements. Though we are thus continuing to take a pragmatic approach, hopefully these further axioms will prove interesting in their own right. At least, their motivations should be clear in the context of this paper. By a *natural model* of KM (Kelley–Morse) we mean one of form $V_{\kappa+1}$, where κ is inaccessible and elements of V_κ are to be the "sets". The axioms are as follows.

A₁(κ). There is a $j: V \to M$ with critical point κ, so that $M^{j(\kappa)} \subseteq M$. ($\kappa$ is huge.)

A₂(κ). There is a $j: V_\alpha \to V_\beta$ with critical point κ, so that $j(\kappa) \leq \alpha$.

A$_3(\kappa)$. There is a $j:\mathrm{V}\to M$ with critical point κ, so that ${}^{\lambda}M\subseteq M$ for every $\lambda<j(\kappa)$.

A$_4(\kappa)$. There is a $\lambda>\kappa$ and a normal ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$ so that if $M\approx \mathrm{V}^{\mathcal{P}_\kappa\lambda}/\mathcal{U}$ and $f\in{}^{\kappa}\kappa$, then $M\models j(f)(\kappa)<\lambda$.

A$_5(\kappa)$. There is a normal ultrafilter $\mathcal{U}$ over κ with the following property. Suppose $\langle \mathcal{M}_\xi \mid \xi<\kappa\rangle$ is a sequence of structures of the same type with each $\mathcal{M}_\xi\in \mathrm{V}_\kappa$. Then for some $X\in\mathcal{U}$, whenever $\alpha,\beta\in X$ and $\alpha<\beta$, there is an elementary embedding $\mathcal{M}_\alpha\to\mathcal{M}_\beta$ which fixes any element of V_α in its domain and moves α if α is in the domain.

A$_6(\kappa)$. $\mathrm{V}_{\kappa+1}$ is a natural model of KM and Vopěnka's Principle: given a proper class of (set) structures of the same type, there exists one that can be elementarily embedded into another.

A$_6^*(\kappa)$. $\mathrm{V}_{\kappa+1}$ is a natural model of KM and the following: there is a stationary class S so that for any integer n and $\alpha_0<\alpha_1<\cdots<\alpha_n<\beta_0<\beta_1<\cdots<\beta_n$ all in S, there is a $j:\mathrm{V}_{\alpha_n}\to\mathrm{V}_{\beta_n}$ with critical point α_0 and $j(\alpha_i)=\beta_i$ for $0\leq i<n$.

A$_7(\kappa)$. κ is extendible and carries a normal ultrafilter containing $\{\alpha<\kappa \mid \alpha$ is extendible$\}$ as a member.

It is convenient in this section to use the following terminology: if $\psi(\cdot)$ and $\varphi(\cdot)$ are both formulas with one free variable, say that $\psi(\kappa)$ *strongly implies* $\varphi(\kappa)$ iff $\psi(\kappa)$ implies $\varphi(\kappa)$ and also that there is a normal ultrafilter over κ containing $\{\alpha<\kappa \mid \varphi(\alpha)\}$ as a member. We shall prove that $\exists\kappa\ \mathrm{A}_1(\kappa)$ implies $\exists\kappa\ \mathrm{A}_2(\kappa)$ in a strong sense, $\mathrm{A}_i(\kappa)$ strongly implies $\mathrm{A}_{i+1}(\kappa)$ for $1<i<5$, and that $\mathrm{A}_6(\kappa)$ and $\mathrm{A}_6^*(\kappa)$ are both strongly implied by $\mathrm{A}_5(\kappa)$. In Section 6, we showed that $\exists\kappa\ \mathrm{A}_6(\kappa)$ implies $\exists\kappa\ \mathrm{A}_7(\kappa)$ in a suitably strong sense; in this section, we show that $\exists\kappa\ \mathrm{A}_6^*(\kappa)$ similarly implies $\exists\kappa\ \mathrm{A}_7(\kappa)$ in a strong sense. However, the exact relationship between $\mathrm{A}_6(\kappa)$ and $\mathrm{A}_6^*(\kappa)$ is as yet unclear. The rest of this section is devoted to the task of establishing these implications.

In terms of relative consistency, we are dealing with very strong principles. However, it should be pointed out that, except for A_7, all of these various assertions about κ, as well as the notion of n-hugeness, are *local* properties, and so do not even imply that κ is supercompact. By 2.6, if any one of these properties hold for some cardinal at all, then it holds for a cardinal less than the least supercompact cardinal. On the other hand, as remarked by Morgenstern [34], a straightforward application of work of Magidor shows that, for example, it is consistent for the least huge cardinal to be larger than the least strongly compact cardinal.

Theorem 8.1. $\mathrm{A}_1(\kappa)$ *implies that there is a normal ultrafilter* $\mathcal{U}$ *over* κ *so that* $\{\alpha<\kappa \mid \mathrm{A}_2(\alpha)\}\in\mathcal{U}$.

Proof. Let $j:\mathrm{V}\to M$ show that κ is huge. As in previous arguments, $\mathrm{V}_{j(\kappa)}\in M$ and $j\restriction \mathrm{V}_{j(\kappa)}\in M$. Hence, as $j\restriction\mathrm{V}_{j(\kappa)}:\mathrm{V}_{j(\kappa)}\to j(\mathrm{V}_{j(\kappa)})$, this is also true in M. Thus, it follows that if $\mathcal{U}$ is the normal ultrafilter over κ corresponding to j, $\{\alpha<\kappa \mid$ there are β, γ, and e so that $e:\mathrm{V}_\beta \to \mathrm{V}_\gamma$ with critical point α and $e(\alpha)\leqslant\beta\}\in\mathcal{U}$. But from this, the result follows. $\square$

For the passage from A_2 to A_3 some auxiliary notions are needed. If $\kappa<\lambda$, say that a sequence $\langle\mathcal{U}_\eta \mid \kappa\leqslant\eta<\lambda\rangle$ is *coherent* iff each $\mathcal{U}_\eta$ is a normal ultrafilter over $\mathcal{P}_\kappa\eta$ and $\kappa\leqslant\eta\leqslant\zeta<\lambda$ implies $\mathcal{U}_\zeta \mid \eta=\mathcal{U}_\eta$. If $\langle\mathcal{U}_\eta \mid \kappa\leqslant\eta<\lambda\rangle$ is coherent, for convenience let $M_\eta=M_{\mathcal{U}_\eta}$, and $j_\eta:\mathrm{V}\to M_\eta$ the canonical elementary embedding. Recall that by 3.7, there exist elementary embeddings $k_{\eta\zeta}:M_\eta\to M_\zeta$ for $\kappa\leqslant\eta\leqslant\zeta<\lambda$ so that $k_{\eta\zeta}\mid\eta$ is the identity. In this notation, consider the following statement.

$\mathrm{A}_3^*(\kappa)$. *There is a strongly inaccessible cardinal $\lambda>\kappa$ and a coherent sequence $\langle\mathcal{U}_\eta \mid \kappa\leqslant\eta<\lambda\rangle$ so that if $\kappa\leqslant\eta<\lambda$ and $\eta\leqslant\rho<j_\eta(\kappa)$, then there is a ζ so that $\eta\leqslant\zeta<\lambda$ and $k_{\eta\zeta}(\rho)=\zeta$.*

$\mathrm{A}_3^*(\kappa)$ is a technical statement that we show is actually equivalent to $\mathrm{A}_3(\kappa)$, and then we prove that $\mathrm{A}_2(\kappa)$ strongly implies $\mathrm{A}_3^*(\kappa)$. Ostensibly, $\mathrm{A}_3^*(\kappa)$ involves proper classes j_η and $k_{\eta\zeta}$, but actually it can be considered a statement in $j_\eta\mid\mathrm{V}_\lambda$ and $k_{\eta\zeta}\mid\mathrm{V}_\lambda$. Hence, $\mathrm{A}_3^*(\kappa)$ can be expressed in the form $\exists\lambda>\kappa\;(\mathrm{V}_{\lambda+1}\models\varphi(\kappa))$ for some suitable set-theoretical formula φ. This aspect of $\mathrm{A}_3^*(\kappa)$ becomes significant in 8.3.

Theorem 8.2. *$\mathrm{A}_3(\kappa)$ is equivalent to $\mathrm{A}_3^*(\kappa)$.*

Proof. Suppose first that $\mathrm{A}_3(\kappa)$ and let $j:\mathrm{V}\to M$ with critical point κ, so that M is closed under $<j(\kappa)$-sequences. If we set $\lambda=j(\kappa)$, then λ is inaccessible. For $\kappa\leqslant\eta<\lambda$ let $\mathcal{U}_\eta$ be the (usual) normal ultrafilter over $\mathcal{P}_\kappa\eta$ corresponding to j. Then it is not hard to see that $\langle\mathcal{U}_\eta \mid \kappa\leqslant\eta<\lambda\rangle$ is coherent.

Assume now that $\kappa\leqslant\eta<\lambda$, $f:\mathcal{P}_\kappa\eta\to\kappa$, and $\eta\leqslant[f]_{\mathcal{U}_\eta}<j_\eta(\kappa)$. (We are using the notation developed for coherent sequences just after 8.1.) Then by definition of $\mathcal{U}_\eta$, we have $\eta=\overline{j''\eta}\leqslant j(f)(j''\eta)<j(\kappa)=\lambda$. Let $\zeta=j(f)(j''\eta)$. But then, $\overline{j''\zeta}=\zeta=j(f)(j''\zeta\cap j(\eta))$ and so $\{x\in\mathcal{P}_\kappa\zeta \mid \bar{x}=f(x\cap\eta)\}\in\mathcal{U}_\zeta$, i.e. $k_{\eta\zeta}([f]_{\mathcal{U}_\eta})=\zeta$ by the definition of $k_{\eta\zeta}$ (see 3.7 and before). Thus, $\mathrm{A}_3^*(\kappa)$ holds.

Suppose now that $\mathrm{A}_3^*(\kappa)$. Continuing to use the notation developed for coherent sequences, note that $\langle M_\eta, k_{\eta\zeta} \mid \kappa\leqslant\eta\leqslant\zeta<\lambda\rangle$ forms a directed system. Since λ is regular, the direct limit of this system is well founded, so let $\tilde{M}$ be its transitive collapse. There are canonical embeddings $k_\eta:M_\eta\to\tilde{M}$ and $\tilde{j}:\mathrm{V}\to\tilde{M}$, where for $\kappa\leqslant\eta<\lambda$, $\tilde{j}=k_\eta\circ j_\eta$.

It suffices to show that $\tilde{M}$ is closed under $<\lambda$-sequences, and that $\tilde{j}(\kappa)=\lambda$. First, note that if $s=\{x_\alpha \mid \alpha<\gamma\}\subseteq\tilde{M}$ where $\gamma<\lambda$ then by regularity of λ there is an $\eta<\lambda$ so that $t=\{y_\alpha \mid \alpha<\gamma\}\subseteq M_\eta$, where for $\alpha<\gamma$, $k_\eta(y_\alpha)=x_\alpha$. We can certainly assume that $\gamma<\eta$ so that $t\in M_\eta$ and hence $k_\eta(t)=s\in\tilde{M}$.

To show that $\bar{j}(\kappa)=\lambda$, note first that if $\eta<\lambda$, since $\eta<j_\eta(\kappa)$,

$$\eta\leqslant k_\eta(\eta)<k_\eta(j_\eta(\kappa))=\bar{j}(\kappa).$$

Hence, $\bar{j}(\kappa)\geqslant\lambda$. Conversely, suppose $\rho<\bar{j}(\kappa)$. There is an η and a $\rho'<j_\eta(\kappa)$ so that $k_\eta(\rho')=\rho$. The case $\rho'<\eta$: since $k_{\eta\xi}$ fixes ρ' for $\eta\leqslant\xi<\lambda$, $\rho=k_\eta(\rho')=\rho'<j_\eta(\kappa)$. The case $\eta\leqslant\rho'$: by $A_3^*(\kappa)$ there is a ζ so that $\eta\leqslant\zeta<\lambda$ and $k_{\eta\zeta}(\rho')=\xi$; thus, whenever $\zeta\leqslant\xi<\lambda$, we have $k_{\eta\xi}(\rho')=\zeta$, so that $\rho=k_\eta(\rho')=\zeta<j_\zeta(\kappa)$. In either case, there is some $\delta<\lambda$ so that $\rho<j_\delta(\kappa)$, and hence,

$$\rho<j_\delta(\kappa)<(2^{\delta^{<\kappa}})^+<\lambda$$

as λ is inaccessible. Thus, $\bar{j}(\kappa)=\lambda$.

The proof is complete.

Theorem 8.3. $A_2(\kappa)$ *strongly implies* $A_3^*(\kappa)$.

Proof. Suppose $j:V_\alpha\to V_\beta$ as for $A_2(\kappa)$. Since $\lambda=j(\kappa)$ is inaccessible and $j(\kappa)\leqslant\alpha$, by using an argument entirely analogous to the first part of 8.2, it can be shown that $A_3^*(\kappa)$ holds — note that j need only be defined on V_λ. It now follows that $A_3^*(\kappa)$ holds within V_β (recall the observation made just before 8.2). Thus, if $\mathscr{U}$ is the normal ultrafilter over κ corresponding to j, $\{\alpha<\kappa\mid A_3^*(\alpha)\}\in\mathscr{U}$.

Theorem 8.4. $A_3(\kappa)$ *strongly implies* $A_4(\kappa)$.

Proof. Let $j:V\to M$ with critical point κ so that M is closed under $<j(\kappa)$-sequences. Since $j(\kappa)$ is inaccessible and $f\in{}^\kappa\kappa$ implies $j(f)(\kappa)<j(\kappa)$, let μ be such that $\sup\{j(f)(\kappa)\mid f\in{}^\kappa\kappa\}<\mu<j(\kappa)$. Let $\mathscr{V}$ be the normal ultrafilter over $\mathscr{P}_\kappa\mu$ corresponding to j. Then

$$\begin{aligned} j_{\mathscr{V}}(f)(\kappa)<\mu \quad &\text{iff}\quad \{x\mid f(\overline{x\cap\kappa})<\bar{x}\}\in\mathscr{V}\\ &\text{iff}\quad j(f)(\overline{j''\mu\cap j(\kappa)})<\overline{j''\mu}\\ &\text{iff}\quad j(f)(\kappa)<\mu.\end{aligned}$$

Thus, $A_4(\kappa)$ holds, but then it also holds in M. That $A_3(\kappa)$ strongly implies $A_4(\kappa)$ follows, with $\mathscr{U}$ the normal ultrafilter over κ corresponding to j. □

Theorem 8.5. $A_4(\kappa)$ *strongly implies* $A_5(\kappa)$.

Proof. Let $j:V\to M$ be determined by a normal ultrafilter over $\mathscr{P}_\kappa\lambda$ as provided by $A_4(\kappa)$. Let $\mathscr{U}$ be normal over κ corresponding to j. We first show that $A_5(\kappa)$ is satisfied with this $\mathscr{U}$.

Suppose $\langle \mathcal{M}_\xi \mid \xi < \kappa \rangle$ are structures of the same type so that each $\mathcal{M}_\xi \in V_\kappa$. We may assume that the underlying set of $\mathcal{M}_\xi$ has the form V_η for η a limit ordinal $> \xi$, and that $\in$ is a predicate of $\mathcal{M}_\xi$. (If not, replace $\mathcal{M}_\xi$ by $\langle V_\eta, \in, \{\mathcal{M}_\xi\}\rangle$ where η is the least limit ordinal greater than $\max(\xi, \operatorname{rank}(\mathcal{M}_\xi))$.)

Claim. *If $\alpha < \kappa$ and $X_\alpha = \{\xi \mid$ there is an elementary embedding $\mathcal{M}_\alpha \to \mathcal{M}_\xi$ with critical point $\alpha\}$, $T = \{\alpha < \kappa \mid X_\alpha \in \mathcal{U}\} \in \mathcal{U}$.*

To show the claim, first set

$$\begin{aligned} j(\langle \mathcal{M}_\xi \mid \xi < \kappa \rangle) &= \langle \mathcal{M}'_\xi \mid \xi < j(\kappa) \rangle, \\ j(\langle \mathcal{M}'_\xi \mid \xi < j(\kappa) \rangle) &= \langle \mathcal{M}''_\xi \mid \xi < j^2(\kappa) \rangle, \\ j(\langle X_\alpha \mid \alpha < \kappa \rangle) &= \langle X'_\alpha \mid \alpha < j(\kappa) \rangle. \end{aligned}$$

Note that $\xi < \kappa$ implies $\mathcal{M}_\xi = \mathcal{M}'_\xi$. If $\alpha < \kappa$, $X_\alpha \in \mathcal{U}$ iff $\kappa \in j(X_\alpha)$ iff $\mathcal{M}_\alpha$ is elementarily embeddable into $\mathcal{M}'_\kappa$ with critical point α. So if $\alpha < j(\kappa)$, $X'_\alpha \in j(\mathcal{U})$ iff in M, $\mathcal{M}'_\alpha$ is elementarily embeddable into $\mathcal{M}''_{j(\kappa)}$ with critical point α. Hence, $T \in \mathcal{U}$ iff $\kappa \in j(T)$ iff $X'_\kappa \in j(\mathcal{U})$ iff in M, $\mathcal{M}'_\kappa$ is elementarily embeddable into $\mathcal{M}''_{j(\kappa)}$ with critical point κ.

Now $j \mid \mathcal{M}'_\kappa : \mathcal{M}'_\kappa \to \mathcal{M}''_{j(\kappa)} = j(\mathcal{M}'_\kappa)$ is elementary, so it suffices to show that $j \mid \mathcal{M}'_\kappa \in M$. Define $f \in {}^\kappa\kappa$ by $f(\xi) = |\mathcal{M}_\xi|$. By the $A_4(\kappa)$ property of j, $j(f)(\kappa) < \lambda$. Thus, since M is closed under λ-sequences and $j \mid \mathcal{M}'_\kappa$ is just a set of ordered pairs of elements of M of cardinality $|\mathcal{M}'_\kappa| = j(f)(\kappa)$, $j \mid \mathcal{M}'_\kappa \in M$. The claim is proved.

Having shown that $T \in \mathcal{U}$, by the diagonal intersection property of normal ultrafilters we have that $\{\alpha \in T \mid \beta < \alpha \;\&\; \beta \in T$ implies $\alpha \in X_\beta\} \in \mathcal{U}$. This set satisfies $A_4(\kappa)$ for $\langle \mathcal{M}_\xi \mid \xi < \kappa \rangle$.

Finally, another application of the $A_4(\kappa)$ property of j shows that $2^\kappa \leq (2^\kappa)^M < \lambda$, so that our $\mathcal{U} \in M$. Thus $M \vDash A_5(\kappa)$, and so $\{\alpha < \kappa \mid A_5(\alpha)\} \in \mathcal{U}$. □

Theorem 8.6. *$A_5(\kappa)$ strongly implies $A_6(\kappa)$.*

Proof. $A_6(\kappa)$ is immediate from $A_5(\kappa)$. But if $j : V \to M$ corresponds to any normal ultrafilter $\mathcal{U}$ over κ, then $M \vDash A_6(\kappa)$, as $V_{\kappa+1} \subseteq M$. Thus, $\{\alpha < \kappa \mid A_6(\alpha)\} \in \mathcal{U}$. □

Theorem 8.7. *$A_5(\kappa)$ strongly implies $A_6^*(\kappa)$.*

Proof. It suffices to show $A_6^*(\kappa)$, for the result would then follow as in the proof of 8.6. Let $\mathcal{U}$ be as in $A_5(\kappa)$. It is sufficient to find an $X_n \in \mathcal{U}$ satisfying the condition for the S of $A_6^*(\kappa)$ for a fixed n, as we can then take a countable intersection. Define a function $F : [\kappa]^{2n+2} \to 2$ by

$$F(\langle \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \rangle) = 0 \quad \text{iff} \quad \alpha_0 < \cdots < \alpha_n < \beta_0 < \cdots < \beta_n$$

and there is a $j: V_{\alpha_n} \to V_{\beta_n}$ with critical point α_0 so that $j(\alpha_i) = \beta_i$ for $i < n$. By the partition property for normal ultrafilters, there is a $Y \in \mathcal{U}$ and an $m < 2$ so that $F''[Y]^{2n+2} = m$. It suffices to get a contradiction from the assumption that $m = 1$.

For α in such a Y, let $\alpha^1, \ldots, \alpha^n$ be the first n members of Y after α, and set

$$\mathcal{M}_\alpha = \langle V_{\alpha^n}; \in, \{\alpha\}, \{\alpha^1\}, \ldots, \{\alpha^{n-1}\}\rangle.$$

By the hypothesis $A_5(\kappa)$, there is an $X \in \mathcal{U}$ so that $\alpha, \beta \in X$ and $\alpha < \beta$ implies there is a $k: \mathcal{M}_\alpha \to \mathcal{M}_\beta$ with critical point α. But this would be contradictory for any $\alpha, \beta \in X \cap Y$ so that $\beta > \alpha^n$. □

Theorem 8.8. *Suppose $A_6^*(\kappa)$ and let S be a stationary class as provided. Then*

(i) *if $\alpha, \beta \in S$ and $\alpha < \beta$, the inclusion map $V_\alpha \to V_\beta$ is elementary,*

(ii) $\alpha \in S \to V_\alpha \prec V_\kappa$,

(iii) $\alpha \in S \to V_\kappa \models A_7(\alpha)$.

Proof. For (i), let $\alpha < \beta < \gamma_1 < \gamma_2 < \gamma_3$ all in S. Let $j: V_{\gamma_1} \to V_{\gamma_3}$ with critical point α and $j(\alpha) = \gamma_2$; and let $k: V_{\gamma_1} \to V_{\gamma_3}$ with critical point β and $k(\beta) = \gamma_2$. Now $j \mid V_\alpha: V_\alpha \to V_{\gamma_2}$ and $k \mid V_\beta: V_\beta \to V_{\gamma_2}$ are elementary, and both are identities. Hence, $V_\alpha \prec V_{\gamma_2}$ and $V_\beta \prec V_{\gamma_2}$, so that $V_\alpha \prec V_\beta$.

(ii) follows from (i) by union of elementary chains.

To show (iii), work within V_κ. Note first that each element of S is an extendible cardinal. Also, if $\alpha < \gamma_1 < \gamma_2 < \gamma_3$ all in S, let $j: V_{\gamma_1} \to V_{\gamma_3}$ with critical point α and $j(\alpha) = \gamma_2$. By 5.10, $V_{\gamma_3} \models \alpha$ is extendible. Hence, if $\mathcal{U}$ is the normal ultrafilter over α corresponding to j, $\{\beta < \alpha \mid \beta \text{ is extendible}\} \in \mathcal{U}$. □

Open Question 8.9. What is the relationship between A_6^* and A_6?

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