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ON P-POINTS OVER A MEASURABLE CARDINAL

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This paper continues the study of κ-ultrafilters over a measurable cardinal κ, following the sequence of papers Ketonen [2], Kanamori [1] and Menas [4]. Much of the concern will be with p-point κ-ultrafilters, which have become a focus of attention because they epitomize situations of further complexity beyond the better understood cases, normal and product κ-ultrafilters.

For any κ-ultrafilter D, let $i_D: V \to M_D \simeq V^\kappa/D$ be the elementary embedding of the universe into the transitization of the ultrapower by D. Situations of $U <_{\mathrm{RK}} D$ will be exhibited when $i_U(\kappa) < i_D(\kappa)$, and when $i_U(\kappa) = i_D(\kappa)$. The main result will then be that if the latter case obtains, then there is an inner model with two measurable cardinals. (As will be pointed out, this formulation is due to Kunen, and improves on an earlier version of the author.) Incidentally, a similar conclusion will also follow from the assertion that there is an ascending Rudin-Keisler chain of κ-ultrafilters of length $\omega + 1$. The interest in these results lies in the derivability of a substantial large cardinal assertion from plausible hypotheses on κ-ultrafilters.

§1 discusses the necessary preliminaries, but also includes a digression on q-point and rapid (semi-q-point) κ-ultrafilters; §2 reviews distinctive cases involving $i_U(\kappa)$ and $i_D(\kappa)$; and §3 contains the main result on large cardinal consequences, with a prefatory discussion which frames the result among related work of Kunen, Ketonen, and Menas.

§1. Preliminaries. My set theory is ZFC, and here follows a litany on basic notation: The letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, whereas $\kappa, \lambda, \mu, \ldots$ are reserved for infinite cardinals. $|x|$ is the cardinality of x, $P(x)$ is its power set, and if f is a function with domain including x, $f''x = \{f(y) | y \in x\}$. ${}^y x$ denotes the collection of functions: $y \to x$, so that λ^κ is the cardinality of ${}^\kappa\lambda$.

A *κ-ultrafilter* is a nonprincipal κ-complete ultrafilter over κ, and hence a witness to the measurability of κ when $\kappa > \omega$. The reader is referred to Kanamori [1]—hereby the reference for any unreferenced facts used in this paper—for a structure theory for κ-ultrafilters over a measurable cardinal κ. This theory is analogous to the extensive one for βN (the Stone-Cech compactification of the integers), which is identifiable with the case $\kappa = \omega$. However, new considerations involving the closed unbounded filter and wellfoundedness of ultrapowers make the $\kappa > \omega$ case quite distinctive.

A function $f \in {}^\kappa\kappa$ is *almost injective iff* $|f^{-1}(\{\alpha\})| < \kappa$ for every $\alpha < \kappa$. If U is an ultrafilter, a function $f \in {}^\kappa\kappa$ is *injective* (mod U) (*almost injective* (mod U), respectively) *iff* f is equal to an injective (almost injective, respectively) function

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$\in {}^{\kappa}\kappa$ on a set in U. A κ-ultrafilter U is a *p-point* iff every $f \in {}^{\kappa}\kappa$ is either constant (mod U), or almost injective (mod U). p-points first arose in the study of the inhomogeneity of βN. The basis of the theory in Kanamori [1] is Puritz' division of ultrapowers of κ into *skies*, and p-points are exactly those κ-ultrafilters which yield ultrapowers of κ with exactly one (nontrivial) sky. Thus, there is a natural interest in the structure of p-points. A well-known example of a p-point is a normal κ-ultrafilter.

A basic tool in the study of ultrafilters is the Rudin-Keisler (RK) partial ordering, which is briefly reviewed here for κ-ultrafilters: If U is a κ-ultrafilter and $f \in {}^{\kappa}\kappa$, let $f_*(U)$ be the ultrafilter defined by: $X \in f_*(U)$ *iff* $f^{-1}(X) \in U$. $f_*(U)$ is also a κ-ultrafilter just in case f is not constant (mod U). Roughly speaking, $f_*(U)$ is computable via a projection from U, and so is of lesser complexity. If U and D are κ-ultrafilters, then $U \leq_{\mathrm{RK}} D$ *iff* there is a function $f \in {}^{\kappa}\kappa$ such that $f_*(D) = U$. $U \simeq_{\mathrm{RK}} D$ iff $U \leq_{\mathrm{RK}} D$, $D \leq_{\mathrm{RK}} U$. It is well known that $U \simeq_{\mathrm{RK}} D$ *iff* there is a bijection $f \in {}^{\kappa}\kappa$ such that $f_*(D) = U$. $U <_{\mathrm{RK}} D$ *iff* $U \leq_{\mathrm{RK}} D$ but $U \neq_{\mathrm{RK}} D$. When $\kappa > \omega$, an RK-minimal κ-ultrafilter is just one RK-equivalent to a normal κ-ultrafilter, and if D is a p-point κ-ultrafilter, there is a unique normal κ-ultrafilter $\leq_{\mathrm{RK}} D$.

Suppose $\kappa > \omega$ and U is a κ-ultrafilter. Then whenever $f: \kappa \to V$, $[f]_U$ will denote the equivalence class of f (mod U) in M_U, the transitization of the ultrapower of V by U. id will be the identity function $\in {}^{\kappa}\kappa$. If $f_*(D) = U$, there is a natural elementary embedding $k: M_U \to M_D$ defined by: $k([g]_U) = [g \cdot f]_D$. This embedding satisfies $i_D = k \cdot i_U$.

From here until the end of this section, I digress with some remarks which are not pertinent to the rest of the paper, but nonetheless provide an interesting point of comparison between the cases $\kappa = \omega$ and $\kappa > \omega$.

DEFINITION 1.1. For a κ-ultrafilter U,

(i) U is a q-point (or rare) *iff* every $f \in {}^{\kappa}\kappa$ almost injective (mod U) is injective (mod U).

(ii) U is *rapid* (or semi-q-point) *iff* whenever $f \in {}^{\kappa}\kappa$ is normal (i.e., strictly increasing and continuous at limits), there is an $X \in U$ such that if $g: \kappa \to X$ is the ascending enumeration of X, then $f(\alpha) \leq g(\alpha)$ for every $\alpha < \kappa$.

(iii) U is an *r-point* *iff* whenever $f \in {}^{\kappa}\kappa$ is almost injective, there is an $X \in U$ such that $|f^{-1}(\{\alpha\}) \cap X| \leq \alpha$ for every $\alpha < \kappa$.

The first concept is due to Choquet, the second to Mokobodzki, and the third to Ketonen. It is easy to see that a q-point is both rapid and an r-point, and that when $\kappa > \omega$, a normal κ-ultrafilter is a q-point. If U and D are κ-ultrafilters, then $U \times D$ (with the standard definition of a product of ultrafilters) is neither a p-point nor a q-point. However, if $\kappa > \omega$ and D is normal, then $U \times D$ is an r-point, and there is an appropriate converse (see Ketonen [2, 2.4]). Rapid κ-ultrafilters for the case $\kappa = \omega$ have recently figured in a consistency result of Miller [5], who established that if ω_2 Laver reals are adjoined to any model of ZFC + CH, there are no rapid ω-ultrafilters in the resulting generic extension.

The following lemma is straightforward.

LEMMA 1.2. (a) (*Mokobodzki*) *U is rapid iff whenever $f \in {}^{\kappa}\kappa$ is almost injective, there is a $g \in {}^{\kappa}\kappa$ such that $\{\alpha | g(\alpha) \leq f(\alpha)\} \in U$ and g is injective* (mod U) *and order-preserving.*

(b) *If U is rapid, then U is an r-point.*

PROOF. Assume first that U is rapid and $f \in {}^\kappa\kappa$ is almost injective. Set $\eta_\alpha = \sup\{\delta + 1 | f(\delta) < \alpha\} < \kappa$, for $\alpha < \kappa$, and let $g \in {}^\kappa\kappa$ be a normal function such that $\eta_\alpha \leq g(\alpha)$ for every $\alpha < \kappa$. Finally, by rapidity let $X \in U$ such that if $h: \kappa \to X$ is the ascending enumeration of X, then $g(\alpha) \leq h(\alpha)$ for every $\alpha < \kappa$. Now $\gamma \in f^{-1}(\{\alpha\})$ implies that $\gamma < h(\alpha + 1)$, so certainly $|X \cap f^{-1}(\{\alpha\})| \leq \alpha$. This establishes (b). Note also that h^{-1} is injective and $h^{-1}(\beta) \leq f(\beta)$ for $\beta \in X$. This establishes the forward direction of (a).

To establish the converse direction of (a), suppose that $F \in {}^\kappa\kappa$ is a normal function. Set $f(\beta) = \alpha$ *iff* $F(\alpha) \leq \beta < F(\alpha + 1)$. By assumption there is a g such that $X = \{\alpha | g(\alpha) \leq f(\alpha)\} \in U$ and g is injective. Let $h: \kappa \to X$ be the ascending enumeration of X. Since f is nondecreasing and g is order-preserving, we have that $\alpha \leq f(h(\alpha))$ for every $\alpha < \kappa$. By definition of f, this implies that $F(\alpha) \leq h(\alpha)$ for every $\alpha < \kappa$. Hence, U is rapid. $\dashv$

Here is the presaged distinction between $\kappa = \omega$ and $\kappa > \omega$:

THEOREM 1.3. (i) *If $\kappa = \omega$, then rapid is equivalent to r-point, and so both are implied by q-point.*

(ii) *If $\kappa > \omega$, then rapid is equivalent to q-point, and so both imply r-point.*

PROOF. For (i), what remains to be established is that an r-point ω-ultrafilter is rapid. So, assume that U is an r-point ω-ultrafilter and $F \in {}^\omega\omega$ is strictly increasing. Define $g \in {}^\omega\omega$ inductively by: $g(0) = 0$, and $g(n + 1) = g(n) + n + 2$. Set $f(m) = 0$ *iff* $m < F(0)$, and $f(m) = n + 1$ *iff* $F(g(n)) \leq m < F(g(n + 1))$. Then by assumption, let $X \in U$ such that $|X \cap f^{-1}(\{n\})| \leq n$ for every $n \in \omega$. If $h: \omega \to X$ is the ascending enumeration of X, then: $F(0) \leq h(0)$; $F(2) \leq h(1), h(2)$; $F(5) \leq h(3), h(4), h(5)$; etc., and certainly $F(n) \leq h(n)$ for each $n \in \omega$.

For (ii), what remains to be established is that when $\kappa > \omega$, then a rapid κ-ultrafilter is a q-point. So, assume that U is a rapid κ-ultrafilter. By wellfoundedness of ultrapowers, there is a *least* almost injective function (mod U). By the previous lemma, this function must be injective (mod U). But by Kanamori [1, 3.2(ii)], this is enough to establish that U is a q-point. $\dashv$

Using CH or MA, a construction of a familiar sort can be used to produce an r-point ω-ultrafilter which is not a q-point. For $\kappa > \omega$, Ketonen's result on products alluded to earlier shows that an r-point κ-ultrafilter is not necessarily a q-point. What is interesting is that among several notions formulated by different people, rapidity seems to be a relative concept between the cases $\kappa = \omega$ and $\kappa > \omega$.

§2. **p-points and embeddings.** From this section on I definitely settle on the case $\kappa > \omega$. Suppose that D is a p-point κ-ultrafilter, and let $U \leq_{\text{RK}} D$, say $f_*(D) = U$. The corresponding elementary embedding $k: M_U \to M_D$ given by $k([g]_U) = [g \cdot f]_D$ is such that $k(i_U(\kappa)) = i_D(\kappa)$, so certainly $i_U(\kappa) \leq i_D(\kappa)$. It is a consequence of general facts about p-points (see [1, 2.3(ii)]) that $k''(i_U(\kappa)) = \{[g \cdot f]_D | g \in {}^\kappa\kappa\}$ is cofinal in $i_D(\kappa)$. Hence, it becomes interesting to look at the nontrivial case $U <_{\text{RK}} D$ and consider when $i_U(\kappa) < i_D(\kappa)$ and when $i_U(\kappa) = i_D(\kappa)$. In this section, known constructions are reviewed to produce examples of both situations. Notice that strong large cardinal hypotheses are employed; that some such assumption is necessary at least in one case will be the topic of the next section.

A case when $i_U(\kappa) < i_D(\kappa)$. Kunen constructed a very canonical sequence of p-points using a relatively strong hypothesis; the following example is taken from his construction: Let κ be 2^κ-supercompact, i.e. there is an elementary embedding $j: V \to M$ where M is an inner model closed under arbitrary sequences of length 2^κ, κ is the first ordinal moved by j, and $2^\kappa < j(\kappa)$. Kunen's main tool was the following:

LEMMA 2.1. *If* $\alpha \leq \beta < (2^\kappa)^+$, *there is an almost injective function* $f \in {}^\kappa\kappa$ *such that* $j(f)(\beta) = \alpha$.

For the proof of this lemma as well as Kunen's construction, see [1, 4.11]. Notice that, in general, whenever $\kappa \leq \theta < j(\kappa)$, if U_θ is defined by:

$$X \in U_\theta \text{ iff } X \subseteq \kappa \;\&\; \theta \in j(X),$$

then it is easy to ascertain that U_θ is a κ-ultrafilter.

Turning to the present goal, first set $U = U_\kappa$. Then U is a normal κ-ultrafilter, and $i_U(\kappa) < (2^\kappa)^+$ is a standard fact. Next, note that as M is closed under sequences of length 2^κ,

$$(2^\kappa)^M = 2^\kappa < (2^\kappa)^+ = (2^\kappa)^{+M} < j(\kappa).$$

So, if we fix a δ such that $i_U(\kappa) < \delta < (2^\kappa)^+$, then $D = U_\delta$ is a κ-ultrafilter. Moreover, with Lemma 2.1, one can check that D is a p-point such that $U <_{\mathrm{RK}} D$. The point is now the following.

Claim. $i_U(\kappa) < i_D(\kappa)$.

This claim follows from a look at embeddings, just as in Menas [4, 2.2]. First, note that if $e: M_D \to M$ is defined by $e([f]_D) = j(f)(\delta)$, then e is an elementary embedding such that $e \cdot i_D = j$. Notice that $e(\kappa) = \kappa$, since $j(f)(\delta) = \kappa$ *iff* f is a least nonconstant function (mod D). Hence, standard arguments show that:

$$(2^\kappa)^{M_D} = 2^\kappa < (2^\kappa)^{+M_D} < i_D(\kappa) < (2^\kappa)^+,$$

so that e first moves the ordinal $(2^\kappa)^{+M_D}$ (to $(2^\kappa)^+$, of course). But $e([\mathrm{id}]_D) = \delta$ by the definition of e. Since $\delta < (2^\kappa)^+$, it follows that $[\mathrm{id}]_D = \delta$, and the conclusion $i_U(\kappa) < \delta < i_D(\kappa)$ now follows.

A case when $i_U(\kappa) = i_D(\kappa)$. Here, an example due to Ketonen [2] is used. Assume that κ is a measurable cardinal and a limit of measurable cardinals. Let U be a normal κ-ultrafilter, and for μ measurable and less than κ, let N_μ be a normal μ-ultrafilter. For $\alpha < \kappa$ set $m(\alpha) =$ the least measurable cardinal $> \alpha$. Finally, define D by:

$$X \in D \text{ iff } X \subseteq \kappa \;\&\; \{\alpha < \kappa \mid X \cap m(\alpha) \in N_{m(\alpha)}\} \in U.$$

If $A \subseteq \kappa$ is the closure (in the order topology) of the set of measurable cardinals below κ, $A \in U$ as U is normal and A is closed unbounded, and also, $B = \{\beta \mid \exists \alpha \in A(\alpha < \beta < m(\alpha))\} \in D$. On B, define a function f by: $f(\beta) = \alpha$ *iff* $\alpha < \beta < m(\alpha)$ & $\alpha \in A$. Then it is straightforward to check that f is a least nonconstant function (mod D), $f_*(D) = U$, and $U <_{\mathrm{RK}} D$. (See [2, 3.6] or [1, 4.6].) The point is now the following.

Claim. $i_U(\kappa) = i_D(\kappa)$.

To establish this claim, recall that as typical of a general fact mentioned at the beginning of this section, $\{[g \cdot f]_D | g \in {}^\kappa\kappa\}$ is cofinal in $i_D(\kappa)$. In other words, the ordinals represented by functions $\in {}^\kappa\kappa$ that are constant on the intervals $(\alpha, m(\alpha))$ for $\alpha \in A$ are cofinal in $i_D(\kappa)$, and the set of these ordinals has order-type $i_U(\kappa)$. Hence, it suffices to prove the following.

LEMMA 2.2. *If $g \in {}^\kappa\kappa$, then there is an order-preserving injection of $[g \cdot f]_D$ into $i_U(\kappa)$.*

The justification here is that since the lemma says that $[g \cdot f]_D \le i_U(\kappa)$ for all $g \in {}^\kappa\kappa$, we have $i_D(\kappa) = \sup\{[g \cdot f]_D | g \in {}^\kappa\kappa\} \le i_U(\kappa)$, thereby establishing the claim.

Turning to the proof of Lemma 2.2, fix $g \in {}^\kappa\kappa$. Notice that $[h]_D < [g \cdot f]_D$ *iff* $\{\alpha \in A | \{\beta \in m(\alpha) | h(\beta) < g(\alpha)\} \in N_{m(\alpha)}\} \in U$. Thus, if a function ϕ with domain $[g \cdot f]_D$ is defined by:

$$\phi([h]_D) = [\langle [h | m(\alpha)]_{N_{m(\alpha)}} | \alpha \in A \rangle]_U,$$

then ϕ is an order-preserving injection of $[g \cdot f]_D$ onto $[\langle \gamma_\alpha | \alpha \in A \rangle]_U$, where γ_α is the order-type of the set $\{[s]_{N_{m(\alpha)}} | s \in {}^{m(\alpha)}g(\alpha)\}$. (Notice that this fact about ϕ relies heavily on the special definition of D.) However, $\gamma_\alpha < \kappa$ for $\alpha \in A$, and so ϕ certainly injects $[g \cdot f]_D$ into $i_U(\kappa)$, proving the lemma.

It turns out that the D in the preceding example is a "two-constellation p-point" (i.e., if $E <_{\mathrm{RK}} D$, then $E \simeq_{\mathrm{RK}}$ the unique normal κ-ultrafilter U below D). For such p-points, is it always the case that $i_U(\kappa) = i_D(\kappa)$?

§3. **Large cardinal consequences.** This section contains the main result of the paper, and the following remarks relate it to the study of p-points. If U is a normal κ-ultrafilter, then Kunen [3, 7.6] established that inside $L[U]$, the universe relatively constructible from U, κ is still measurable, yet the only p-points are those RK-isomorphic to the (unique) normal κ-ultrafilter, $U \cap L[U]$. Thus, stronger hypotheses than mere measurability are necessary to produce non-RK-minimal p-points. As indicated in §2, Ketonen constructed such κ-ultrafilters assuming κ to be a measurable limit of measurable cardinals. Menas [4, §3] then established that the κ in Ketonen's example can be made the *least* measurable cardinal by forcing, and still retain a non-RK-minimal p-point D, which by remarks in §2 has a $U <_{\mathrm{RK}} D$ such that $i_U(\kappa) = i_D(\kappa)$.

It will be shown (see 3.2) that in general whenever $\kappa > \omega$ and there are κ-ultrafilters $U <_{\mathrm{RK}} D$ such that $i_U(\kappa) = i_D(\kappa)$, then there is an inner model with two measurable cardinals. Thus, the large cardinal strength Menas used is necessary in the following sense. From the existence of an inner model with two measurable cardinals (or the weaker assertion that the set of integers $0^\dagger$ exists), it follows that below any measurable cardinal there are arbitrarily many δ such that δ is measurable in an inner model (see Kunen [3, §9]). This is certainly true for the κ in Menas' model, and now one sees that it was a necessary state of affairs. That p-points have even this much structure is somewhat unexpected in view of past experience in the theory of ultrafilters.

I shall first establish a technical result as a sort of preview, but it is interesting in its own right. The proof is a typical application of Kunen's technique of iterated ultrapowers, and the notation and substance of his paper [3] will be used. In particular, if M is any class and E is an "M-ultrafilter", then $\mathrm{Ult}_\alpha(M, E)$ denotes the αth iterated ultrapower of M by E (with $\mathrm{Ult}_0(M, E) = M$). If E is actually a countably complete ultrafilter in the real world V, then for $\alpha \leq \beta$, $i^E_{\alpha\beta}$: $\mathrm{Ult}_\alpha(V, E) \to \mathrm{Ult}_\beta(V, E)$ denotes the usual embedding of the αth iterated ultrapower of V into its βth iterated ultrapower.

THEOREM 3.1. *Suppose that $U \leq_{\mathrm{RK}} D$ are κ-ultrafilters with $f_*(D) = U$, and $k: M_U \to M_D$ is the corresponding natural embedding. Then if there is a least ordinal moved by k, say ρ, then ρ is measurable in an inner model.*

PROOF. Fix a strictly increasing sequence of cardinals $\langle \lambda_n | n \in \omega \rangle$ so that each λ_n is a strong limit cardinal with $\mathrm{cf}(\lambda_n) > \kappa$, and $\rho < \lambda_0$. The salient features of the λ_n's are that for any κ-ultrafilter E: (a) $i^E_{0\lambda_n}(\kappa) = \lambda_n$, and (b) $i^E_{0\alpha}(\lambda_n) = \lambda_n$ for $\alpha < \lambda_n$ (see [3, 3.8, 3.9]).

Let F be the filter over $\lambda = \sup \lambda_n$ generated by $\langle \lambda_n | n \in \omega \rangle$, i.e.

$$X \in F \textit{ iff } X \subseteq \lambda \;\&\; \exists m \forall n > m(\lambda_n \in X).$$

The aforementioned properties of the λ_n's can now be used in connection with embeddings $i^E_{0\alpha}$ for $\alpha \leq \lambda$ and E any κ-ultrafilter to establish that *in $L[F]$, $F \cap L[F]$ is a normal λ-ultrafilter* (see [3, 10.10] for the analogue).

Turning next to our particular situation, define $W \subseteq P(\rho)$ by:

$$X \in W \quad \textit{iff} \quad X \in P(\rho) \cap L[F] \;\;\&\;\; \rho \in k(X).$$

Since i_U and i_D both fix $\langle \lambda_n | n \in \omega \rangle$, it must be the case that $i_U(L[F]) = L[F] = i_D(L[F])$. Thus, $k \restriction L[F]$: $L[F] \to L[F]$ is elementary. This is enough to show that *W is an $L[F]$-ultrafilter over ρ*, in the sense of Kunen (see [3, 4.6]).

Next, it is a standard fact that M_U is closed under arbitrary sequences of length κ. So, a straightforward argument shows that arbitrary countable intersections of elements of W are nonempty. Hence, *$\mathrm{Ult}_\alpha(L[F], W)$ is wellfounded for every α* (see [3, 3.6]).

Finally, all these ingredients can be put together: $L[F]$ is an inner model for the measurability of $\lambda > \rho$, and W is an $L[F]$-ultrafilter over ρ such that $\mathrm{Ult}_\alpha(L[F], W)$ is wellfounded for every α. Thus by [3, 6.9], W is a normal ultrafilter inside $L[W]$, and hence ρ is measurable in an inner model. $\dashv$

There are nontrivial situations as hypothesized in the theorem (involving product κ-ultrafilters) when ρ turns out to be $i_U(\kappa)$. But then the theorem does not affirm anything new, as $i_U(\kappa)$ is already known to be measurable in an inner model, to wit, M_U. On the other hand, suppose that $U <_{\mathrm{RK}} D$, say with $f_*(D) = U$, and yet $i_U(\kappa) = i_D(\kappa)$. Then the corresponding elementary embedding $k: M_U \to M_D$ first moves some ordinal $< i_U(\kappa)$.

For example, the ordinal $[\mathrm{id}]_D$ is not in the range of k; otherwise, if $[g \cdot f]_D = [\mathrm{id}]_D$ for some g, then surely f is injective (mod D), contradicting $U <_{\mathrm{RK}} D$. Since $[\mathrm{id}]_D < i_D(\kappa)$ and $k(i_U(\kappa)) = i_D(\kappa) = i_U(\kappa)$, this means that there must be an ordinal $< i_U(\kappa)$ moved by k.

This prefatory remark sets the stage for the main result. The author is grateful to Kunen for pointing out the present formulation; originally, the author had been preoccupied with p-points, and had stopped at the weaker conclusion that the set of integers $0^\dagger$ exists.

THEOREM 3.2. *Suppose that $U <_{\mathrm{RK}} D$ are κ-ultrafilters such that $i_U(\kappa) = i_D(\kappa)$. Then there is an inner model with two measurable cardinals.*

PROOF. As before, let $f_*(D) = U$, $k\colon M_U \to M_D$ the corresponding embedding, and ρ the first ordinal moved by k. Then, as pointed out, $\rho < i_U(\kappa)$. If $\rho_n = k^n(\rho)$ for $n \in \omega$, clearly $\rho_n < \rho_m$ for $n < m$. But $k(i_U(\kappa)) = i_D(\kappa) = i_U(\kappa)$ by hypothesis, so it must be the case that each $\rho_n < i_U(\kappa)$. Let G be the filter over $\bar{\rho} = \sup\{\rho_n \mid n \in \omega\}$ generated by the sequence $\langle \rho_n \mid n \in \omega\rangle$, i.e.,

$$X \in G \quad \textit{iff} \quad X \subseteq \bar{\rho}\ \&\ \exists m \forall n > m(\rho_n \in X).$$

Next, let $\langle \lambda_n \mid n \in \omega\rangle$ be exactly as in the proof of 3.1, with F the filter over $\lambda = \sup \lambda_n$ generated by this sequence. Setting $\bar{G} = G \cap L[G, F]$ and $\bar{F} = F \cap L[G, F]$, the proof will now be complete, once we establish the following.

Claim. Inside $L[G, F]$, $\bar{G}$ is a normal $\bar{\rho}$-ultrafilter and $\bar{F}$ is a normal λ-ultrafilter.

To establish this claim, note first that since M_U and M_D are closed under arbitrary countable sequences, $\langle \rho_n \mid n \in \omega\rangle$ and $\langle \lambda_n \mid n \in \omega\rangle$ are members of both inner models. Thus, $G \cap M_U \in M_U$ and $F \cap M_U \in M_U$ so that $L[G, F] \subseteq M_U$, and similarly for M_D. Next, notice that $k(L[G, F]) = L[G, F]$ since: (a) k preserves final segments of $\langle \rho_n \mid n \in \omega\rangle$ and hence preserves membership in G, and (b) k fixes $\langle \lambda_n \mid n \in \omega\rangle$, as both i_U and i_D do. Now, standard arguments using iterations of k show that in $L[G, F]$, $\bar{G}$ is a normal $\bar{\rho}$-ultrafilter ([3, 10.10] is the analogue).

The demonstration that $\bar{F}$ is a normal λ-ultrafilter inside $L[G, F]$ is analogous to the argument for 3.1, but there are some adjustments. First, observe that $\mathrm{Ult}_1(V, D)$ is just another notation for M_D, and we already saw that $L[G, F] \subseteq M_D$. Now for $\alpha \geq 1$, the usual embedding $i^D_{1\alpha}\colon M_D \to \mathrm{Ult}_\alpha(V, D)$ of the first ultrapower into the αth ultrapower first moves the ordinal $i_D(\kappa)$. Of course, since the sequence $\langle \rho_n \mid n \in \omega\rangle \in M_D$ yet $i_D(\kappa)$ is inaccessible in M_D, it must be the case that $\bar{\rho} = \sup \rho_n < i_D(\kappa)$. Thus, $\bar{G}$ is fixed by $i^D_{1\alpha}$ for any $\alpha \geq 1$.

Going on, by properties of the λ_n's, whenever $\alpha < \lambda$, $i^D_{1\alpha}$ fixes a final segment of $\langle \lambda_n \mid n \in \omega\rangle$, and hence fixes $L[G, F]$. Having realized this, standard arguments using these embeddings now conclude the proof by establishing that $\bar{F}$ is a normal λ-ultrafilter inside $L[G, F]$. $\dashv$

The following corollary shows that any substantial length in the RK-ordering leads to a substantial large cardinal assertion. $0^\dagger$ is an analogue of $0^\#$ for inner models of measurability; see [3, 9.2] for some informative characterizations.

COROLLARY 3.3. *If there is an ascending* RK-*chain of κ-ultrafilters of length $\omega + 1$, then the set of integers $0^\dagger$ exists.*

PROOF. Suppose that $U_0 <_{\mathrm{RK}} U_1 <_{\mathrm{RK}} U_2 <_{\mathrm{RK}} \cdots <_{\mathrm{RK}} D$, so that $i_{U_0}(\kappa) \leq i_{U_1}(\kappa) \leq i_{U_2}(\kappa) \leq \cdots \leq i_D(\kappa)$. One can assume that strict inequality holds everywhere, or else by the previous theorem there is an inner model with two measurable cardinals, and the existence of $0^\dagger$ follows by a standard argument. But if strict inequality holds everywhere, there are infinitely many ordinals δ between κ and $i_D(\kappa)$ such that δ is measurable in an inner model. So again, the Kunen technology can be applied to conclude that $0^\dagger$ exists (see [3, 6.9, 9.2*D*]). $\dashv$

I conjecture that the assumption $i_U(\kappa) = i_D(\kappa)$ is not needed in 3.2, when D is a p-point.

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