Annals of Mathematical Logic 19 (1980) 97–114

PERFECT-SET FORCING FOR UNCOUNTABLE CARDINALS

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Received 5 December 1979

Perfect-set forcing has been around for a long time. Sacks [10] himself had made substantial use of it to get important minimality results both in set theory and in recursion theory, and the fusion idea that he popularized has become an integral part of several notions of forcing. After Laver [8] developed the idea of adding reals iteratively with countable support, Baumgartner and Laver [2] applied it to the case of perfect-set forcing to produce interesting consistency results about Ramsey ultrafilters over ω and the tree property for ω_2. Since then, work of Shelah, Baumgartner, and others has considerably systematized countable support iterated forcing. As a first step in generalization, I develop in this paper a notion of perfect-set forcing for regular uncountable cardinals κ and its iteration with κ size supports. An application of an effective version of this forcing has already been made in recent work by Sacks and Slaman [11] in the study of abstract E-recursion and sideways extensions of E-closed structures.

In Section 1 the notion of forcing and its iteration are formulated, and their basic properties established. In particular, the appropriate fusion lemmas are stated and proved. Section 2 is dominated by the long proof of a key technical theorem, one of whose many consequences is that κ^+ is preserved as a cardinal by the iterated forcing. The use of a $\diamond_\kappa$ sequence in the ground model is an essential feature of this fusion argument. There is much less control over the forcing machinery in the uncountable case as compared to the ω case considered in [2], but $\diamond_\kappa$ gives us just enough structural information about subsets of κ to allow more economical procedures to work. In fact, it will be clear that this paper owes an obvious debt to [2], with the new modulations arising primarily from limit stage constructions and the use of $\diamond_\kappa$.

In Section 3 it is shown that if $2^\kappa = \kappa^+$, then $\leq\kappa^{++}$ iterations of the forcing still preserves κ^{++}, but that, in general, κ^+ iterations adds a $\diamond_{\kappa^+}$ sequence (in fact, a κ^+-Suslin tree) and hence collapses κ^{++} if $2^\kappa > \kappa^+$ had been satisfied in the ground model. In Section 4, the result on Aronszajn trees in [2] is lifted: Using $\diamond_\kappa$, a closure property for the iterated forcing is established, and this implies, as Silver first showed in Mitchell's model (see [9]), that if the forcing is iterated λ times, where λ is a weakly compact cardinal $>\kappa$, then there are no κ^{++}-Aronszajn trees in the resulting extension.

Section 5 makes some brief remarks about the side-by-side, or product, version of the forcing, and Section 6 is devoted to the special case of κ being inaccessible, where the analogies to the ω case are much stronger. A consistency result is established here which answers a question of Baumgartner and Taylor [3] negatively: There is a model of ZFC where 2^{ω_1} is large yet the non-stationary ideal over ω_1 is ω_2-generated but not 2^{ω_1}-saturated.

The set theoretical notation is standard, and the following litany should take care of any possible variations: The letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, whereas $\kappa, \lambda, \mu, \ldots$ are reserved for infinite cardinals. If x is a set, $P(x)$ denotes its power set and $|x|$ its cardinality. If f is a function, then $f''x = \{f(y) \mid y \in x\}$ and $f \restriction x = f \cap (x \times V)$. ${}^y x$ denotes the collection of functions: $y \to x$, so that λ^κ is the cardinality of ${}^\kappa\lambda$. If s and t are sequences, then $s^\frown t$ denotes their concatenation $s^\frown 0$ and $s^\frown 1$ will be shorthand for $s^\frown\langle 0\rangle$ and $s^\frown\langle 1\rangle$ respectively. Concerning the forcing formalism, $p \leq q$ will mean that p gives more information than q, $\Vdash \phi$ will mean that any condition forces ϕ, and it is convenient to take V as a relative term for the ground model and "construct" generic extensions $V[G]$.

1. The notion of forcing

In this section the basic notion of forcing and its iteration are formulated, and their main properties established. Any experience with perfect-set forcing for ω (see [10] or [2]) should make the motivating ideas here familiar. For the duration of the paper let κ denote a regular uncountable cardinal such that $2^{<\kappa} = \kappa$, and set $\mathrm{Seq} = \bigcup_{\alpha<\kappa} {}^\alpha 2$.

Definition 1.1. (a) If $p \subseteq \mathrm{Seq}$ and $s \in p$, say that s *splits in* p *iff* $s^\frown 0 \in p$ and $s^\frown 1 \in p$.

(b) Say that $p \subseteq \mathrm{Seq}$ is *perfect iff*

(i) If $s \in p$, then $s \restriction \alpha \in p$ for every α.

(ii) If $\alpha < \kappa$ is a limit ordinal, $s \in {}^\alpha 2$, and $s \restriction \beta \in p$ for every $\beta < \alpha$, then $s \in p$. 'p is closed.'

(iii) If $s \in p$, then there is a $t \in p$ with $t \supseteq s$ such that t splits in p.

(iv) If $\alpha < \kappa$ is a limit ordinal, $s \in {}^\alpha 2$, and for arbitrarily large $\beta < \alpha$, $s \restriction \beta$ splits in p, then s splits in p. 'The splitting nodes of p are closed.'

(c) If p is perfect and $s \in \mathrm{Seq}$, set $p_s = \{t \in p \mid s \subseteq t \text{ or } t \subseteq s\}$. (So p_s is perfect *iff* $s \in p$.)

(d) Set $P = \{p \subseteq \mathrm{Seq} \mid p \text{ is perfect}\}$ and order P by: $p \leq q$ *iff* $p \subseteq q$.

If $p_s = p$, then s is an initial segment of what can be called the 'stem' of p. Evidently, forcing with P adds a generic filter G which is identifiable with a new function $f \in {}^\kappa 2$, where $f(\alpha) = 0$ *iff* for some $p \in G$ and $s \in p$, we have $p = p_s$ and $s(\alpha) = 0$. Variants of P were known to Baumgartner, Laver, and perhaps others.

The key clause in the above definition is (iv), the exact form of which seems necessary for the coming use of $\diamondsuit_\kappa$. In the presence of (i)–(iii), an alternate, second-order formulation of (iv) is: if $f \in {}^\kappa 2$ is a branch through p, i.e. $f \restriction \alpha \in p$ for every α, then $\{\alpha \mid f \restriction \alpha$ splits in $p\}$ is closed unbounded in κ.

Lemma 1.2. *If $\beta < \kappa$ and $\langle p_\alpha \mid \alpha < \beta \rangle$ is a decreasing sequence in P, then $p = \bigcap_{\alpha<\beta} p_\alpha \in P$. Hence, P is a $<\kappa$-closed notion of forcing.*

Proof. It suffices to check condition (iii). But if $s \in p$, it is straightforward to find a cofinal branch $f \in {}^\kappa 2$ through p such that s is an initial segment of f. Then define an increasing sequence of ordinals $\langle \eta_\alpha \mid \alpha \leq \beta \rangle$ so that: $s \subseteq f \restriction \eta_0$; if δ is a limit ordinal, $\eta_\delta = \bigcup_{\alpha<\delta} \eta_\alpha$; and $f \restriction \eta_{\alpha+1}$ splits in p_α. Thus, $f \restriction \eta_\beta$ splits in p.

Definition 1.3. If $\alpha < \kappa$ and $p, q \in P$, set $p \leq_\alpha q$ *iff* $p \leq q$ and $p \cap {}^{\alpha+1}2 = q \cap {}^{\alpha+1}2$.

That it is ${}^{\alpha+1}2$ rather than ${}^{\alpha}2$ seems necessary, although this will cause technical complications later on. The key property of P is isolated in the following lemma:[1]

Lemma 1.4 (Fusion Lemma). *Suppose that $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a decreasing sequence in P such that: $p_{\alpha+1} \leq_\alpha p_\alpha$ for every α, and if δ is a limit ordinal, then $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$. Then $p = \bigcap_{\alpha<\kappa} p_\alpha \in P$.*

Proof. Again, it suffices to check condition (iii), and again if $s \in p$, we can suppose that there is a branch $f \in {}^\kappa 2$ through p such that s is an initial segment of f. Then define an increasing sequence of ordinals $\langle \eta_i \mid i \in \omega \rangle$ so that: $s \subseteq f \restriction \eta_0$, and $f \restriction \eta_{i+1}$ splits in p_{η_i}. Hence, if $\eta = \sup \eta_i$, then $f \restriction \eta$ splits in p_η. But also, if $\gamma \geq \eta$, then by hypothesis $p_\gamma \cap {}^{\eta+1}2 = p_\eta \cap {}^{\eta+1}2$, so that $f \restriction \eta$ splits in p_γ. Thus, $f \restriction \eta$ splits in p.

Using this lemma and $2^{<\kappa} = \kappa$, the following theorem follows much as in [10]:

Theorem 1.5. *If G is P-generic over V, then $(\kappa^+)^{V[G]} = (\kappa^+)^V$, and G is a minimal degree of constructibility over V (i.e. if $X \in V[G]$, then $X \in V$ or $G \in V[X]$).*

Hence, if we assume that $2^\kappa = \kappa^+$ holds in V, then by the κ^{++}-c.c. and Lemma 1.2, P preserves all cardinals and adds a 'minimal' subset of κ without adding any bounded subsets.

It is interesting to note that unless κ is strongly inaccessible, some aspects of perfect-set forcing on ω do not lift to the uncountable case. For instance, it can be

[1] Lemma 1.4 is called the Fusion Lemma in order to be consistent with [2]. The analogous lemma in [10] was called the Sequential Lemma. Mathias had formulated this lemma more abstractly and called the result the Fusion Lemma. Shoenfield had invoked the term 'splitting' to describe its proof. I have restrained myself from calling Lemma 1.4 the Fission Lemma.

shown by the same argument as in the ω case that when κ is inaccessible and G is P-generic over V, then: for every $X \in V[G]$ with $X \subseteq \kappa$, there is a $Y \in V$ with $|Y| = \kappa$ so that either $Y \subseteq X$ or $Y \subseteq \kappa - X$. However, Laver has pointed out that this is not true if in V there is a λ so that $2^\lambda = \kappa$ (e.g. if $\kappa = \omega_1$ and the Continuum Hypothesis holds):

In V, one can define a partition $\mathrm{Seq} = A \cup B$ such that for any $p \in P$ there is an $\alpha < \kappa$ so that $\alpha \leq \beta < \kappa$ implies that $p \cap {}^\beta 2 \cap A \neq \emptyset$ and $p \cap {}^\beta 2 \cap B \neq \emptyset$. This can be done as follows: Let λ be the least cardinal such that $2^\lambda = \kappa$. Since $2^{<\lambda} < \kappa$, one can enumerate as $\langle H_\alpha \mid \alpha < \kappa \rangle$ those subsets of Seq $\subseteq$-isomorphic to the tree $\bigcup_{\xi<\lambda} {}^\xi 2$. Then partition Seq level by level: to take care of ${}^\beta 2$, for every $\alpha < \beta$ such that $H_\alpha \subseteq \bigcup_{\xi<\beta} {}^\xi 2$, find two cofinal branches $b^A_{\alpha\beta}$ and $b^B_{\alpha\beta}$ through H_α satisfying: (a) if $\gamma < \alpha < \beta$, then $\bigcup b^A_{\alpha\beta} \neq \bigcup b^B_{\gamma\beta}$ and $\bigcup b^B_{\alpha\beta} \neq \bigcup b^A_{\gamma\beta}$; and (b) if either $b^A_{\alpha\beta} \subseteq s$ or $b^B_{\alpha\beta} \subseteq s$, then $s \notin \bigcup_{\gamma<\beta} H_\gamma$. Note that (b) can be easily satisfied since $|\bigcup_{\gamma<\beta} H_\gamma| < \kappa$. Thus, we can now color ${}^\beta 2$ so as to render $\{s \in {}^\beta 2 \mid b^A_{\alpha\beta} \subseteq s\} \subseteq A$ and $\{s \in {}^\beta 2 \mid b^B_{\alpha\beta} \subseteq s\} \subseteq B$ for every $\alpha < \beta$. It is not difficult to see that this partition satisfies our requirements, since any $p \in P$ has some H_α as an initial segment.

Suppose now that G is P-generic over V, and $X = \{\alpha < \kappa \mid \exists p \in G\, \exists s \in p\, (p = p_s \,\&\, s \in {}^\alpha 2 \cap A)\}$. A simple density argument establishes that there can be no $Y \in V$ with $|Y| = \kappa$ such that either $Y \subseteq X$ or $Y \subseteq \kappa - X$.

As another example, just as in the ω case it can be shown (see Theorem 6.2) that when κ is inaccessible and G is P-generic over V, then: for every $f \in {}^\kappa\kappa \cap V[G]$, there is a $g \in {}^\kappa\kappa \cap V$ which eventually dominates f, i.e. for some $\alpha < \kappa$, $f(\beta) \leq g(\beta)$ whenever $\alpha \leq \beta$. This too has a counterexample when κ is not inaccessible:

Let λ and $\langle H_\alpha \mid \alpha < \kappa \rangle$ be as before, and for each H_α, enumerate its cofinal branches in type κ. If G is P-generic over V, then define $f \in {}^\kappa\kappa$ by: $f(\alpha) = \beta$ *iff* $\exists p \in G\, \exists s \in p\, (p = p_s \,\&\, s$ extends the βth branch through H_α), and $\beta = 0$ otherwise. Again, a density argument establishes that for any $g \in {}^\kappa\kappa \cap V$ and $\alpha < \kappa$, there is a $\beta \geq \alpha$ such that $g(\beta) < f(\beta)$.

Similarly, we can show that there is a regressive function in ${}^\kappa\kappa \cap V[G]$ not eventually dominated by any regressive function in ${}^\kappa\kappa \cap V$, by using for each H_α a surjection of its cofinal branches onto α.

Let us now turn to the iteration of P.

Definition 1.6. (a) P_ξ for $\xi \geq 1$ is defined by induction as follows: $P_1 = P$; $P_{\xi+1} = P_\xi * \tau_\xi$, where τ_ξ is a canonical term denoting the partial order P as defined in the extension via P_ξ, and $*$ is the usual conglomeration of forcing twice; and P_δ for δ a limit ordinal is the inverse limit of $\langle P_\xi \mid \xi < \delta \rangle$ if $\mathrm{cf}(\delta) \leq \kappa$ and the direct limit otherwise. As there will be no reason for confusion, just $\leq$ will denote the partial order of P_ξ, and finally, $\Vdash_\xi$ its corresponding forcing relation.

(b) Under a standard identification, P_ξ will be considered, as a well-defined set, the collection of functions p so that domain(p) is a $\leq\kappa$ size subset of ξ, and for every $\beta \in \text{domain}(p)$, $p \upharpoonright \beta \Vdash_\beta p(\beta) \in \tau_\beta$. With this identification, for p and q in P_ξ, $p \leqslant q$ *iff* $\text{domain}(p) \supseteq \text{domain}(q)$ and for every $\beta \in \text{domain}(q)$, $p \upharpoonright \beta \Vdash_\beta p(\beta) \leqslant q(\beta)$.

P_ξ is the bona fide 'upward' iteration of P through ξ steps with $\leqslant\kappa$ size supports. The side-by-side or product forcing of ξ ground model copies of P with $\leqslant\kappa$ size supports shares many properties with P_ξ, but the verifications are simpler (see Section 5). Of course, the difficulty in dealing with P which does not arise in side-by-side forcing is that $p \upharpoonright \beta$ does not in general decide all the members of the term $p(\beta)$.

Basic properties of P are now lifted over to P_ξ.

Definition 1.7. For any $\xi \geqslant 1$,

(a) If $\{p_\alpha \mid \alpha < \beta\} \subseteq P_\xi$, then the 'meet' $p = \bigwedge_{\alpha<\beta} p_\alpha$ is defined so that: $\text{domain}(p) = \bigcup_{\alpha<\beta} \text{domain}(p_\alpha)$ and for every $\gamma \in \text{domain}(p)$, $p \upharpoonright \gamma \Vdash_\gamma p(\gamma) = \bigcap \{p_\alpha(\gamma) \mid \gamma \in \text{domain}(p_\alpha)\}$. (That $p \upharpoonright \gamma \in P_\gamma$ for $\gamma \in \text{domain}(p)$ is assumed here; if this ever fails, or $|\text{domain}(p)| > \kappa$, then $\bigwedge_{\alpha<\beta} p_\alpha$ is left undefined.)

(b) If $p, q \in P_\xi$, $\alpha < \kappa$, and $F \subseteq \text{domain}(q)$ with $|F| < \kappa$, then $p \leqslant_{F,\alpha} q$ *iff* $p \leqslant q$ and for every $\beta \in F$, we have $p \upharpoonright \beta \Vdash_\beta p(\beta) \leqslant_\alpha q(\beta)$.

Lemma 1.8. *For any $\xi \geqslant 1$, if $\beta < \kappa$ and $\langle p_\alpha \mid \alpha < \beta\rangle$ is a decreasing sequence in P_ξ, then $\bigwedge_{\alpha<\beta} p_\alpha \in P_\xi$. Hence, P_ξ is a $<\kappa$-closed notion of forcing.*

Lemma 1.9 (Generalized Fusion Lemma). *For any $\xi \geqslant 1$, if $p_\alpha \in P_\xi$ and F_α for $\alpha < \kappa$ are such that:*

(a) *$p_{\alpha+1} \leqslant_{F_\alpha,\alpha} p_\alpha$, and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ for limit δ.*

(b) *$F_\alpha \subseteq F_{\alpha+1}$; $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for limit δ; and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} \text{domain}(p_\alpha)$.*

Then $\bigwedge_{\alpha<\kappa} p_\alpha \in P_\xi$.

The proofs of these lemmata proceed by a straightforward induction using Lemmas 1.2 and 1.4, and are left to the reader. Lemma 1.8 is a particular instance of a general property of iterated forcing when sufficiently many inverse limits are taken, and Lemma 1.9 is the appropriate generalization of the Fusion Lemma where the F_α's are a standard strategem for ultimately covering all the domains of the p_α's.

2. The use of $\diamondsuit_\kappa$

This section is dominated by the long proof of a technical theorem. This important result has an analogue in the ω case (see Theorem 2.3(i) of [2]) and has as a direct consequence the fact that κ^+ is preserved as a cardinal in forcing with any P_ξ. However, the analogy is not exact unless κ is inaccessible (see Section 6), and the use of $\diamondsuit_\kappa$ is the distinctive new feature in the general case which allows a more involved argument to work. The use of $\diamondsuit_\kappa$ was suggested by Baumgartner, who was the first to apply it in forcing arguments. (See [1, Theorem 6.7].)

To scamper up many trees all at once, a generalization of p_s is now defined:

Definition 2.1. Suppose that $p \in P_\xi$, $F \subseteq \text{domain}(p)$ with $|F| < \kappa$, and $\sigma : F \to {}^\alpha 2$. Then $p \mid \sigma$ is a function with the same domain as p, given by:

$$p \mid \sigma(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F, \\ p(\beta)_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}$$

Thus, $p \mid \sigma \in P_\xi$ just in case for every $\beta \in F$, we have $(p \mid \sigma) \restriction \beta \Vdash_\beta \sigma(\beta) \in p(\beta)$. To complete the preliminaries, let us recall that Jensen's well-known principle $\diamondsuit_\kappa$ can easily be made to hold by a preliminary generic extension via a $<\kappa$-closed notion of forcing, and let us make explicit the following bipartite formulation of $\diamondsuit_\kappa$, easily seen to be equivalent to the usual one:

There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that $S_\alpha \subseteq \alpha \times \alpha$ and for every $X \subseteq \kappa \times \kappa$, the set $\{\alpha < \kappa \mid X \cap \alpha \times \alpha = S_\alpha\}$ is stationary in κ.

We can now state the main theorem, which serves to delimit the range of possible values for a term denoting a member of the standard universe V.

Theorem 2.2. *Assume $\diamondsuit_\kappa$ and that $p \in P_\xi$ with $p \Vdash_\xi \tau \in V$. Suppose also that $F \subseteq \text{domain}(p)$ with $|F| < \kappa$, and that $\gamma < \kappa$. Then there is a $q \leqslant_{F,\gamma} p$ and an $x \in V$ with $|x| \leqslant \kappa$ such that $q \Vdash_\xi \tau \in x$.*

Proof. To produce q, we shall construct a fusion sequence $\langle \langle p_\alpha, F_\alpha \rangle \mid \alpha < \kappa \rangle$ with $p_0 = p$ and $F_0 = F$ appropriate for using the Generalized Fusion Lemma 1.9. Along the way, some sets $x_\alpha \in V$ for some successor ordinals $\alpha < \kappa$ will also be defined. We want in addition that $p_{\alpha+1} \leqslant_{F_\alpha, \gamma+\alpha} p_\alpha$, but this can be arranged by just starting the coming construction at some indecomposable ordinal $\alpha > \gamma$ and setting $p_\beta = p_0$ for $\beta < \alpha$. Finally let us decide ahead of time on an efficient bookkeeping device to insure $|F_\alpha| \geqslant |\alpha|$ and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} \text{domain}(p_\alpha)$, and also keep track explicitly of bijections $g_\alpha : F_\alpha \leftrightarrow \eta_\alpha$ where $\eta_\alpha \geqslant \alpha$, so that $\alpha \leqslant \bar{\alpha}$ implies $g_\alpha \subseteq g_{\bar{\alpha}}$ and $g_\delta = \bigcup_{\alpha<\delta} g_\alpha$ for limit δ.

The limit step of the construction is obvious, so it only remains to explicate the successor step. So, suppose that $\alpha < \kappa$ and that p_α and F_α are already given.

Assume that $g_\alpha : F_\alpha \leftrightarrow \alpha$, i.e. $\eta_\alpha = \alpha$. Then first define $\sigma_\alpha : F_\alpha \to {}^{\alpha+1}2$ from the $\diamondsuit_\kappa$ sequence given above by: if $\beta \in F_\alpha$, then

$$(\sigma_\alpha(\beta))(\delta) = \begin{cases} 1 & \text{if } \delta < \alpha \text{ and } \langle g_\alpha(\beta), \delta \rangle \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that $\sigma_\alpha(\beta)(\alpha) = 0$ by this definition; we need $\sigma_\alpha : F_\alpha \to {}^{\alpha+1}2$ rather than ${}^\alpha 2$ because of the definition of $\leqslant_\alpha$, and there is an arbitrariness here in that we could just as well have uniformly required, say, the other split $\sigma_\alpha(\beta)(\alpha) = 1$ everywhere, and proceed accordingly.)

Now *assume* that there is an $r \leqslant p_\alpha$ so that $r = r \mid \sigma_\alpha$ and for every $\beta \in F_\alpha$, we have

$$r \restriction \beta \Vdash_\beta \sigma_\alpha(\beta) \restriction \alpha \quad \text{splits in } p_\alpha(\beta).$$

If either of these assumptions do not hold, set $p_{\alpha+1} = p_\alpha$ and $F_{\alpha+1} = F_\alpha$. Otherwise, produce by hypothesis an $r_\alpha \leqslant r$ and an $x_\alpha \in V$ so that $r_\alpha \Vdash_\xi \tau = x_\alpha$. Finally, we formulate $p_{\alpha+1}$ to be an 'amalgamation' of p_α and r_α, whose definition insures $p_{\alpha+1} \leqslant_{F_\alpha,\alpha} p_\alpha$, as follows.

(a) $\text{domain}(p_{\alpha+1}) = \text{domain}(r_\alpha)$.

(b) If $\beta \in F_\alpha$, then $p_{\alpha+1}(\beta)$ is a term such that:

$$r_\alpha \restriction \beta \Vdash_\beta p_{\alpha+1}(\beta) = (p_\alpha(\beta) - p_\alpha(\beta)_{\sigma_\alpha(\beta)}) \cup r_\alpha(\beta),$$

and for any condition $c \leqslant p_{\alpha+1} \restriction \beta$ incompatible with $r_\alpha \restriction \beta$,

$$c \Vdash_\beta p_{\alpha+1}(\beta) = p_\alpha(\beta).$$

(c) If $\beta \notin F_\alpha$, then $p_{\alpha+1}(\beta)$ is a term such that:

$$r_\alpha \restriction \beta \Vdash_\beta p_{\alpha+1}(\beta) = r_\alpha(\beta),$$

and for any condition $c \leqslant p_{\alpha+1} \restriction \beta$ incompatible with $r_\alpha \restriction \beta$,

$$c \Vdash_\beta p_{\alpha+1}(\beta) = \begin{cases} p_\alpha(\beta) & \text{if } \beta \in \text{domain}(p_\alpha), \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

The formulation of $p_{\alpha+1}(\beta)$ corresponds to a use of the Maximal Principle in the Boolean algebraic setting, and insures that $p_{\alpha+1} \restriction \beta \Vdash_\beta p_{\alpha+1}(\beta) \in \tau_\beta$. It underscores the fact that we are really in an iterated, rather than side-by-side, forcing situation.

This completes the inductive definition. Let $q = \bigwedge_{\alpha<\kappa} p_\alpha$, $x = \{x_\alpha \mid x_\alpha \text{ is defined}\}$, and $g = \bigcup_{\alpha<\kappa} g_\alpha$, so that $g : \text{domain}(q) \leftrightarrow \kappa$. Thus, $q \in P_\xi$ by Lemma 1.9, with $q \leqslant_{F,\gamma} p$, and it is claimed that:

$$q \Vdash_\xi \tau \in x.$$

This claim is established through the following several lemmata.

Sublemma 1. *Suppose that $t \leqslant q$. Then there is a sequence $\langle t_\alpha \mid \alpha < \kappa \rangle$ with $t_0 = t$, and functions $s^\beta_\alpha \in V$ with $s^\beta_\alpha : \rho^\beta_\alpha \to 2$ for some $\rho^\beta_\alpha \geqslant \alpha$ for every $\beta \in F_\alpha$ such that:*

(a) *$\alpha \leqslant \bar\alpha$ implies $t_{\bar\alpha} \leqslant t_\alpha$.*

(b) *$\alpha < \bar\alpha$ implies $s^\beta_\alpha{}^\frown 0 \subseteq s^\beta_{\bar\alpha}$ when $\beta \in F_\alpha$, and $s^\beta_\delta = \bigcup_{\alpha<\delta} s^\beta_\alpha$ for limit δ.*

(c) *For every $\beta \in F_\alpha$, we have $t_\alpha \restriction \beta \Vdash_\beta t_\alpha(\beta) = t_\alpha(\beta)_{s^\beta_\alpha{}^\frown 0}$ and s^β_α splits in $q(\beta)$. (So notice that $t_\alpha = t_\alpha \mid \langle s^\beta_\alpha{}^\frown 0 \mid \beta \in F_\alpha \rangle$.)*

Proof. The construction proceeds by induction, carrying along the additional hypothesis:

(*) For every $\beta \in F_{\alpha+1}$, we have $t_{\alpha+1} \restriction \beta \Vdash_\beta t_{\alpha+1}(\beta) = t_{\alpha+1}(\beta)_{s^\beta_{\alpha+1}{}^\frown 0}$ and $s^\beta_{\alpha+1}$ splits in $t_\alpha(\beta)$.

At limit steps δ, simply set $s^\beta_\delta = \bigcup_{\alpha<\delta} s^\beta_\alpha$, and $u = \bigwedge_{\alpha<\delta} t_\alpha$. Then by (*),

$$\text{For every } \beta \in F_\delta, \text{ we have } u \restriction \beta \Vdash_\beta u(\beta) = u(\beta)_{s^\beta_\delta} \text{ and } s^\beta_\delta \text{ splits in } u(\beta).$$

(s^β_δ splits in $u(\beta)$ since by (*) s^β_δ splits in $t_\alpha(\beta)$, for every $\alpha < \delta$.) Now it is easy to produce a $t_\delta \leq u$ which satisfies (c) above with $\alpha = \delta$.

At successor steps $\alpha + 1$ with t_α already given, enumerate $F_{\alpha+1}$ ascendingly as $\langle \beta_\nu \mid \nu < \eta \rangle$. Then find a decreasing sequence $\langle u_\nu \mid \nu < \eta \rangle$ with $u_0 = t_\alpha$ as follows: Set $u_\delta = \bigwedge_{\nu<\delta} u_\nu$ for δ a limit. Now given u_ν, find $v \leq u_\nu \restriction \beta_\nu$ and $s^{\beta_\nu}_{\alpha+1} : \rho^{\beta_\nu}_{\alpha+1} \to 2$, for some $\rho^{\beta_\nu}_{\alpha+1} \geq \alpha + 1$, so that $s^{\beta_\nu}_{\alpha+1} \supseteq s^{\beta_\nu}_\alpha{}^\frown 0$ when $\beta_\nu \in F_\alpha$, and:

$$v \Vdash_{\beta_\nu} s^{\beta_\nu}_{\alpha+1} \in t_\alpha(\beta_\nu) \quad \text{and} \quad s^{\beta_\nu}_{\alpha+1} \text{ splits in } t_\alpha(\beta_\nu).$$

Set

$$u_{\nu+1} = v^\frown \widehat{t_\alpha(\beta)_{s^{\beta_\nu}_{\alpha+1} {}^\frown 0}\, t_\alpha} \restriction (\text{domain}(t_\alpha) - \beta_\nu + 1)).$$

With $\langle u_\nu \mid \nu < \eta \rangle$ thus defined, we can set $t_{\alpha+1} = \bigwedge_{\nu<\eta} u_\nu$, so that $t_{\alpha+1}$ satisfies (*). This completes the inductive construction, hence the proof of the sublemma.

Sublemma 2. *$C_1 = \{\alpha < \kappa \mid \rho^\beta_\alpha = \alpha$ for every $\beta \in F_\alpha\}$ and $C_2 = \{\alpha < \kappa \mid g \restriction \alpha : F_\alpha \leftrightarrow \alpha\}$ are both closed unbounded in κ.*

Proof. This is immediate since for each β, $\langle \rho^\beta_\alpha \mid \alpha < \kappa \rangle$ is continuous at limits, and $\langle F_\alpha \mid \alpha < \kappa \rangle$ is also continuous at limits.

The proof of Theorem 2.2 can now be completed as follows: Suppose that $t \leq q$ is arbitrary. Carry out the construction of Sublemma 1 for this t. Set $s^\beta = \bigcup_{\alpha<\kappa} s^\beta_\alpha$ for $\beta \in \text{domain}(q)$, and then set $X = \{\langle g(\beta), \delta \rangle \mid \beta \in \text{domain}(q)$ and $s^\beta(\delta) = 1\}$. By $\diamondsuit_\kappa$, $S = \{\alpha \in C_1 \cap C_2 \mid X \cap (\alpha \times \alpha) = S_\alpha\}$ is stationary.

Fix $\alpha \in S$, so that by tracing through the definitions we have $t_\alpha = t_\alpha \mid \sigma_\alpha$. Note also that $t_\alpha \leq t \leq q \leq p_\alpha$, and for $\beta \in F_\alpha$, $t_\alpha \restriction \beta \Vdash_\beta s^\beta_\alpha$ splits in $p_\alpha(\beta)$. Thus, the assumptions for the non-trivial construction of $p_{\alpha+1}$ were satisfied, and $t_\alpha \leq p_{\alpha+1} \mid \sigma_\alpha = r_\alpha$ where $r_\alpha \Vdash_\xi \tau = x_\alpha$. Hence, to any $t \leq q$, a condition $v \leq t$ and an x_α have been found so that $v \Vdash_\xi \tau = x_\alpha$. Thus, $q \Vdash_\xi \tau \in x$, as was claimed, and the proof of Theorem 2.2 is now complete.

The following self-refinement of Theorem 2.2, a sort of covering property for sets of cardinality $\leq\kappa$ in the generic extension, is now a direct consequence.

Theorem 2.3. *Assume $\diamondsuit_\kappa$, and that $p \in P_\xi$ with $p \Vdash_\xi \tau \subseteq V$ with $|\tau| \leq \kappa$. Then there is a $q \leq p$ and an $x \in V$ with $|x| \leq \kappa$ such that $q \Vdash_\xi \tau \subseteq x$. Hence, forcing with P_ξ preserves κ^+ as a cardinal.*

Proof. We might as well assume that $p \Vdash_\xi \tau = \{\tau_\alpha \mid \alpha < \kappa\} \subseteq V$. Then using Theorem 2.2 repeatedly, it is easy to define a fusion sequence $\langle\langle p_\alpha, F_\alpha \rangle \mid \alpha < \kappa \rangle$ such that for every $\alpha < \kappa$, there is an $x_\alpha \in V$ with $|x_\alpha| \leq \kappa$ such that $p_{\alpha+1} \Vdash_\xi \tau_\alpha \in x_\alpha$. Thus, if $q = \bigwedge_{\alpha<\kappa} p_\alpha$ and $x = \bigcup \{x_\alpha \mid \alpha < \kappa\}$, then $q \Vdash_\xi \tau \subseteq x$.

3. Preserving and collapsing κ^{++}

We saw in the last section that for any $\xi \geqslant 1$ forcing with P_ξ always preserves κ^+. In this section we shall investigate what happens to κ^{++} in different cases. In analogy to the ω case, it will first be shown that if $2^\kappa = \kappa^+$ (and $\diamondsuit_\kappa$), then for every $\xi \leqslant \kappa^{++}$, P_ξ has the κ^{++}-chain condition and hence preserves all cardinals. Then the section is concluded with the observation that in general, forcing with P_{κ^+} adds a $\diamondsuit_{\kappa^+}$ sequence, so that in many situations κ^{++} is collapsed.

The following lemma is directly analogous to [2, 3.1].

Lemma 3.1. *Assume $\diamondsuit_\kappa$ and that $\xi < \kappa^{++}$. Then there is a dense subset $W_\xi \subseteq P_\xi$ such that $|W_\xi| = 2^\kappa$.*

Proof. Let W_ξ be the collection of those $p \in P_\xi$ such that:

(i) there is a sequence $\langle F_\alpha \mid \alpha < \kappa \rangle$ such that each $F_\alpha \subseteq \text{domain}(p)$ with $|F_\alpha| < \kappa$, and: $\alpha \leqslant \bar{\alpha}$ implies $F_\alpha \subseteq F_{\bar{\alpha}}$; $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for δ a limit; and $\bigcup_{\alpha<\kappa} F_\alpha = \text{domain}(p)$.

(ii) there is a sequence $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ such that $\sigma_\alpha : F_\alpha \to {}^{\alpha+1}2$ satisfying: whenever $t \leqslant p$ and $\gamma < \kappa$, there is an $\alpha \geqslant \gamma$ so that t and $p \mid \sigma_\alpha$ are compatible.

It is not difficult to see from the proof of Theorem 2.2 that W_ξ is dense in P_ξ. To establish that $|W_\xi| = 2^\kappa$ for $\xi < \kappa^{++}$, note that there are 2^κ possible $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$'s, so it suffices to show that any $p \in W_\xi$ is characterized by its $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$:

Suppose that p and q are both in W_ξ and have the same $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ (and hence the same $\langle F_\alpha \mid \alpha < \kappa \rangle$). Assume by way of contradiction that there is a β so that $p \restriction \beta = q \restriction \beta$, yet for some $r \leqslant p \restriction \beta$ and s and γ, we have

$$r \Vdash_\beta s \in (p(\beta) - q(\beta)) \cap {}^\gamma 2.$$

Consider

$$t = r^\frown p(\beta)_s{}^\frown p \restriction (\text{domain}(p) - \beta + 1).$$

Then for some $\alpha \geqslant \gamma$, $\beta \in F_\alpha$ and there is a $u \leqslant t$ with $u \leqslant p \mid \sigma_\alpha$. This follows from $p \in W_\xi$. It then follows that $\sigma_\alpha(\beta) \in {}^{\alpha+1}2$ and $\sigma_\alpha(\beta) \supseteq s$. But also $p \restriction \beta = q \restriction \beta$, so that

$$(q \restriction \beta) \mid (\sigma_\alpha \restriction \beta) \Vdash_\beta \sigma_\alpha(\beta) \in q(\beta),$$

as p and q have the same $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$. This is a contradiction of $s \subseteq \sigma_\alpha(\beta)$, and $u \restriction \beta \leqslant (q \restriction \beta) \mid (\sigma_\alpha \restriction \beta)$ and $u \restriction \beta \leqslant r$ and $r \Vdash_\beta s \notin q(\beta)$.

This lemma leads directly to the theorem on preservation of κ^{++}:

Theorem 3.2. *Assume $\diamondsuit_\kappa$ and $2^\kappa = \kappa^+$. Then for every $\xi \leqslant \kappa^{++}$, P_ξ has the κ^{++}-chain condition, and so preserves all cardinals.*

Proof. For $\xi < \kappa^{++}$ this is immediate from Lemma 3.1, and for $\xi = \kappa^{++}$ there is a standard Δ-system argument.

I have been unable to determine whether, in analogy to the ω case, (see [2, Section 5]), $\diamondsuit_\kappa$ and $2^\kappa = \kappa^+$ implies that forcing with $P_{\kappa^{++}+1}$ collapses κ^{++}. This sharp result does hold with the further assumption that κ is inaccessible. However, the following result will imply that κ^{++} is collapsed quickly enough in many cases: an iteration of length κ^+ suffices. Suggested by P. Dordal, it is really a general fact about how the support structure of conditions adds what looks like a Cohen subset of κ^+.

Theorem 3.3. *If* $\diamondsuit_\kappa$, *then* $\Vdash_{\kappa^+} \diamondsuit_{\kappa^+}$.

Proof. Let us note at the beginning that the term κ^+ will be unambiguous in the following, as it is always preserved as a cardinal by Theorem 2.3. For any ξ, let G_ξ be the canonical generic filter over P_ξ. The desired $\diamondsuit_{\kappa^+}$ sequence $\langle S_\eta \mid \eta < \kappa^+ \rangle$ will be recovered from G_{κ^+} as follows: $\beta \in S_\eta$ *iff* $\beta < \eta$ and there is a $p \in G_{\kappa^+}$ with $\eta + \beta \in \text{domain}(p)$ and

$$p \restriction (\eta + \beta) \Vdash_{\kappa^+} p(\eta + \beta) \subseteq \{s \in \text{Seq} \mid s(0) = 1\}.$$

In other words, a generic subset of κ^+ is naturally decoded from the initial splits along the κ^+ fibers of G_{κ^+}, and S_η is to consist simply of those $\beta < \eta$ so that $\eta + \beta$ is in this generic set.

To establish that this sequence has the $\diamondsuit_{\kappa^+}$ property in $V[G_{\kappa^+}]$, suppose that $q \Vdash_{\kappa^+} X \subseteq \kappa^+$ and $C \subseteq \kappa^+$ is closed unbounded. We must show that there is a $p \leqslant q$ and an $\eta \in \kappa^+$ so that $p \Vdash_{\kappa^+} X \cap \eta = S_\eta$ and $\eta \in C$.

Say that p *determines* X *up to* ζ *iff* $p \leqslant q$ with $p \in W_{\kappa^+}$, as defined in Lemma 3.1, with the corresponding $\langle F_\alpha \mid \alpha < \kappa \rangle$ and $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ so that for some bijection $\phi : \kappa \leftrightarrow \zeta$ and some sequence $\langle y_\alpha \mid \alpha < \kappa \rangle$ of sets in V, we have $p \mid \sigma_\alpha \Vdash_{\kappa^+} X \cap \phi''\alpha = y_\alpha$. For any $r \leqslant q$ and $\kappa \leqslant \zeta < \kappa^+$ the proof of Theorem 2.2 with $\diamondsuit_\kappa$ indicates that there is a $p \leqslant r$ which determines X up to ζ. The salient point is that if p determines X up to ζ, then $p \Vdash_{\kappa^+} X \cap \zeta \in V[G_\eta]$, whenever $\text{domain}(p) \subseteq \eta$, since it can be seen that $X \cap \zeta$ is completely determined by p and its corresponding $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ and $\langle y_\alpha \mid \alpha < \kappa \rangle$.

Let us now construct a sequence of conditions $\langle p_n \mid n \in \omega \rangle$ and a sequence of ordinals $\langle \zeta_n \mid n \in \omega \rangle$ as follows: Set $p_0 = q$ and $\zeta_0 = \kappa$. Given p_n and ζ_n, first find some $r \leqslant p_n$ and a $\zeta_{n+1} > \max(\zeta_n, \bigcup \text{domain}(p_n))$ such that $r \Vdash_{\kappa^+} \zeta_{n+1} \in C$. Then produce a $p_{n+1} \leqslant r$ with $\text{domain}(p_{n+1}) \supseteq \zeta_{n+1}$ which determines X up to ζ_{n+1}. Finally, set $\bar{p} = \bigwedge p_n$ and $\eta = \sup \zeta_n$. It is not difficult to see from the construction that $\text{domain}(\bar{p}) = \eta$ and $\bar{p} \Vdash_{\kappa^+} X \cap \eta \in V[G_\eta]$ and $\eta \in C$. Thus, it is permissible to define a $p \leqslant \bar{p}$ with $\text{domain}(p) \subseteq \eta + \eta$ by $p \restriction \eta = \bar{p}$, and:

$$p(\eta + \beta) \text{ is defined and equals } t_{\eta+\beta} \quad \textit{iff } \bar{p} \Vdash_\eta \beta \in X \cap \eta,$$

where t_ξ is a term so that

$$\Vdash_\xi t_\xi = \{s \in \mathrm{Seq} \mid s(0) = 1\}.$$

From the definitions, $p \Vdash_{\kappa^+} X \cap \eta = S_\eta$ and $\eta \in C$.

Corollary 3.4. *If $2^\kappa > \kappa^+$ in V, then forcing with P_{κ^+} collapses κ^{++}.*

Proof. This is immediate since $\diamondsuit_{\kappa^+}$ implies $2^\kappa = \kappa^+$.

Let us recall that if λ is regular and $E \subseteq \lambda$ is stationary in λ, then $\diamondsuit_\lambda(E)$ is the sharpened principle: There is a sequence $\langle S_\eta \mid \eta < \lambda \rangle$ such that $S_\eta \subseteq \eta$ and for every $X \subseteq \lambda$, the set $\{\eta \in E \mid X \cap \eta = S_\eta\}$ is stationary in λ. Then the conclusion of Theorem 3.3 can actually be $\Vdash_{\kappa^+} \diamondsuit_{\kappa^+}(E_\mu)$ for every regular $\mu < \kappa$, where $E_\mu = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \mu\}$, since the sequence $\langle p_n \mid n \in \omega \rangle$ can be extended to length μ by the $<\kappa$-closure of P_{κ^+}. However, note that the proof of Theorem 3.3 does not apply to perfect-set forcing for ω to add a $\diamondsuit$ $(= \diamondsuit_{\omega_1})$ sequence, since the countable closure of the forcing was used. Thus, it is worthwhile to prove an amplified version of Theorem 3.3 which relies on a further fusion argument and will apply to the ω case. The extra feature that we shall need is the following lemma:

Lemma 3.5. *Assume $\diamondsuit_\kappa$, and that $p \in P_\xi$ with $p \Vdash_\xi C \subseteq \kappa^+$ is closed unbounded. Suppose also that $F \subseteq \mathrm{domain}(p)$ with $|F| < \kappa$ and that $\gamma < \kappa$. Then for any $\rho < \kappa^+$, there is a $\zeta \geq \rho$ and a $q \leq_{F,\gamma} p$ such that $q \Vdash_\xi \zeta \in C$.*

Proof. To produce q and ζ, we shall construct a sequence of conditions $\langle p_n \mid n \in \omega \rangle$ and a sequence of ordinals $\langle \zeta_n \mid n \in \omega \rangle$ as follows: Set $p_0 = p$ and $\zeta_0 = \rho$. Given p_n and ζ_n, let τ_n be a term such that $p_n \Vdash_\xi \tau_n$ is the least element of $C - \zeta_n$. Using Theorem 2.2, let $p_{n+1} \leq_{F,\gamma} p_n$ be such that for some $x_n \in V$ with $|x_n| \leq \kappa$, we have $p_{n+1} \Vdash_\xi \tau_n \in x_n$. Then set $\zeta_{n+1} = \sup x_n$. Finally, set $q = \bigwedge p_n$ and $\zeta = \sup \zeta_n$. It is not difficult to see that this works.

Theorem 3.6. *If $\diamondsuit_\kappa$, then $\Vdash_{\kappa^+} \diamondsuit_{\kappa^+}(E_\kappa)$.*

Proof. We can define the desired sequence $\langle S_\eta \mid \eta < \kappa^+ \rangle$ exactly as in Corollary 3.4. Assuming that $q \Vdash_{\kappa^+} X \subseteq \kappa^+$ and $C \subseteq \kappa^+$ is closed unbounded, we must now show that there is a $p \leq q$ and an $\eta < \kappa^+$, with the additional proviso that $\mathrm{cf}(\eta) = \kappa$, so that $p \Vdash_{\kappa^+} X \cap \eta = S_\eta$ and $\eta \in C$.

To produce p, we construct a fusion sequence $\langle \langle p_\alpha, F_\alpha \rangle \mid \alpha < \kappa \rangle$ which interlaces Theorem 3.3 and Lemma 3.5, defining an increasing sequence of ordinals $\langle \zeta_\alpha \mid \alpha < \kappa \rangle$ along the way. Set $p_0 = q$ and $\zeta_0 = \kappa$, and decide beforehand as usual on a procedure for determining the F_α's. The limit steps δ of the construction are obvious, with $\zeta_\delta = \sup_{\alpha < \delta} \zeta_\alpha$, so it remains to explicate the successor step. Given

p_α and ζ_α and F_α, first use Lemma 3.5 to find some $r \le_{F_\alpha,\alpha} p_\alpha$ and a $\zeta_{\alpha+1} > \max(\zeta_\alpha, \bigcup \mathrm{domain}(p_\alpha))$ such that $r \Vdash_{\kappa^+} \zeta_{\alpha+1} \in C$. Then produce a $p_{\alpha+1} \le_{F_\alpha,\alpha} r$ with $\mathrm{domain}(p_{\alpha+1}) \supseteq \zeta_{\alpha+1}$ which determines X up to $\zeta_{\alpha+1}$. This completes the inductive definition.

Set $\bar{p} = \bigwedge_{\alpha<\kappa} p_\alpha$ and $\eta = \sup_{\alpha<\kappa} \zeta_\alpha$. Then $\bar{p} \in P_{\kappa^+}$ and $\mathrm{cf}(\eta) = \kappa$, and it is not difficult to see from the construction that $\mathrm{domain}(\bar{p}) = \eta$ and $\bar{p} \Vdash_{\kappa^+} X \cap \eta \in V[G_\eta]$ and $\eta \in C$. Thus, we can complete the proof exactly as in Theorem 3.3 to produce a $p \le \bar{p}$ such that $p \Vdash_{\kappa^+} X \cap \eta = S_\eta$ and $\eta \in C$.

Corollary 3.7. *If $\diamondsuit_\kappa$, then $\Vdash_{\kappa^+}$. There is a κ^+-Suslin tree.*

Proof. The following is a known theorem of ZFC: If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit_{\lambda^+}(\{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) = \lambda\})$, then there is a λ^+-Suslin tree. (See [6, p. 336].)

The proof of Theorem 3.6 has the heralded application that if perfect-set forcing for ω is iterated ω_1 times over any ground model, then $\diamondsuit$ holds in the extension, since the analogue of Lemma 3.5 for the ω case goes through by a straightforward fusion argument.

4. Aronszajn trees

This section is devoted to the proof of a strong closure property for P_ξ, and its consequence on the possibility of κ^{++} having no κ^{++}-Aronszajn trees. Mitchell and Silver (see [9]) had achieved definitive results about various tree properties for accessible cardinals. For one case of their results, the consistency of ω_2 having no ω_2-Aronszajn trees, [2, Section 6] showed that the same result can be achieved by iterating perfect-set forcing for ω up to a weakly compact cardinal. The result is here extended to κ^{++}-Aronszajn trees, and this provided one of the initial motivations for investigating perfect-set forcing for uncountable cardinals.

The following theorem affirms an important closure property of P_ξ, and is analogous to [2, Theorem 6.2], although the proof is somewhat modulated by $\diamondsuit_\kappa$:

Theorem 4.1. *Assume $\diamondsuit_\kappa$. If $p \in P_\xi$ and $\mathrm{cf}(\rho) > \kappa$ and $p \Vdash_\xi (f : \rho \to V$ and $f \restriction \eta \in V$ for every $\eta < \rho)$, then $p \Vdash_\xi f \in V$.*

Proof. It will be convenient to use $\diamondsuit_\kappa$ in another equivalent form: There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that $S_\alpha \subseteq 2 \times \alpha \times \alpha$ and for every $X \subseteq 2 \times \kappa \times \kappa$, the set $\{\alpha < \kappa \mid X \cap (2 \times \alpha \times \alpha) = S_\alpha\}$ is stationary in κ.

To establish the theorem, argue by contradiction and assume that there is a $\bar{p} \le p$ such that $\bar{p} \Vdash f \notin V$. We shall now construct a fusion sequence similar to the proof of Theorem 2.2, adopting the same general notaton and attending to the necessary adjustments. We start with $p_0 = \bar{p}$ and moreover for later convenience, arrange $0 \in \text{domain}(p_0)$ and $0 \in F_0$. At the successor step with p_α and F_α already given and $g_\alpha : F_\alpha \leftrightarrow \alpha$ assumed, we now define $\sigma^i_\alpha : F_\alpha \leftrightarrow {}^{\alpha+1}2$ for both $i = 0$ and $i = 1$ by: if $\beta \in F_\alpha$, then

$$(\sigma^i_\alpha(\beta))(\delta) = \begin{cases} 1 & \text{if } \delta < \alpha \text{ and } \langle i, g_\alpha(\beta), \delta\rangle \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\sigma^0_\alpha(0) \neq \sigma^1_\alpha(0)$. Next assume that for both $i = 0$ and $i = 1$, there is an $r^i \leq p_\alpha$ so that $r^i = r^i \mid \sigma^i_\alpha$ and for every $\beta \in F_\alpha$,

$$r^i \restriction \beta \Vdash_\beta \sigma^i_\alpha(\beta) \restriction \alpha \quad \text{splits in } p_\alpha(\beta).$$

As before, if any of these assumptions do not hold, set $p_{\alpha+1} = p_\alpha$ and $F_{\alpha+1} = F_\alpha$. Otherwise, for $i = 0$ and $i = 1$, produce $r^i_\alpha \leq r^i$ and a function $h^i_\alpha \in V$ so that $r^i_\alpha \Vdash_\xi \exists \eta (f \restriction \eta = h^i_\alpha)$, such that h^0_α and h^1_α differ at some ordinal common to both their domains. (This is possible since $\bar{p} \Vdash_\xi f \notin V$, so that we can first find $r \leq r^0$ and $\bar{r} \leq r^0$ and an η with $r \Vdash_\xi f \restriction \eta = h$, and $\bar{r} \Vdash_\xi f \restriction \eta = \bar{h}$, where h and $\bar{h}$ differ at some ordinal. Then let $\langle r^1_\alpha, h^1_\alpha\rangle$ be any pair such that $r^1_\alpha \leq r^1$ and $r^1_\alpha \Vdash_\xi f \restriction \eta = h^1_\alpha$. Since at least one of h or $\bar{h}$ differs from h^1_α, we can take $\langle r^0_\alpha, h^0_\alpha\rangle$ to be at least one of $\langle r, h\rangle$ and $\langle \bar{r}, \bar{h}\rangle$.)

We can now formulate $p_{\alpha+1}$ to be an amalgamation of p_α, r^0_α, and r^1_α, whose definition insures $p_{\alpha+1} \leq_{F_\alpha, \alpha} p_\alpha$, as follows:

(a) $\text{domain}(p_{\alpha+1}) = \text{domain}(r^0_\alpha) \cup \text{domain}(r^1_\alpha)$.

(b) $p_{\alpha+1}(0) = (p_\alpha(0) - p_\alpha(0)_{\sigma^0_\alpha(0)} - p_\alpha(0)_{\sigma^1_\alpha(0)}) \cup r^0_\alpha(0) \cup r^1_\alpha(0)$.

(c) If $\beta > 0$ and $\beta \in F_\alpha$, then $p_{\alpha+1}(\beta)$ is a term such that for both $i = 0$ and $i = 1$, we have:

$$r^i_\alpha \restriction \beta \Vdash_\beta p_{\alpha+1}(\beta) = (p_\alpha(\beta) - p_\alpha(\beta)_{\sigma^i_\alpha(\beta)}) \cup r^i_\alpha(\beta),$$

and for any $c \leq p_{\alpha+1} \restriction \beta$ incompatible with both $r^0_\alpha \restriction \beta$ and $r^1_\alpha \restriction \beta$,

$$c \Vdash_\beta p_{\alpha+1}(\beta) = p_\alpha(\beta).$$

(Notice that $p_{\alpha+1}(\beta)$ is well (enough) defined, since $\sigma^0_\alpha(0) \neq \sigma^1_\alpha(0)$ implies that $r^0_\alpha \restriction \beta$ and $r^1_\alpha \restriction \beta$ are incompatible.)

(d) If $\beta \notin F_\alpha$, $p_{\alpha+1}(\beta)$ is a term such that for both $i = 0$ and $i = 1$, we have

$$r^i_\alpha \restriction \beta \Vdash_\beta p_{\alpha+1}(\beta) = \begin{cases} r^i_\alpha(\beta) & \text{whenever } \beta \in \text{domain}(r^i_\alpha), \\ \mathbf{1} & \text{otherwise;} \end{cases}$$

and for any $c \leq p_{\alpha+1} \restriction \beta$ incompatible with both $r^0_\alpha \restriction \beta$ and $r^1_\alpha \restriction \beta$,

$$c \Vdash_\beta p_{\alpha+1}(\beta) = \begin{cases} p_\alpha(\beta) & \text{if } \beta \in \text{domain}(p_\alpha), \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

This completes the inductive definition. Let $q = \bigwedge_{\alpha<\kappa} p_\alpha \in P_\xi$, and $g = \bigcup_{\alpha<\kappa} g_\alpha$, so that $g : \text{domain}(q) \leftrightarrow \kappa$. Since $\text{cf}(\rho) > \kappa$, there must be some $\bar{\eta} < \rho$ larger than the domains of all the h^i_α's appearing in the construction. Let $t \leq q$ so that for some

$h \in V$, we have $t \Vdash_\xi f \restriction \check{\eta} = h$. In order to derive the ultimate contradiction, we continue to follow the pattern of the proof of Theorem 2.2:

Let $\langle {}^0t_\alpha \mid \alpha < \kappa \rangle$ and $\{{}^0s^\beta_\alpha \mid \alpha, \beta < \kappa\}$ be the result of carrying out the construction of Sublemma 1 of Theorem 2.2 for $t \leq q$, and let $\langle {}^1t_\alpha \mid \alpha < \kappa \rangle$ and $\{{}^1s^\beta_\alpha \mid \alpha, \beta < \kappa\}$ be the result of carrying out another such construction with some ${}^1s^0_0$ which differs from ${}^0s^0_0$ at some ordinal. (Recall that we specified $0 \in F_0$.) For both $i = 0$ and $i = 1$, set ${}^is^\beta = \bigcup_{\alpha<\kappa} {}^is^\beta_\alpha$ for $\beta \in \text{domain}(q)$, and then set

$$X = \{\langle 0, g(\beta), \delta \rangle \mid \beta \in \text{domain}(q) \text{ and } {}^0s^\beta(\delta) = 1\}$$
$$\cup \{\langle 1, g(\beta), \delta \rangle \mid \beta \in \text{domain}(q) \text{ and } {}^1s^\beta(\delta) = 1\}.$$

By $\diamondsuit_\kappa$, there is an α such that $X \cap (2 \times \alpha \times \alpha) = S_\alpha$. Recall that ${}^0s^0_0$ differs from ${}^1s^0_0$, so that $\sigma^0_\alpha(0) \neq \sigma^1_\alpha(0)$. Thus, since we can also require α to be in certain relevant closed unbounded sets as in Theorem 2.2, we can suppose that all of the assumptions for the non-trivial construction of $p_{\alpha+1}$ were satisfied. Hence, for both $i = 0$ and $i = 1$, we have ${}^it_\alpha \leq p_{\alpha+1} \mid \sigma^i_\alpha = r^i_\alpha$, where $r^i_\alpha \Vdash_\xi \exists\eta(f \restriction \eta = h^i_\alpha)$. But then $h^0_\alpha \subseteq h$ and $h^1_\alpha \subseteq h$, contradicting the fact that h^0_α and h^1_α differ at some ordinal. This completes the proof of the theorem.

The following consistency result now follows from Corollary 3.4 and Theorem 4.1 exactly as [2, Lemma 6.4] is established. Familiarity is assumed with the rather well-known concepts and terms involved, as well as their significance in the context of the theory of large cardinals (see [6] or [7]). In brief, a *λ-Aronszajn tree* is a tree of height λ all of whose levels have cardinality $<\lambda$, which has no branches of length λ. A λ^+-Aronszajn tree is *special* if there is an order preserving injection of it into the tree

$$\left\langle \bigcup_{\alpha<\lambda^+} \{f \mid f : \alpha \to \lambda \text{ is injective}\}, \subseteq \right\rangle.$$

Theorem 4.2. (i) *Assume $\diamondsuit_\kappa$, and that λ is a weakly compact cardinal such that $\lambda > \kappa$. Then $\Vdash_\lambda \lambda = \kappa^{++}$ and there are no λ-Aronszajn trees.*

(ii) *Assume $\diamondsuit_\kappa$, and that λ is a Mahlo cardinal such that $\lambda > \kappa$. Then $\Vdash_\lambda \lambda = \kappa^{++}$ and there are no special λ-Aronszajn trees.*

This result shows that perfect-set forcing provides an alternative method of establishing many cases of some consistency results of Mitchell and Silver (see [9]). Admittedly, their work is technically more general, in that their method also works in a case not covered here: the consistency of κ being a regular limit cardinal and there being no κ^+-Aronszajn trees.

5. Side-by-side forcing

Although the cardinal collapsing result of Section 3 was used to advantage in Theorem 4.2, it is an unfortunate feature of the P_ξ's if we want to render 2^κ large by adding many subsets of κ through the iteration of P. In this section, side-by-side or product forcing, with $\leqslant\kappa$ size support, of ground model copies of P is briefly considered, forcing which it turns out will preserve all cardinals.

Definition 5.1. For $\xi \geqslant 1$, Q_ξ is the collection of functions p such that $\text{domain}(p) \subseteq \xi$ with $|\text{domain}(p)| \leqslant \kappa$, and $p(\beta) \in P$ for every $\beta \in \text{domain}(p)$. Order Q_ξ by: $p \leqslant q$ *iff* $\text{domain}(p) \supseteq \text{domain}(q)$ and for every $\beta \in \text{domain}(q)$, $p(\beta) \leqslant q(\beta)$.

Just as for P_ξ, one can define $\bigwedge_{\alpha<\beta} p_\alpha$ and $p \leqslant_{F,\alpha} q$ for Q_ξ, and show that: (a) Q_ξ is a $<\kappa$-closed notion of forcing, and (b) the Generalized Fusion Lemma 1.9 works for Q_ξ. Of course, the difference between P_ξ and Q_ξ is that for $p \in Q_\xi$, $p(\beta)$ is a definite member of P as in the ground model, not just a term for a member of P as defined in some partial generic extension. Hence, the following result holds via an easier proof that avoids the complications introduced by the necessity of forcing more and more definite members of Seq into conditions.

Theorem 5.2. *Theorem 2.2 holds with Q_ξ replacing P_ξ. Hence, assuming $\diamondsuit_\kappa$, forcing with Q_ξ for any $\xi \geqslant 1$ preserves every cardinal $\leqslant\kappa^+$.*

P_ξ and Q_ξ now part ways. If $2^\kappa = \kappa^+$, then P obviously has the κ^{++}-chain condition since $|P| = \kappa^+$. The following is a direct consequence of this and a standard fact about product forcing (see e.g. [6, p. 190]).

Theorem 5.3. *If $2^\kappa = \kappa^+$, then Q_ξ has the κ^{++}-chain condition for every $\xi \geqslant 1$. Hence, if also $\diamondsuit_\kappa$ holds, then forcing with Q_ξ for any $\xi \geqslant 1$ preserves all cardinals.*

We shall have the opportunity to use Q_ξ to render 2^κ large in the next section.

6. The inaccessible case

When κ is (strongly) inaccessible, $\diamondsuit_\kappa$ is unnecessary in deriving the salient properties of the iterated forcing, and what is more, distinctly new results are possible. In particular, a consistency result about the closed unbounded filter can be established which answers a question raised by Baumgartner and Taylor. This section is devoted to such ramifications afforded by the inaccessibility of κ.

Our first task is to attend to the analogue for Theorem 2.2, and typically a more straightforward argument is available which also yields a stronger conclusion. (It is interesting to note that this argument also works for the ω case to provide an alternative proof for [2, 2.3(i)].)

Theorem 6.1. *Assume that κ is inaccessible and that $p \in P_\xi$ with $p \Vdash_\xi \tau \in V$. Suppose also that $F \subseteq \text{domain}(p)$ with $|F| < \kappa$, and that $\gamma < \kappa$. Then there is a $q \leqslant_{F,\gamma} p$ and an $x \in V$ with $|x| < \kappa$ such that $q \Vdash_\xi \tau \in x$. (The new feature here is $|x| < \kappa$.)*

Proof. Let $\langle\sigma_\alpha \mid \alpha<\eta\rangle$ enumerate the collection of all functions $\sigma: F \to {}^{\gamma+1}2$. By the inaccessibility of κ, we can assume that $\eta<\kappa$. To produce q, we shall construct by induction a decreasing sequence of conditions $\langle p_\alpha \mid \alpha<\eta\rangle$ such that $\alpha<\bar{\alpha}$ implies $p_{\bar{\alpha}} \leqslant_{F,\gamma} p_\alpha$, and also some sets $x_\alpha \in V$ for some successor ordinals $\alpha<\eta$ along the way.

If $\delta<\eta$ is a limit, set $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$. For the successor step with p_α given, set $p_{\alpha+1} = p_\alpha$ if there is no $r \leqslant p_\alpha$ such that $r = r \mid \sigma_\alpha$. Otherwise, for such an r, produce by hypothesis an $r_\alpha \leqslant r$ and an $x_\alpha \in V$ so that $r_\alpha \Vdash_\xi \tau = x_\alpha$. We can now formulate $p_{\alpha+1}$ to be an amalgamation of p_α and r_α exactly as in the corresponding construction of Theorem 2.2, to insure that $p_{\alpha+1} \leqslant_{F,\gamma} p_\alpha$. This completes the inductive definition.

Let $q = \bigwedge_{\alpha<\eta} p_\alpha$, and $x = \{x_\alpha \mid x_\alpha \text{ is defined}\}$. Then $q \leqslant_{F,\gamma} p$, and the proof will be complete once we establish that: $q \Vdash_\xi \tau \in x$. So, suppose that $t \leqslant q$. Surely there is a $\bar{t} \leqslant t$ so that for some $\alpha<\eta$, we have $\bar{t} = \bar{t} \mid \sigma_\alpha$. The condition for the non-trivial construction of $p_{\alpha+1}$ was thus satisfied, and so $\bar{t} \leqslant q \mid \sigma_\alpha \leqslant p_{\alpha+1} \mid \sigma_\alpha = r_\alpha$, and hence $\bar{t} \Vdash_\xi \tau = x_\alpha$. The proof is complete.

The following now follows from Theorem 6.1 exactly as Theorem 2.3 follows from Theorem 2.2:

Theorem 6.2. *Assume that κ is inaccessible and that $p \in P_\xi$ with $p \Vdash_\xi \tau: \kappa \to V$. Then there is a $q \leqslant p$ and a sequence $\langle x_\alpha \mid \alpha<\kappa\rangle$ of $<\kappa$ size sets in V such that $q \Vdash_\xi \forall\alpha(\tau(\alpha) \in x_\alpha)$. Hence, forcing with P_ξ preserves κ^+ as a cardinal.*

Theorem 6.2 has the consequence heralded at the end of Section 1 that if κ is inaccessible and G is P-generic over V, then every $f \in {}^\kappa\kappa \cap V[G]$ is eventually dominated by a $g \in {}^\kappa\kappa \cap V$.

With Theorems 6.1 and 6.2 in hand, one can follow the pattern of previous sections, often using arguments more closely modelled on those of [2], to check that all of the results of those sections hold with the hypothesis $\diamondsuit_\kappa$ replaced by the inaccessibility of κ. However, there seems little need to dwell on this, since $\diamondsuit_\kappa$ is such a mild hypothesis to assume. One new aspect which does deserve mention is that it can be shown (by roughly following [2, Section 5] for the ω case) that if κ is inaccessible and $2^\kappa = \kappa^+$, then forcing with $P_{\kappa^{++}+1}$ collapses κ^{++}; as mentioned in Section 3, I do not know whether this sharp result holds with $\diamondsuit_\kappa$ replacing the inaccessibility of κ.

Let us now turn to some new considerations involving the closed unbounded filter afforded by the inaccessibility of κ. The following result shows that the closed unbounded filter in the generic extension is generated by the closed unbounded filter in V.

Theorem 6.3. *Assume that κ is inaccessible and that $p \in P_\xi$ with $p \Vdash_\xi C \subseteq \kappa$ is closed unbounded. Then there is a $q \leqslant p$ and a $D \in V$ which is closed unbounded such that $q \Vdash_\xi D \subseteq C$.*

Proof. We might as well assume that $p \Vdash_\xi \tau: \kappa \to \kappa$ enumerates C in increasing order. Then use Theorem 6.2 to find a $q \leq p$ and a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ of $<\kappa$ size sets of ordinals in V, such that $q \Vdash_\xi \forall \alpha (\tau_\alpha \in x_\alpha)$. Let D be the set of limit ordinals $\beta < \kappa$ such that whenever $\alpha < \beta$, then $x_\alpha \subseteq \beta$. Then D is closed unbounded, and it is not difficult to see that $q \Vdash_\xi D \subseteq C$. For if $\beta \in D$, then

$$q \Vdash_\xi \tau(\beta) = \bigcup_{\alpha<\beta} \tau(\alpha) \leq \bigcup_{\alpha<\beta} \bigcup x_\alpha \leq \beta \leq \tau(\beta),$$

i.e. $q \Vdash_\xi \tau(\beta) = \beta \in C$.

I state without proof that a more involved argument establishes that if κ is inaccessible and $\diamond^*_\kappa$ holds (see e.g. [4] for a definition), then a $\diamond^*_\kappa$ sequence in V is also a $\diamond^*_\kappa$ sequence in any generic extension via P_ξ

Typically, when κ is not inaccessible, the conclusion to Theorem 6.3 no longer holds. To see this, note that for uncountable cardinals κ, the question whether every closed unbounded subset of κ contains a closed unbounded subset from the ground model is equivalent to the question whether every $f: \kappa \to \kappa$ is dominated by a function from the ground model. The failure of the latter was demonstrated after Theorem 1.5 for κ not inaccessible.

Let us recall that an ideal $I \subseteq P(\kappa)$ is λ-*saturated* if the Boolean algebra $P(\kappa)/I$ has the λ-chain condition, i.e. whenever $\{A_\alpha \mid \alpha < \mu\} \subseteq P(\kappa) - I$ are such that $\alpha \neq \beta$ implies $A_\alpha \cap A_\beta \in I$, then $\mu < \lambda$. Also, an ideal $I \subseteq P(\kappa)$ is λ-*generated* if there is an $X \subseteq I$ with $|X| = \lambda$ such that I is the smallest ideal extending X. Finally, NS_λ denotes the non-stationary ideal over λ, the dual to the closed unbounded filter. Theorem 6.3 leads to the following consistency results:

Theorem 6.4. Con(ZFC *and* κ *is inaccessible*) *implies* Con(ZFC *and* κ *is inaccessible and* 2^κ *is large and* NS_κ *is* κ^+*-generated but not* 2^κ*-saturated*).

Proof. By either relativizing to L or performing a preliminary generic extension, we can assume that κ is inaccessible, $2^\kappa = \kappa^+$, and $\diamond_\kappa$ holds in V. Then force with the side-by-side Q_ξ for ξ as large as desired. It is easy to check that Theorems 6.1–6.3 hold with Q_ξ replacing P_ξ, and so we have in the generic extension that: NS_κ is κ^+-generated and $2^\kappa \geq \xi$. Finally, it is well-known that a $<\kappa$-closed notion of forcing preserves $\diamond_\kappa$ (see [4]), and that $\diamond_\kappa$ implies that NS_κ is not 2^κ-saturated. (For this last assertion, let $\langle S_\alpha \mid \alpha < \kappa \rangle$ be a $\diamond_\kappa$ sequence and for each $X \subseteq \kappa$, set $T_X = \{\alpha \mid X \cap \alpha = S_\alpha\}$. Then $\{T_X \mid X \subseteq \kappa\} \subseteq P(\kappa) - \mathrm{NS}_\kappa$, yet $X \neq Y$ implies $|T_X \cap T_Y| < \kappa$.)

The following answers Question 7.2 of Baumgartner and Taylor [3], relative to the consistency strength of the existence of an inaccessible cardinal.

Theorem 6.5. Con(ZFC *and there is an inaccessible*) *implies* Con(ZFC *and* 2^{ω_1} *is large and* NS_{ω_1} *is* ω_2*-generated but not* 2^{ω_1}*-saturated*).

Proof. Simply perform the standard Lévy collapse of κ to ω_1 on the model of Theorem 6.4. By a standard argument, the κ-chain condition of the collapse insures that any closed unbounded subset of κ in this further generic extension contains a closed unbounded subset in the given model, and so in particular stationary sets in the given model are also stationary in the further extension.

Added in proof: T. Jech (in: On the number of generators of an ideal, to appear) has established that the conclusion of Theorem 6.5 follows from just Con(ZF). His model is very different, since he first insures that 2^{ω_1} is large and NS_{ω_1} is not 2^{ω_1}-saturated, and then generically adds ω_2 closed unbounded sets which generate NS_{ω_1} in the extension. The model of Theorem 6.5 has the distinction of having the closed unbounded sets of the ground model still generating NS_{ω_1} in the extension.

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