ON VOPĚNKA'S AND RELATED PRINCIPLES*

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The n-huge cardinals were given cursory treatment in [SRK], and this paper is a brief venture into their further study. Specifically, Vopěnka's Principle, which is also discussed in [SRK] and can be construed as a universal-algebraic principle, is generalized in direct analogy to Baumgartner's n-subtle and n-ineffable cardinals to yield new principles which cascade up alongside the n-huge cardinals. That Baumgartner's work lifts so neatly into a context of elementary embeddings is quite interesting. It should be mentioned that the direct ω-ary analogues of both his generalized ineffability concepts and the principles to be discussed in this paper are refuted in ZFC for similar reasons: roughly, the Axiom of Choice imposes a structural restriction on the universe by disallowing infinite exponent partition relations.

The results of this paper are indeed to be theorems of ZFC, although some initial discussion of the background material is better carried out formally in Kelley-Morse set theory. In general, the letters $\alpha,\beta,\gamma,\ldots$ denote ordinals whereas $\kappa,\lambda,\mu,\ldots$ are reserved for cardinals. V_α denotes the collection of sets of rank $< \alpha$. By $j : V \to M$ is meant an elementary embedding of the set theoretical universe into some inner model M. All elementary embeddings will be non-trivial (i.e. will shift some ordinal), and in the context of a fixed embedding j, κ_0 denotes the critical point (i.e. the least ordinal moved); for each integer n $\kappa_{n+1} = j(\kappa_n)$ whenever defined; and $\kappa_\omega = \sup\{ \kappa_n \mid n \in \omega \}$.

In the paper [SRK] we discuss several strong properties of large cardinal character formulated via elementary embeddings. Various natural motivating ideas are involved, and as a result a satisfying picture is presented of a superstructure of stronger and stronger principles over ZFC, ultimately delimited by a well-known result of Kunen. Kunen[K1]established that whenever $j : V \to M$, there is an $X \subseteq \kappa_\omega$ so that $X \notin M$. In particular, there is no $j : V \to V$. Define κ to be <u>n-huge</u> <u>iff</u> there is a $j : V \to M$ with critical point κ, so that ${}^{\kappa_n}M \subseteq M$, i.e. M is closed under arbitrary sequences of length κ_n. Thus, κ is measurable <u>iff</u> κ is 0-huge, and the main interest lies in the cases $n > 0$. The n-huge cardinals form into a hierarchy of weaker principles approaching Kunen's result, but not yet known to be refutable in ZFC. They are consistency-wise

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*This was not the topic of the author's talk at the conference.

much stronger than supercompactness (see [SRK]), and can be considered its generalizations through the imposition of stronger closure conditions on the range inner models of elementary embeddings. 1-huge cardinals have already been used to establish the consistency of interesting set theoretical phenomena: (a) the existence of a non-trivial ω_2-saturated ideal over ω_1 (Kunen[K2]; Laver has amplifications); and (b) the G.C.H. holding below ω_ω, yet failing there (Magidor [M]). With so little experience involving the n-huge cardinals (especially for $n > 1$), one should perhaps be duly cautious about asserting their consistency; but at any rate, it would seem that if they are inconsistenct, an entirely new argument beyond Kunen's is needed to show this. It is to be pointed out that n-hugeness is equivalent to the existence of an ultrafilter of a certain kind, and thus can be reduced formally to an existential statement in ZFC. Indeed, the methods of [SRK] easily establish that such an ultrafilter can be retreived from an elementary embedding of a sufficiently large initial portion of the universe:

Lemma 1: If there is a $j : V_\alpha \to V_\beta$ with critical point κ so that $\kappa_n < \alpha$, then κ is n-huge.

Let us now turn to the other thematic source for the present paper. The subtle and ineffable cardinals were first isolated in the context of Jensen's work on the Generalized Kurepa's Hypothesis in L. Baumgartner[B] formulated the following generalizations: For any positive integer n, κ is n-subtle (respectively, n-ineffable) whenever $\langle S_{\beta_1 \ldots \beta_n} \mid \beta_1 < \ldots < \beta_n < \kappa \rangle$ is such that $S_{\beta_1 \ldots \beta_n} \subseteq \beta_1$, then for every closed unbounded $C \subseteq \kappa$, there is a $Y \subseteq C$ so that $|Y| = n+1$ (respectively, stationary in κ) so that given any $\alpha_0 < \ldots < \alpha_n$ in Y, $S_{\alpha_0 \ldots \alpha_{n-1}} = \alpha_0 \cap S_{\alpha_1 \ldots \alpha_n}$. (Actually, Baumgartner's original definitions were slightly different, but are equivalent to the ones given; this is made explicit in 4.2 of [B] for subtlety, and the argument for n-ineffability is similar.) The subtle cardinals are just the 1-subtle cardinals, and the ineffable cardinals, the 1-ineffable cardinals. All these various cardinals are inaccessible, but compatible with $V = L$.

Let us temporarily indicate by $\kappa \overset{*}{\to} (\lambda)^n_\delta$ the proposition that whenever $f : [\kappa]^n \to \delta$, there is a stationary subset S of λ of order type λ so that S is homogeneous for f, i.e. $|f''[S]^n| = 1$. Kunen established that κ is ineffable iff $\kappa \overset{*}{\to} (\kappa)^2_2$. It is well-known that, deleting $*$, $\kappa \to (\kappa)^2_2$ is a characterization of weak compactness, and that it implies the ostensibly stronger principle: for every $n \in \omega$ and $\delta < \kappa$, $\kappa \to (\kappa)^n_\delta$. Baumgartner showed that κ is n-ineffable iff $\kappa \overset{*}{\to} (\kappa)^{n+1}_2$; that the n-ineffable cardinals below an n+1-subtle cardinal κ form a stationary subset of κ; and hence, in contrast to the weak compactness case, that $\kappa \overset{*}{\to} (\kappa)^n_2$ does not imply $\kappa \overset{*}{\to} (\kappa)^{n+1}_2$.

In his further tural study, Baumgartner goes on to show in a natural way that subtlety rather than ineffability seems to be the proper generalization of inaccessibility. Recent results of Stavi also support this view.

Let us immediately proceed to some definitions. Throughout, κ is to be an inaccessible cardinal, so V_κ models ZFC .

Definition: A sequence $< M_\alpha \mid \alpha < \kappa >$ of structures is called natural iff each $M_\alpha = < V_{f(\alpha)}, \varepsilon, \{\alpha\}, R_\alpha >$, where $R_\alpha \subseteq V_{f(\alpha)}$, and $\alpha < \beta < \kappa$ implies $\alpha < f(\alpha) \leq f(\beta) < \kappa$.

Through coding, we will also construe as natural those sequences where the R_α is replaced by a finite number of relations. The specification of $\{\alpha\}$ in M_α insures that whenever $\alpha < \beta$ and $j : M_\alpha \to M_\beta$ is elementary, j moves some ordinal, since $j(\alpha) = \beta$. In what follows, among all sequences $< M_\alpha \mid \alpha < \kappa >$ of structures of the same similarity type so that each $M_\alpha \in V_\kappa$, only the natural sequences are considered, to give more definite form to the discussion; however, it will be clear that this restriction is not an essential one. As in [B] and §6 of [SRK] a large cardinal property is defined not only for cardinals, but for subsets of a cardinal generally:

Definitions: If n is a positive integer and $X \subseteq \kappa$, X is Vopěnka-n-subtle (respectively, Vopěnka-n-ineffable) iff whenever $< M_\alpha \mid \alpha < \kappa >$ is a natural sequence, there is a $Y \subseteq X$ so that $|Y| = n+1$ (respectively, Y is stationary in κ) and given any $\alpha_0 < \ldots < \alpha_n$ in Y , there is an elementary embedding $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ so that α_0 is the critical point and $j(\alpha_i) = \alpha_{i+1}$ for every $i < n$.

An initial comment is that this definition is already a self-refinement, that the concept remains unchanged if "$j : M_{\alpha_{n-1}} \to M_{\alpha_n}$" is replaced by "$j : M_\gamma \to M_\delta$ for some $\gamma < \delta < \kappa$" : If the given sequence $< M_\alpha \mid \alpha < \kappa >$ is such that $M_\alpha = < V_{f(\alpha)}, \varepsilon, \{\alpha\}, R_\alpha >$, define an auxilary sequence $< N_\alpha \mid \alpha < \kappa >$ by $N_\alpha = < V_{f(\alpha)+\omega}, \varepsilon, \{\alpha\}, R_\alpha, < M_\beta \mid \beta < \alpha > >$. Then if $j : N_\gamma \to N_\delta$ is such that α_0 is the critical point with $j(\alpha_i) = \alpha_{i+1}$ for $i < n$, then $j|M_{\alpha_{n-1}} : M_{\alpha_{n-1}} \to M_{\alpha_n}$ has these same properties.

It is clear that in the terminology of [SRK], κ is Vopěnka-1-subtle iff $V_\kappa \models$ Vopěnka's Principle, and that an $X \subseteq \kappa$ is Vopěnka-1-subtle iff $\kappa - X$ is not enforceable. Hence, these concepts can be considered generalizations, in analogy to the n-subtle and n-ineffable cardinals. There is also a medium notion of n-almost ineffability which can be appropriately cast into the present context.

A technical remark may be in order here: For n-ineffability, the stationary

$Y \subseteq \kappa$ could be required to have the property that whenever $\beta_1 < \ldots < \beta_n$ and $\bar{\beta}_1 < \ldots < \bar{\beta}_n$ all in Y with $\beta_1 \leq \bar{\beta}_1$, then $S_{\beta_1 \ldots \beta_n} = \beta_1 \cap S_{\bar{\beta}_1 \ldots \bar{\beta}_n}$. No such freedom is possible in the present context. For $n = 2$ for instance, if $\beta_1 < \bar{\beta}_1 < \beta_2 = \bar{\beta}_2$, then to require $M_{\beta_2} \to M_{\bar{\beta}_2}$ with critical point β_1, $i(\beta_1) = \bar{\beta}_1$, and $i(\beta_2) = \bar{\beta}_2$ would be impossible: The least fixed point above β_1 for such an i is $\eta = \sup\{ i^n(\beta_1) \mid n \in \omega \}$, but $i(\beta_2) = \bar{\beta}_2 = \beta_2$ implies that $\eta \leq \beta_2$. We are thus in a situation where η corresponds exactly to κ_ω as in the introduction, and Kunen's argument can be used to derive a contradiction.

The following development owes an obvious debt to [B].

Definitions: For each $n > 0$,

$$\mathcal{F}_n = \{ X \subseteq \kappa \mid \kappa - X \text{ is not Vopěnka-n-subtle} \}, \text{ and}$$
$$\mathcal{G}_n = \{ X \subseteq \kappa \mid \kappa - X \text{ is not Vopěnka-n-ineffable} \}.$$

These are filters which are naturally related to the introduced concepts. $\mathcal{F}_n$ is improper, i.e. $\emptyset \in \mathcal{F}_n$, iff κ is not Vopěnka-n-subtle, and similarly for $\mathcal{G}_n$. In any case, if we admit improper filters, the following is always true:

Theorem 2: For each positive n, both $\mathcal{F}_n$ and $\mathcal{G}_n$ are normal κ-complete filters over κ. (Hence, $\mathcal{F}_n$ and $\mathcal{G}_n$ contain all closed unbounded subsets of κ.)

Proof: The last remark follows simply from the fact that a closed unbounded subset of κ is in every normal filter over κ. For definiteness, let us consider $\mathcal{F}_n$; the argument for $\mathcal{G}_n$ is entirely similar. It suffices to show that if $\{ X_\gamma \mid \gamma < \kappa \} \subseteq \mathcal{F}_n$, then the diagonal intersection $Y = \{ \delta < \kappa \mid \gamma < \delta$ implies $\delta \in X_\gamma \} \in \mathcal{F}_n$. (This in part would also establish κ-completeness.)

Assume to the contrary that $Y \notin \mathcal{F}_n$. Thus, $\kappa - Y$ is Vopěnka-n-subtle. Fix a function $F : (\kappa - Y) \to \kappa$ so that $F(\delta) < \delta$ and $\delta \notin X_{F(\delta)}$. For each $\gamma < \kappa$, since $X_\gamma \in \mathcal{F}_n$, choose a natural sequence $\langle M_\alpha^\gamma \mid \alpha < \kappa \rangle$ so that: whenever $\alpha_0 < \ldots < \alpha_n$ and there is a $j : M^\gamma_{\alpha_{n-1}} \to M^\gamma_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$, then $\{\alpha_0, \ldots, \alpha_n\} \subseteq X_\gamma$.

Now define a natural sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ by

$$N_\alpha = \langle V_{g(\alpha)+\omega}, \varepsilon, \{\alpha\}, \langle M^\gamma_\alpha \mid \gamma < \alpha \rangle, F|(\alpha - Y) \rangle,$$

where $V_{g(\alpha)}$ is the union of the domains of M^γ_α for $\gamma < \alpha$. Since by assumption $\kappa - Y$ is Vopěnka-n-subtle, let $\alpha_0 < \ldots < \alpha_n$ in $\kappa - Y$ so that there is a $j : N_{\alpha_{n-1}} \to N_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$. Using F, we have that if $\rho = F(\alpha_0) < \alpha_0$, then $F(\alpha_1) = F(j(\alpha_0)) = j(F(\alpha_0)) = F(\alpha_0) = \rho$, as α_0 was the first ordinal moved. One can now inductively proceed to show that $F(\alpha_i) = \rho$ for each $i \leq n$. Thus, $\{\alpha_0, \ldots, \alpha_n\} \subseteq \kappa - X_\rho$ yet $j|M^\rho_{\alpha_{n-1}} : M^\rho_{\alpha_{n-1}} \to M^\rho_{\alpha_n}$ is clearly elementary, a contradiction. ⫞

The next several theorems connect the new concepts with the old.

Theorem 3: For any $n > 0$, $\{ \alpha < \kappa \mid \alpha \text{ is } (n-1)\text{-huge} \} \in F_n$.

Proof: This is immediate: Let $H_n = \{ \alpha < \kappa \mid \alpha \text{ is } (n-1)\text{-huge} \}$. If to the contrary $H_n \notin F_n$, then $\kappa - H_n$ would be Vopěnka-n-subtle. Letting $M_\alpha = \langle V_{\alpha+\omega} , \varepsilon , \{\alpha\} \rangle$ for $\alpha < \kappa$, there must be $\alpha_0 < \ldots < \alpha_n$ in $\kappa - H_n$ so that there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$. However, by Lemma 1 this implies that α_0 is $(n-1)$-huge, a contradiction. ⊣

Concerning the above for the case $n = 1$, the main thrust of §6 of [SRK] was to prove something much stronger in the context of that paper: $\{ \alpha < \kappa \mid V_\kappa \models \alpha \text{ is extendible} \} \in F_1$.

Theorem 4: If κ is Vopěnka-n-subtle (respectively, Vopěnka-n-ineffable), then κ is n-subtle (respectively, n-ineffable).

Proof: Assume κ is Vopěnka-n-subtle; the ineffable case is similar. Given a sequence $\langle S_{\beta_1\ldots\beta_n} \mid \beta_1 < \ldots < \beta_n < \kappa \rangle$ so that $S_{\beta_1\ldots\beta_n} \subseteq \beta_1$ and a closed unbounded $C \subseteq \kappa$, we must find $\alpha_0 < \ldots < \alpha_n$ so that $S_{\alpha_0\ldots\alpha_{n-1}} = \alpha_0 \cap S_{\alpha_1\ldots\alpha_n}$.

To this end, let $M_\alpha = \langle V_{\alpha+\omega} , \varepsilon , \{\alpha\} , \langle S_{\beta_1\ldots\beta_n} \mid \beta_1 < \ldots < \beta_n \leq \alpha \rangle \rangle$. Then $\langle M_\alpha \mid \alpha < \kappa \rangle$ is natural, so by hypothesis and Theorem 2, let $\alpha_0 < \ldots < \alpha_n$ in C so that there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$. Thus $j(S_{\alpha_0\ldots\alpha_{n-1}}) = S_{\alpha_1\ldots\alpha_n}$. Also since $S_{\alpha_0\ldots\alpha_{n-1}} \subseteq \alpha_0$, a standard argument shows that $j(S_{\alpha_0\ldots\alpha_{n-1}}) \cap \alpha_0 = S_{\alpha_0\ldots\alpha_{n-1}}$. Hence, $S_{\alpha_0\ldots\alpha_{n-1}} = \alpha_0 \cap S_{\alpha_1\ldots\alpha_n}$, which was to be proved. ⊣

The following subsumes A_5 in §8 of [SRK] .

Theorem 5: If $n > 0$ and κ is n-huge, then there is a normal ultrafilter U over κ so that whenever $\langle M_\alpha \mid \alpha < \kappa \rangle$ is a natural sequence, there is an $X \in U$ so that for any $\alpha_0 < \ldots < \alpha_n$ in X there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$. In particular, κ is Vopěnka-n-ineffable.

Proof: The proof works by a sort of unravelling; we take the case $n = 2$ for notational definiteness, though the proof is completely general. Thus, let $j : V \to M$ with critical point κ so that ${}^{\kappa_2}M \subseteq M$. Define U by:

$$X \in U \ \underline{\text{iff}} \ X \subseteq \kappa \ \& \ \kappa \in j(X) .$$

Standard arguments show that U is a normal ultrafilter over κ .

Now let $\langle M_\alpha \mid \alpha < \kappa \rangle$ be a natural sequence. For $\alpha < \beta < \kappa$, define $X_{\alpha\beta} = \{ \gamma < \kappa \mid$ there is an $i : M_\beta \to M_\gamma$ with critical point α , $i(\alpha) = \beta$, and

$i(\beta) = \gamma \}$, and $Y_\alpha = \{ \beta \mid X_{\alpha\beta} \in U \}$. Finally, set:

$$< M^*_\alpha \mid \alpha < \kappa_1 > \; = j(< M_\alpha \mid \alpha < \kappa >)$$
$$< M^{**}_\alpha \mid \alpha < \kappa_2 > \; = j^2(< M_\alpha \mid \alpha < \kappa >)$$
$$< M^{***}_\alpha \mid \alpha < \kappa_3 > \; = j^3(< M_\alpha \mid \alpha < \kappa >)$$
$$< X^*_{\alpha\beta} \mid \alpha < \beta < \kappa_1 > \; = j(< X_{\alpha\beta} \mid \alpha < \beta < \kappa >)$$
$$< Y^*_\alpha \mid \alpha < \kappa_1 > \; = j(< Y_\alpha \mid \alpha < \kappa >)$$

<u>Claim</u>: $T = \{ \alpha < \kappa \mid Y_\alpha \in U \} \in U$.

Firstly, note that for $\alpha < \beta < \kappa$, we have $X_{\alpha\beta} \in U$ <u>iff</u> $\kappa \in j(X_{\alpha\beta})$ <u>iff</u> $M \models$ "there is an $i : M^*_\beta \to M^*_\kappa$ with critical point α , $i(\alpha) = \beta$, and $i(\beta) = \kappa$ " <u>iff</u> there is an $i : M_\beta \to M^*_\kappa$ with critical point α , $i(\alpha) = \beta$, and $i(\beta) = \kappa$. (This last equivalence follows since $M^*_\beta = M_\beta$ for $\beta < \kappa$, and such an i is just a set (of ordered pairs of elements in M) of cardinality $< \kappa < \kappa_2$, and thus would exist in M just in case it (really) exists.) Hence, by elementarity, for $\alpha < \beta < \kappa_1$, we have $X^*_{\alpha\beta} \in j(U)$ <u>iff</u> there is an $i : M^*_\beta \to M^{**}_{\kappa_1}$ with critical point α , $i(\alpha) = \beta$, and $i(\beta) = \kappa_1$. (Again, an absoluteness remark applies here, since such an i has cardinality $< \kappa_2$.)

Secondly, note that it follows for $\alpha < \kappa$, that $Y_\alpha \in U$ <u>iff</u> $\kappa \in j(Y_\alpha)$ <u>iff</u> $X^*_{\alpha\kappa} \in j(U)$ <u>iff</u> there is an $i : M^*_\kappa \to M^{**}_{\kappa_1}$ with critical point α , $i(\alpha) = \kappa$, and $i(\kappa) = \kappa_1$. Hence, by elementarity, for $\alpha < \kappa_1$, we have $Y^*_\alpha \in j(U)$ <u>iff</u> there is an $i : M^{**}_{\kappa_1} \to M^{***}_{\kappa_2}$ with critical point α , $i(\alpha) = \kappa_1$, and $i(\kappa_1) = \kappa_2$. (Once again, an absoluteness remark applies.)

Finally, $T \in U$ <u>iff</u> $\kappa \in j(T)$ <u>iff</u> $Y^*_\kappa \in j(U)$ iff there is an $i : M^{**}_{\kappa_1} \to M^{**}_{\kappa_2}$ with critical point κ , $i(\kappa) = \kappa_1$, and $i(\kappa_1) = \kappa_2$. But there is such an i, to wit $j|M^{**}_{\kappa_1}$. Thus, the claim is proved.

The normality of U is now applied twice. For any $\alpha \in T$, since $Y_\alpha = \{ \beta \mid X_{\alpha\beta} \in U \} \in U$, the diagonal intersection $\bar{Y}_\alpha = \{ \gamma \in Y_\alpha \mid \beta < \gamma \;\&\; \beta \in Y_\alpha$ implies $\gamma \in X_{\alpha\beta} \} \in U$. Again by normality, it follows that

$$\bar{T} = \{ \beta \in T \mid \alpha < \beta \;\&\; \alpha \in T \text{ implies } \beta \in \bar{Y}_\alpha \} \in U .$$

But now we are done, since if $\alpha_0 < \alpha_1 < \alpha_2$ in $\bar{T}$, then $\alpha_2 \in \bar{Y}_{\alpha_0}$ and $\alpha_1 \in Y_{\alpha_0}$ so that $\alpha_2 \in X_{\alpha_0\alpha_1}$ by definition, i.e. there is an $i : M_{\alpha_1} \to M_{\alpha_2}$ with critical point α_0, $i(\alpha_0) = \alpha_1$, and $i(\alpha_1) = \alpha_2$. ⊣

The next technical theorem is similar to 4.1 of [B] .

<u>Theorem 6</u>: Suppose $X \subseteq \kappa$ is Vopěnka-n-subtle, $< M_\alpha \mid \alpha < \kappa >$ a natural sequence, and $Y = \{ \alpha \in X \mid$ there is a normal ultrafilter U_α over α and an $X_\alpha \in U_\alpha$ so that whenever $\alpha_0 < \ldots < \alpha_n$ in X_α , there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with

critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$ }. Then $X - Y$ is not Vopěnka-n-subtle.

Proof: Assume to the contrary that $X - Y$ is Vopěnka-n-subtle. Define $< N_\alpha \mid \alpha < \kappa >$ by $N_\alpha = < V_{f(\alpha)+\omega}, \varepsilon, \{\alpha\}, < M_\beta \mid \beta \leq \alpha >, \{< M_\beta \mid \beta \leq \alpha >\}>$, where $V_{f(\alpha)}$ is the domain of M_α. Then by Vopěnka-n-subtlety, there are $\alpha_0 <\ldots< \alpha_n$ in $X - Y$ and a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$.

We are now in an analogous position to that of Theorem 5, and can conclude the proof by showing that $\alpha_0 \in Y$, yielding a contradiction. The main features for the reduction to the proof of Theorem 5 for the illustrative case $n = 2$ are as follows:

(i) Let α_i correspond to κ_i for $i \leq 2$, and define U, $X_{\alpha\beta}$, Y_α, and T as before.

(ii) Get the characterizations of membership of these sets in U in terms of the existence of embeddings. A crucial point is that if $j(< M_\alpha \mid \alpha \leq \alpha_1 >) = < M^*_\alpha \mid \alpha \leq \alpha_2 >$, then by elementarity on the last, singleton component of the N_α's we have $< M^*_\alpha \mid \alpha \leq \alpha_2 > = < M_\alpha \mid \alpha \leq \alpha_2 >$, so that "starring" the natural sequence is not necessary. Also, the absoluteness considerations about the various embeddings i still apply.

(iii) Conclude that $T \in U$ and as before, complete the proof with two applications of normality. ⊣

For the notion of a Π^m_n-indescribable cardinal, see Lévy[L]. Again following [B], let us call an $X \subseteq \kappa$ Π^m_n-indescribable iff whenever $R \subseteq V_\kappa$, ϕ is Π^m_n, and $< V_\kappa, \varepsilon, R > \models \phi$, then there is an $\beta \in X$ so that $< V_\beta, \varepsilon, R \cap V_\beta > \models \phi$. Lévy showed that for each $m,n > 0$,

$$H^m_n = \{ X \subseteq \kappa \mid \kappa - X \text{ is not } \Pi^m_n\text{-indescribable} \}$$

is a normal κ-complete filter over κ (improper just in case κ is not Π^m_n-indescribable). The next corollary is the exact analogue of 7.2 in [B], and the proof is essentially unchanged; two filters are said to be coherent if the filter generated by their union is a (proper) filter.

Corollary 7: For any $n > 0$, κ is Vopěnka-n-ineffable iff H^1_2 and F_n are coherent.

Proof: If κ is Vopěnka-n-ineffable, then G_n is a (proper) filter extending F_n. Also, it is well known that if $Y \subseteq \kappa$ is n-ineffable, then Y is Π^1_2-indescribable (see 7.1 of [B]). Suppose now that $X \subseteq \kappa$ is in H^1_2, i.e. $\kappa - X$ is not Π^1_2-indescribable. Then $\kappa - X$ is not n-ineffable, and by the relativized version of Theorem 4, $\kappa - X$ is not Vopěnka-n-ineffable. Hence, $X \in G_n$. Thus H^1_2 and F_n are coherent, both being extended by G_n.

For the converse, first note that κ is both Vopěnka-n-subtle and Π^1_2-indescribable, else either H^1_2 or F_n would already be improper and hence they would be incoherent. Assume now that $< M_\alpha \mid \alpha < \kappa >$ is a natural sequence. Then since κ is Vopěnka-n-subtle, it follows easily from Theorem 6 that $Y = \{ \alpha < \kappa \mid \alpha$ is inaccessible and there is a set X_α stationary in α so that whenever $\alpha_0 < \ldots < \alpha_n$ in X_α, there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n \}$ is a member of F_n.

By the assumption of coherence, we must have that Y is Π^1_2-indescribable. Let ϕ be : $\forall X \exists C$(whenever $\alpha_0 < \ldots < \alpha_n$ in X, there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$, then C is closed unbounded and $C \cap X = \emptyset$). Clearly, ϕ is Π^1_2 in $< M_\alpha \mid \alpha < \kappa >$, and $< V_\kappa , \varepsilon , < M_\alpha \mid \alpha < \kappa > > \models \phi$ <u>iff</u> $< M_\alpha \mid \alpha < \kappa >$ is a counterexample to the Vopěnka-n-ineffability of κ. Now if $M_\alpha = < V_{f(\alpha)} , \ldots >$ for $\alpha < \kappa$, then $D = \{ \alpha < \kappa \mid f : \alpha \to \alpha \}$ is closed unbounded in κ, as κ is inaccessible. Hence, $D \in H^1_2$, so that $Y \cap D$ is still a Π^1_2-indescribable subset of κ. But if $< V_\kappa , \varepsilon , < M_\alpha \mid \alpha < \kappa >> \models \phi$, there must be a $\beta \in Y \cap D$ so that $< V_\beta , \varepsilon , < M_\alpha \mid \alpha < \kappa > \cap V_\beta > \models \phi$. As $\beta \in D$, $< M_\alpha \mid \alpha < \kappa > \cap V_\beta = < M_\alpha \mid \alpha < \beta >$, thus contradicting the existence of the stationary X_α. Hence, $< V_\kappa , \varepsilon , < M_\alpha \mid \alpha < \kappa > > \models \neg\phi$. Since $< M_\alpha \mid \alpha < \kappa >$ was arbitrary, it follows that κ is Vopěnka-n-ineffable. $\dashv$

The paper is now concluded with some observations which first motivated it. The very strong principle "there is a $j : V_{\kappa_\omega} \to V_{\kappa_\omega}$", though very close to the proposition proved inconsistent by Kunen, has thus far evaded all attempts to prove its inconsistency. With κ_0 the critical point of j and U the normal ultrafilter over κ_0 corresponding to j as in Theorem 5, the proof of that theorem indicates that whenever $n \leq m < \omega$, $G_n \subseteq G_m \subseteq U$. In fact, by taking a countable intersection, we have that: whenever $< M_\alpha \mid \alpha < \kappa >$ is a natural sequence, there is an $S \in U$ so that <u>for any n</u> and $\alpha_0 < \ldots < \alpha_n$ in S, there is a $j : M_{\alpha_{n-1}} \to M_{\alpha_n}$ with critical point α_0 and $j(\alpha_i) = \alpha_{i+1}$ for $i < n$. We can appropriately abstract this property and call a cardinal κ <u>Vopěnka-Ramsey iff</u> the above property holds for κ with "$S \in U$" replaced by "S is stationary in κ". Then a Vopěnka-Ramsey cardinal would in particular be Ramsey, and we have the version of Ramseyness appropriate for the present context. The author has so far been unsuccessful in pushing these methods far enough to show the inconsistency of such ω-ary properties with ZFC .

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