PATRAS LOGIC SYMPOSION
G. Metakides (ed.)

MORASS-LEVEL COMBINATORIAL PRINCIPLES

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This paper is devoted to an investigation of some combinatorial principles in set theory of about the strength of a morass. As with well-known combinatorial propositions like $\Diamond$ which have wide application, these principles are useful and not difficult to state. However, that they hold in L seems to be a consequence only of a deep structural analysis of the sort afforded by morasses.

In §1 the principles and their applications are described, the historical context established, and acknowledgements made. Then §2 and §3 deal respectively with the consistency question for a strong principle available at successor cardinals, and the general case. The main theme is a forcing technique, and in §3 a new kind of density argument is presented.

The set theoretical notation is standard; for example $[x]^\kappa$ denotes the collection of subsets of x of cardinality κ. Concerning the forcing formalism, $p \leq q$ will mean that p gives more information than q. Finally, let me quickly review the relevant cases of the polarized partition symbol of Erdös, Hajnal, and Rado [EHR].

$$\binom{\lambda}{\kappa} \longrightarrow \binom{\mu}{\nu}_\gamma$$

means that whenever $F: \lambda\times\kappa \to \gamma$, there are $X \in [\lambda]^\mu$ and $Y \in [\kappa]^\nu$ such that $|F"(X\times Y)| = 1$. Also,

$$\binom{\lambda}{\kappa} \longrightarrow \begin{bmatrix}\mu\\ \nu\end{bmatrix}_\gamma$$

means that whenever $F: \lambda\times\kappa \to \gamma$, there are $X \in [\lambda]^\mu$ and $Y \in [\kappa]^\nu$ such that $F"(X\times Y) \neq \gamma$. To denote the negation of these propositions, $\to$ is replaced by $\not\to$.

Besides [EHR], see the secondary source Williams[W], Chapter 4, for further information.

§1. THE COMBINATORIAL PRINCIPLES

Let me formulate forthwith the main versions of the combinatorial principles to be investigated. $\Delta(\kappa,\lambda)$ will denote the following proposition:

There is an $F \subseteq {}^\kappa\kappa$ with $|F| \geq \lambda$ such that:

$$\forall s \varepsilon [F]^\kappa \forall \phi \varepsilon {}^s\kappa (|\{\xi < \kappa | \ \forall f \varepsilon s (f(\xi) \neq \phi(f))\}| < \kappa).$$

Roughly, there are λ functions from κ into κ such that: if guesses are made at possible values for any κ many of them, then for sufficiently large $\xi < \kappa$ at least one guess is rendered correct at ξ. When κ is a <u>successor</u> cardinal, it will be natural to consider a more stringent form. $\Delta^+(\kappa,\lambda)$ will denote the following proposition, when κ is a successor cardinal, with κ^- its predecessor:

There is an $F \subseteq {}^\kappa\kappa$ with $|F| \geq \lambda$ such that:

$$\forall s \varepsilon [F]^{\kappa^-} \forall \phi \varepsilon {}^s\kappa (|\{\xi < \kappa | \ \forall f \varepsilon s (f(\xi) \neq \phi(f))\}| < \kappa).$$

Thus, $\Delta^+(\kappa,\lambda)$ is a stronger version of $\Delta(\kappa,\lambda)$ where $\forall s \varepsilon [F]^\kappa$ has been replaced by $\forall s \varepsilon [F]^{\kappa^-}$. General, many-cardinal versions of these propositions could have been written down initially and Δ and Δ^+ formulated as special cases, but there is no need to obfuscate the situation with many parameters; in any case, Δ and Δ^+ are the strongest versions possible, and natural weakenings in a couple of directions will be explicitly discussed.

It is clear that the larger the λ gets, the stronger these propositions become. The main case of interest for consistency results involving L will be $\lambda = \kappa^+$, but we shall see that λ can be made arbitrarily large by forcing. Turning to a concrete case, it is already a consequence of the Continuum Hypothesis that $\Delta^+(\omega_1,\omega_2)$ holds (see §2). Prikry[P] first formulated the stronger $\Delta^+(\omega_1,\omega_2)$ and established its consistency with ZF + GCH by forcing. I rediscovered his proof in the course of answering a question of Szymański (see below), and it is presented in §2. Prikry's work was focused on a question of Erdös, Hajnal, and Rado [EHR] about polarized partition relations; the following easy proposition establishes the connection:

PROPOSITION 1.1: $\binom{\lambda}{\kappa} \not\rightarrow \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{\gamma}$ holds iff there is an $F \subseteq {}^{\kappa}\gamma$ with $|F| = \lambda$ such that: $\forall s \in [F]^{\mu} \forall \eta \in \gamma \; (|\{\xi < \kappa | \; \forall f \in s(f(\xi) \neq \eta)\}| < \nu$.

PROOF: Let $\{f_\alpha | \; \alpha < \lambda\}$ enumerate F, and set $G(\alpha,\xi) = f_\alpha(\xi)$. Then G is a counterexample to the partition relation.

Hence, note that

$$\Delta(\kappa,\lambda) \quad \text{implies} \; \binom{\lambda}{\kappa} \not\rightarrow \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}_{\kappa} \quad \text{and}$$

$$\Delta^{+}(\kappa,\lambda) \quad \text{implies} \; \binom{\lambda}{\kappa} \not\rightarrow \begin{bmatrix}\kappa^{-}\\ \kappa\end{bmatrix}_{\kappa} ,$$

as the negative polarized partition relations are equivalent to the weak versions of Δ and Δ^{+} where the ϕ in the definitions range over just the constant functions. Thus for example, the aforementioned result with CH shows that under this hypothesis, we have

$$\binom{\omega_1}{\omega_1} \not\rightarrow \begin{bmatrix}\omega_0\\ \omega_1\end{bmatrix}_{\omega_1} .$$

The Erdös-Hajnal-Rado question was whether the ω_1 in the top left can be raised to ω_2, and Prikry's result shows just that. This was the first of several examples of the phenomenon of consistency results, rather than outright demonstrations, in the partition calculus. Jensen soon established that Prikry's negative partition relation follows from $V = L$ by showing in fact that $\Delta^{+}(\omega_1,\omega_2)$ does.

In the other direction, Laver (unpublished) has established the relative consistency of

$$\binom{\omega_2}{\omega_1} \rightarrow \binom{\omega_1}{\omega_1}_{\omega_0}$$

by forcing over a ground model satisfying ZFC and a strong large cardinal hypothesis, the existence of a huge cardinal. He achieved this result by first forging an ideal over ω_1 with a particularly strong saturation property, improving upon some work of Kunen[Ku]. Working instead from ZF plus very strong determinacy hypotheses, Woodin (unpublished) produced an ideal over ω_1 with an even stronger property, that of having an ω_1-dense subset.

These are the only known positive consistency results, but both start with assumptions which are considerably more problematic than say the existence of a measurable cardinal.

In general, the proposition

$$(*) \qquad \binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_\gamma \quad \text{for every } \gamma > \kappa$$

seems to hold very rarely, even when κ is not a successor cardinal. The earliest result along these lines was due to Erdös and Rado [ER] (Theorem 48), who established that (*) is true for $\kappa = \omega$. Hajnal [H2] then established that (*) is true for κ a measurable cardinal; see also Chudnovsky[C] and Kanamori[Ka] for some refinements. Chudnovsky[C], p.295, stated without proof that (*) is true for κ a weakly compact cardinal. (Wolfsdorf[Wo] and Shelah have since provided proofs.) Moreover, Shelah showed that this is about as far as one can go, by developing a notion of forcing which adds a counterexample to (*) but preserves, say, the Mahloness of κ.

What is the situation in L? This question as well as more general motives ultimately led Shelah and Stanley [SS1][SS2] to provide a new characterization of morasses in terms of Martin's Axiom-type principles. (The work of Velleman[V] is closely intertwined.) In particular, this showed that Shelah's forcing can be transformed into a construction in L, to establish: If $V = L$, then κ is weakly compact <u>iff</u> (*) holds for κ and κ is regular and uncountable.

Unaware of this work, I came upon the key feature of Shelah's notion of forcing in a different formalism. In §3 my consistency result about a strong form of $\Delta(\kappa,\lambda)$ which incorporates a strong $\Diamond$-type property is presented, which subsumes Shelah's more ad hoc construction and makes transparent the natural progression from the $\Delta^+(\kappa,\lambda)$ result first discovered by Prikry.

To formulate natural weakenings of Δ and Δ^+ in another direction, a third cardinal can be introduced which delimits the range of the functions. Let $\Delta(\mu,\kappa,\lambda)$ denote the following proposition:

There is an $F \subseteq {}^{\kappa}\mu$ with $|F| \geq \lambda$ such that:

$$\forall s \varepsilon [F]^{\kappa} \forall \phi \varepsilon {}^{s}\mu(|\{\xi < \kappa | \ \forall f \varepsilon s(f(\xi) \neq \phi(f))\}| < \kappa).$$

Thus, $\Delta(\kappa,\lambda)$ is $\Delta(\kappa,\kappa,\lambda)$; define $\Delta^{+}(\mu,\kappa,\lambda)$ analogously. As expected, there are some implications:

PROPOSITION 1.2:

(a) Suppose that either $\kappa < \lambda$, or $\kappa = \lambda$ is regular. If $1 < \nu \leq \mu \leq \kappa$, and $\Delta(\mu,\kappa,\lambda)$ holds, then so does $\Delta(\nu,\kappa,\lambda)$.

(b) Suppose that either $\kappa^{-} < \lambda$, or $\kappa^{-} = \lambda$ is regular. If $1 < \nu \leq \mu \leq \kappa$, and $\Delta^{+}(\mu,\kappa,\lambda)$ holds, then so does $\Delta^{+}(\nu,\kappa,\lambda)$.

PROOF: I shall establish (a); (b) is quite analogous. Suppose that $\{f_\alpha | \alpha < \lambda\} \subseteq {}^{\kappa}\mu$ satisfies $\Delta(\mu,\kappa,\lambda)$, where it is understood that the f_α's are to be taken to be mutually distinct. Let $k\colon \mu \to \nu$ be surjective. Then it is easy to see that $F = \{k \cdot f_\alpha | \alpha < \lambda\}$ would satisfy $\Delta(\nu,\kappa,\lambda)$, if only $|F| \geq \lambda$. That is, we must make sure that k has not identified too many of the f_α's. If we assume to the contrary that $|F| < \lambda$, then by the hypotheses on the relationship of κ to λ, there must be an $s \ \varepsilon \ [\lambda]^{\kappa}$ such that $\alpha,\beta \ \varepsilon \ s$ implies $k \cdot f_\alpha = k \cdot f_\beta$. Since $\nu > 1$ and k is surjective, $\{\xi < \kappa | \ \forall \alpha \varepsilon s(k \cdot f_\alpha(\xi) \neq 0\}$ and $\{\xi < \kappa | \ \forall \alpha \varepsilon s(k \cdot f_\alpha(\xi) \neq 1)\}$ both have cardinality $< \kappa$. Thus, there is a $\xi < \kappa$ such that for some $\alpha \ \varepsilon \ s$, $k \cdot f_\alpha(\xi) = 0$ and also, for some $\beta \ \varepsilon \ s$, $k \cdot f_\beta(\xi) = 1$. This is a contradiction, and hence $|F| \geq \lambda$.

I do not know whether converses exist, e.g. does $\Delta(2,\kappa,\kappa^{+})$ imply $\Delta(\kappa,\kappa^{+})$? The apparent independence of these three cardinal versions make them more distinctive, and they are directly applicable.

Balcar, Simon, and Vojtáš ask ([BSV], Problem 20b) whether the following is consistent: whenever κ is regular and uncountable and U is a uniform ultrafilter over κ, then there are κ^{+} sets in U such that the intersection of any infinitely many of then has cardinality $< \kappa$. Probably, this is true in L, and the proof will depend heavily on the structure of ultrafilters. But at least, one can affirm the case $\kappa = \omega_1$ (by results in §2) from a combinatorial proposition about sets using the following:

PROPOSITION 1.3: Suppose $\Delta(2,\kappa,\lambda)$ (respectively, $\Delta^+(2,\kappa,\lambda)$). Then whenever U is a uniform ultrafilter over κ, there are λ sets in U such that the intersection of any κ (respectively, κ^-) of them has cardinality $< \kappa$.

PROOF: Immediate.

My initial interest in this whole area of research was kindled by another problem. In developing some Baire Category-type theorems for $U(\omega_1)$, the space of uniform ultrafilters over ω_1, Szymański [Sz] formulated the following concept: For any infinite cardinal λ, a matrix $\{A^n_\alpha \mid n < \omega, \alpha < \lambda\}$ is a <u>λ-matrix</u> <u>iff</u>

(i) if $m < n$ and $\alpha < \lambda$, then $A^m_\alpha \subseteq A^n_\alpha$,

(ii) $\cup\{A^n_\alpha \mid n < \omega\} = \omega_1$ for each $\alpha < \lambda$, and

(iii) for each infinite $s \subseteq \lambda$ and $\phi \in {}^s\omega$,
$|\cap\{A^{\phi(\alpha)}_\alpha \mid \alpha \in s\}| \leq \omega$.

A basic clopen set for $U(\omega_1)$ is a set of form $\{u \in U(\omega_1) \mid A \in u\}$ for some $A \subseteq \omega_1$; and a G_δ closed set is a countable intersection of basic clopen sets. Szymański established the following equivalence: A λ-matrix exists <u>iff</u> there exists a family of λ G_δ closed and nowhere dense subsets of $U(\omega_1)$ such that the union of any infinite subfamily is dense in $U(\omega_1)$. Typically, a topological property has been reduced to its set theoretical essence.

At the end of his paper, Szymański asked whether the existence of λ-matrices for $\lambda > \omega$ is consistent with ZFC. It was in response to this question that I first formulated $\Delta^+(\kappa,\lambda)$, and rediscovered Prikry's consistency proof (see §2). The connection to Szymański's matrices is clear:

PROPOSITION 1.4: If $\Delta^+(\omega,\omega_1,\lambda)$ holds, then there is a λ-matrix.

PROOF: If $\{f_\alpha \mid \alpha < \lambda\} \subseteq {}^{\omega_1}\omega$ satisfies $\Delta^+(\omega,\omega_1,\lambda)$, just set $A^n_\alpha = \{\xi \mid f_\alpha(\xi) \leq n\}$.

I do not know whether the converse holds, and actually, there are some nice equivalences other than the topological one, just concerning matrices: The following result is due to Baumgartner and included with his permission; for $f,g \in {}^{\omega}\omega$, write $f < g$ iff $\{n \mid g(n) \le f(n)\}$ is finite.

THEOREM 1.5: The following are equivalent:

(a) An ω_1-matrix exists.

(b) An ω-matrix exists.

(c) There is a subset of ${}^{\omega}\omega$ with cardinality ω_1 without an upper bound in ${}^{\omega}\omega$ under $<$.

PROOF: (a) $\to$ (b) is immediate; for (b) $\to$ (c), suppose that $\{A_i^n \mid n,i \in \omega\}$ is an ω-matrix, and define f_α for $\alpha < \omega_1$ by: $f_\alpha(i) = n$ iff $\alpha \in A_i^{n+1} - A_i^n$. If $f \in {}^{\omega}\omega$ were an upper bound under $<$ for all the f_α's, then there would be a fixed n so that $i > n$ implies $f_\alpha(i) < f(i)$ for uncountably many α's. But this is a contradiction, since $\{\alpha \mid f_\alpha(i) < f(i)$ for every $i > n\} = \bigcap\{A_i^{f(i)} \mid i > n\}$ is countable by hypothesis. Thus, $\{f_\alpha \mid \alpha < \omega_1\}$ is as desired.

For (c) $\to$ (a), suppose that $\{g_\xi \mid \xi < \omega_1\}$ has no upper bound in ${}^{\omega}\omega$ under $<$. By making adjustments if necessary, we can assume that the g_ξ's are strictly increasing and that $\eta < \xi$ implies $g_\eta < g_\xi$. For each $\xi < \omega_1$, let $\pi_\xi \colon \xi \to \omega$ be injective, and define $h_\xi \colon \xi \to \omega$ by: $h_\xi(\alpha) = \max\{g_\xi(n) \mid n < \pi_\xi(\alpha)\}$. Set $A_\alpha^n = \{\xi \mid \xi \le \alpha$ or $h_\xi(\alpha) < n\}$. To show that $\{A_\alpha^n \mid n < \omega,\ \alpha < \omega_1\}$ is an ω_1-matrix, suppose that $s \in [\omega_1]^\omega$ and $\phi \in {}^{s}\omega$. Then $\bigcap\{A_\alpha^{\phi(\alpha)} \mid \alpha \in s\} = \{\xi \mid h_\xi(\alpha) < \phi(\alpha)$ for every $\alpha \in s\}$, and call this set T. Let $\langle \alpha_i \mid i \in \omega \rangle$ enumerate s, and define $\bar{\phi} \in {}^{\omega}\omega$ by: $\bar{\phi}(n) = \max\{\phi(\alpha_i) \mid i \le n+1\}$. Notice that if $\xi \in T$ with $\xi > \bigcup s$, then for every n, we have $g_\xi(n) < \max\{h_\xi(\alpha_0),\dots,h_\xi(\alpha_{n+1})\} < \bar{\phi}(n)$. Thus, if T were uncountable, then given any $\eta < \omega_1$ there would be a $\xi \in T$ with $\eta < \xi$ and so $g_\eta < g_\xi < \bar{\phi}$, contradicting the hypothesis that the g_ξ's have no upper bound. Hence, T is countable, and $\{A_\alpha^n \mid n < \omega,\ \alpha < \omega_1\}$ is an ω_1-matrix.

In view of this result, Baumgartner asked the following question: If there is an ω-matrix, is there a 2^ω-matrix? An independence result may be hard to achieve, since the standard ways of adding reals also seem to add a 2^ω-matrix.

With this, the survey of the variations of $\Delta(\kappa,\lambda)$ is at an end; in the next sections, consistency results are discussed.

§2. *CONSISTENCY OF* $\Delta^+(\kappa,\lambda)$

This section discusses the question of consistency for $\Delta^+(\kappa,\lambda)$, with first a direct construction, then a forcing argument, and finally the situation in L. The following enumeration argument is a natural one in the present context, but it has an antecedent already in Braun and Sierpinski [BS], Proposition (Q).

THEOREM 2.1: If κ is a successor cardinal and $2^{\kappa^-} = \kappa$, then $\Delta^+(\kappa,\kappa)$. In particular, if the Continuum Hypothesis holds, then $\Delta^+(\omega_1,\omega_1)$.

PROOF: Using $2^{\kappa^-} = \kappa$, let $\langle\langle s_\delta,\phi_\delta\rangle \mid \delta < \kappa\rangle$ be an enumeration of all pairs $\langle s,\phi\rangle$ such that $s \in [\kappa]^{\kappa^-}$ and $\phi \in {}^s\kappa$. To build a collection $\{f_\alpha \mid \alpha < \kappa\} \subseteq {}^\kappa\kappa$ satisfying $\Delta^+(\kappa,\kappa)$, given a $\xi < \kappa$, a value for $f_\alpha(\xi)$ will be determined for every $\alpha < \kappa$. This will be done by a diagonal construction so that for every $\delta < \xi$, there is an $\alpha \in s_\delta$ such that $f_\alpha(\xi) = \phi_\delta(\alpha)$. The proof would then be complete, since for each $\delta < \kappa$, we would have $\{\xi \mid \forall\alpha\varepsilon s_\delta(f_\alpha(\xi) \neq \phi_\delta(\alpha))\} \subseteq \delta+1$.

So, fix $\xi < \kappa$, and let $\langle \xi_\gamma \mid \gamma < \kappa^-\rangle$ be an enumeration of ξ. Proceed by induction on $\gamma < \kappa^-$, at each step determining for some α a value for $f_\alpha(\xi)$, as follows: At the γth step, since s_{ξ_γ} has cardinality κ^-, there must be some $\alpha \in s_{\xi_\gamma}$ such that $f_\alpha(\xi)$ has not yet been defined; set $f_\alpha(\xi) = \phi_{\xi_\gamma}(\alpha)$. Finally, for any α such that $f_\alpha(\xi)$ had been left undetermined by this construction, set $f_\alpha(\xi) = 0$.

That the converse is not true will be evident from the comments after the next theorem. To meet more requirements than can be handled by a direct enumeration, the foregoing argument can be converted into a forcing scheme. The next theorem is essentially due to Prikry [P]; I rediscovered his argument in the notationally simpler form now presented.

THEOREM 2.2: If the ground model V satisfies that κ is a successor cardinal and $2^{\kappa^-} = \kappa$, then for any cardinal $\lambda > \kappa$ in V, there is a $<\kappa$-closed, κ^+-c.c. forcing extension in which $\Delta^+(\kappa,\lambda)$ holds.

PROOF: Define a notion of forcing as follows: P^κ_λ is to be the collection of pairs $\langle F,S\rangle$ where

(a) F is a function: $a\times\gamma \to \kappa$ for some $\gamma < \kappa$ and some $a \in [\lambda]^{\kappa^-}$, and

(b) S is a subset of $\{\langle s,\phi\rangle \mid s \in [\lambda]^{\kappa^-}$ and $\phi \in {}^s\kappa\}$ of cardinality κ^-.

For $\langle F,S\rangle, \langle G,T\rangle \in P^{\kappa}_{\lambda}$, define $\langle F,S\rangle \leq \langle G,T\rangle$ iff

(i) $F \supseteq G$ and $S \supseteq T$, and

(ii) if $F: a\times\gamma \to \kappa$ and $G: b\times\delta \to \kappa$ say, then for every $\xi \in \gamma-\delta$ and $\langle s,\phi\rangle \in T$, there is an $\alpha \in s$ such that $\alpha \in a$ and $F(\alpha,\xi) = \phi(\alpha)$.

Intuitively, a condition consists of a κ^- size approximation to a $\Delta^+(\kappa,\lambda)$ family, and κ^- many requirements $\langle s,\phi\rangle$ which must thence be met by any extension of the condition.

This notion of forcing is $<\kappa$-closed, since if $\eta < \kappa$ and $\alpha < \beta < \eta$ implies $\langle F_\beta,S_\beta\rangle \leq \langle F_\alpha,S_\alpha\rangle$, then there is a common extension, namely $\langle \bigcup_{\alpha<\eta} F_\alpha, \bigcup_{\alpha<\eta} S_\alpha\rangle \in P^{\kappa}_{\lambda}$. Also, for any $\langle s,\phi\rangle$ such that $s \in [\lambda]^{\kappa^-}$ and $\phi \in {}^{s}\kappa$, $D_{\langle s,\phi\rangle} = \{\langle F,S\rangle \in P^{\kappa}_{\lambda} \mid \langle s,\phi\rangle \in S\}$ is obviously dense. Finally, for any $\gamma < \kappa$ and $a \in [\lambda]^{\kappa^-}$, $D_{\langle a,\gamma\rangle} = \{\langle F,S\rangle \in P^{\kappa}_{\lambda} \mid \textbf{domain}(F) \supseteq a\times\gamma\}$ is also dense. (To see this, note that given any $\langle G,T\rangle \in P^{\kappa}_{\lambda}$, say with $G: b\times\delta \to \kappa$, the argument of the last paragraph of the proof of Theorem 2.1 shows that there is an $\langle F,T\rangle \leq \langle G,T\rangle$ such that $F: c\times(\delta+1) \to \kappa$ for some $c \supseteq b$. Thus, one step extensions are always possible, and the rest follows from $<\kappa$-closure.)

It now follows that if G is any generic filter over V, then one can define $\{f^{G}_{\alpha} \mid \alpha < \lambda\} \subseteq {}^{\kappa}\kappa$ in $V[G]$ by: $f^{G}_{\alpha}(\xi) = \beta$ iff $\exists\langle F,S\rangle \in G(F(\alpha,\xi) = \beta)$. To see that this collection satisfies $\Delta^+(\kappa,\lambda)$, let $t \in V[G]$ with $t \in [\lambda]^{\kappa^-}$, and $\phi \in {}^{t}\kappa$. By $<\kappa$-closure, we can assume that $\langle t,\phi\rangle \in V$. But then, by genericity there is some $\langle F,S\rangle \in G$ such that $\langle t,\phi\rangle \in S$, say with $F: a\times\gamma \to \kappa$. It now follows from the nature of the forcing that: $\{\xi \mid \forall\alpha\in t(f^{G}_{\alpha}(\xi) \neq \phi(\alpha))\} \subseteq \gamma$.

Finally, a Δ-system argument using $2^{\kappa^-} = \kappa$ shows that P^{κ}_{λ} has the κ^+-c.c.. (This sort of argument is standard; see Jech[J1], p.248 for an example. Incidentally, this is the only place where $2^{\kappa^-} = \kappa$ is used.)

P^{κ}_{λ} has several notable properties. First of all, it is actually <u>$<\kappa$-complete</u>, i.e. if $\eta < \kappa$, then any decreasing η sequence in P^{κ}_{λ} has a <u>greatest</u> lower bound, namely the condition formed by taking unions of both coordinates. Also, when $\lambda \leq 2^{\kappa}$, it is actually possible to show that in addition to having the κ^+-c.c., P^{κ}_{λ} is <u>κ-linked</u>, i.e. is the union of κ many subsets each of which consists of pairwise compatible elements. Finally, instead of P^{κ}_{λ} one could just as well have used the dense suborder $\bar{P}^\kappa_\lambda$ consisting of those $\langle F,S\rangle \in P^\kappa_\lambda$ such that: if $F: a\times\gamma \to \kappa$ say, then for each $\langle s,\phi\rangle \in S$ we have $s \subseteq a$. $\bar{P}^\kappa_\lambda$ has the basic properties of P^κ_λ, but in addition is <u>well-met</u>, i.e. any two compatible conditions have a <u>greatest</u> lower bound. Thus, taking the concrete case $\kappa = \omega_1$, $\bar{P}^{\omega_1}_\lambda$ meets the requirements for Baumgartner's [B] generalization of Martin's Axiom. Hence, $\Delta^+(\omega_1,\lambda)$ holds in any model of CH & $\lambda^\omega < 2^{\omega_1}$ & Baumgartner's Axiom, and such models exist with 2^{ω_1} arbitrarily large.

Forcing with P^κ_λ is demonstrably different from adding many Cohen subsets of κ with κ^- size approximations, since conditions are bound together by the lateral constraints $\langle s,\phi\rangle$. However, there is complete freedom in choosing the values of any particular f^G_α on arbitrarily large κ^- size intervals. For example, for any $\alpha < \lambda$ and $E \subseteq \kappa^-$, $\{\langle F,S\rangle \in P^\kappa_\lambda \mid \exists\beta<\kappa\forall\xi<\kappa\ (F(\alpha,\beta+\xi)$ is defined and $= 0$ <u>iff</u> $\xi \in E)\}$ is a dense set of conditions. Thus, whether $2^{\kappa^-} = \kappa$ was satisfied in the ground model or not, f^G_0 already codes an enumeration Γ of the power set of κ^- in type κ given by: $\Gamma(\beta) = \{\xi \in \kappa^- \mid f^G_0(\beta+\xi) = 0\}$. In fact, it can be shown that f^G_0 codes a $\diamondsuit_\kappa$ sequence.

If $2^{\kappa^-} = \kappa$ is not to be retained, one can force $\Delta^+(\kappa,\lambda)$ to hold with $<\kappa^-$ size approximations when κ^- is regular. Taking the concrete case $\kappa = \omega_1$, several people including Baumgartner, Cichon, and A. Miller, noticed that if $\lambda \geq \omega_1$ Cohen reals are added with finite conditions, then $\Delta^+(\omega_1,\lambda)$ holds. The point here is that the λ reals can be recast as functions $f_\alpha\colon \omega_1 \to \omega_1$ for $\alpha < \lambda$. Now if a finite condition $p \Vdash \dot{s} \in [\lambda]^\omega$ & $\dot{\phi} \in {}^{\dot{s}}\omega_1$, then $p \Vdash \exists\alpha\in\dot{s}(\dot{f}_\alpha(\xi) = \dot{\phi}(\alpha))$ for any $\xi < \omega_1$ not yet influenced by p, since $\dot{s}$ is forced to be infinite.

Of course, adding λ Cohen reals with finite conditions preserves all cardinals and renders $2^\omega \geq \lambda$. Hajnal and Juhasz noticed that one can get $\Delta^+(\omega_1,\omega_2)$ to hold together with the GCH by just adding ω_1 Cohen reals; the idea is to use the $\Delta^+(\omega_1,\omega_1)$ example as described above, and stretch it to a $\Delta^+(\omega_1,\omega_2)$ example by using ω_2 almost disjoint subsets of ω_1. Thus, this is an alternate way to get Prikry's consistency result (see Theorem 2.2.) of $\Delta^+(\omega_1,\omega_2)$ together with the GCH.

Finally, let me turn to the situation in the constructible universe. The first result, as one might have guessed, was due to Jensen, who upon seeing Prikry's result established that his principle $\Delta^+(\omega_1,\omega_2)$ holds in L by a direct morass-type argument. More recently, Shelah and Stanley[SS1] and Velleman[V] have made the formidable apparatus of a $(\kappa,1)$-morass (gap-1 morass at κ) more tractable by providing a Martin's Axiom-type characterization. That is, certain partial orders and collections of dense sets are described, and the existence of a morass is shown to be equivalent to the proposition that for every such partial order and every such collection F of dense sets, there is a F-generic filter in the usual sense. In this way, morasses can be better understood through intuitions developed with experience in forcing. It is noteworthy that Shelah-Stanley and Velleman came up with quite distinctive formulations, with Velleman's more compact. On the other hand, the Shelah-Stanley formulation also led (see [SS2]) to a characterization of $(\kappa,1)$-morasses "with built-in $\Diamond$-principle" which exist in L when κ is not weakly compact, and this will be crucial to the considerations in §3.

With some care, the partial order described in Theorem 2.2 can be recast to fit both the Shelah-Stanley and Velleman schemes, and if $2^{\kappa^-} = \kappa$, there are enough dense sets. Thus, we have

<u>THEOREM 2.3</u>: If κ is a successor cardinal, $2^{\kappa^-} = \kappa$, and there is a $(\kappa,1)$-morass, then $\Delta^+(\kappa,\kappa^+)$. In particular, if V = L, then for every successor cardinal κ,

$$\binom{\kappa^+}{\kappa} \not\to \begin{bmatrix}\kappa^-\\ \kappa\end{bmatrix}_\kappa .$$

<u>§3. CONSISTENCY OF $\Delta(\kappa,\lambda)$</u>

This section is devoted to consistency results which encompass the case κ is inaccessible. We saw in §2 that when there is a greatest cardinal below κ, special enumeration possibilities exist for constructions with $<\kappa$ size approximations which establish the consistency of $\Delta^+(\kappa,\lambda)$ with the GCH. However, if it is no longer assumed that κ is a successor cardinal, then we must be content with $\Delta(\kappa,\lambda)$ and a more delicate situation. The thematic evolution of the following result from Theorem 2.2 should be obvious. The proof involves a new and rather elegant kind of density argument discovered independently (and much earlier) by Shelah in his direct approach to the partition relation consistency problem. The $\Delta(\kappa,\lambda)$ family provided by the proof has further strong properties, and subsumes Shelah's more ad hoc construction.

THEOREM 3.1: If the ground model V satisfies $\kappa^{<\kappa} = \kappa$, then for any cardinal $\lambda > \kappa$ in V, there is a κ^+-c.c. forcing extension in which $\Delta(\kappa,\lambda)$ holds. Further, this forcing adds no new η sequences of ordinals for any $\eta < \kappa$; also, properties like inaccessibility or Mahloness of κ are preserved.

PROOF: In order to formulate the appropriate generalization of P^{κ}_{λ}, let us use the following concept: If $S \subseteq \{\langle s,\phi\rangle | s \in [\lambda]^{<\kappa} \ \&\ \phi \in {}^{s}\kappa\}$, then $h: \cup\{s| \langle s,\phi\rangle \in S\} \to \kappa$ is a consistent map for S iff for every $\langle s,\phi\rangle \in S$, there are infinitely many elements $\alpha \in s$ so that $h(\alpha) = \phi(\alpha)$. Now define a notion of forcing as follows: Q^{κ}_{λ} is to be the collection of pairs $\langle F,S\rangle$ where

(a) F is a function: $a \times \gamma \to \kappa$ for some $\gamma < \kappa$ and $a \in [\lambda]^{<\kappa}$.

(b) S is a subset of $\{\langle s,\phi\rangle| s \in [\lambda]^{<\kappa} \ \&\ \phi \in {}^{s}\kappa\}$ of cardinality less than κ possessing a consistent map.

For $\langle F,S\rangle, \langle G,T\rangle \in Q^{\kappa}_{\lambda}$, define $\langle F,S\rangle \leq \langle G,T\rangle$ iff

(i) $F \supseteq G$ and $S \supseteq T$.

(ii) if $F: a\times\gamma \to \kappa$ and $G: b\times\delta \to \kappa$ say, then for every $\xi \in \gamma-\delta$ and $\langle s,\phi\rangle \in T$, there is an $\alpha \in s$ such that $\alpha \in a$ and $F(\alpha,\xi) = \phi(\alpha)$.

(iii) any consistent map for T can be extended to a consistent map for S.

It follows from (iii) that this notion of forcing is $<\kappa$-continuously closed, i.e. if $\eta < \kappa$ and $\alpha < \beta < \eta$ implies $\langle F_\beta,S_\beta\rangle \leq \langle F_\alpha,S_\alpha\rangle$ and $\langle F_\delta,S_\delta\rangle = \langle \bigcup_{\alpha<\delta} F_\alpha, \bigcup_{\alpha<\delta} S_\alpha\rangle$ for limit ordinals $\delta < \eta$, then there is a common extension, namely $\langle \bigcup_{\alpha<\eta} F_\alpha, \bigcup_{\alpha<\eta} S_\alpha\rangle \in Q^{\kappa}_{\lambda}$. Hence, this notion of forcing does not add new η sequences of ordinals for any $\eta < \kappa$, and for example preserves the Mahloness of κ by standard arguments. Also, for any $a \in [\lambda]^{<\kappa}$ and $\gamma < \kappa$, $D_{\langle a,\gamma\rangle} = \{\langle F,S\rangle \in Q^{\kappa}_{\lambda}| \text{ domain}(F) \supseteq a\times\gamma\}$ is dense. (To see this, note that given any $\langle G,T\rangle \in Q^{\kappa}_{\lambda}$, say with $G: b\times\delta \to \kappa$, a consistent map for T can be employed to produce a $\langle F,T\rangle \leq \langle G,T\rangle$ such that $F: c\times(\delta+1) \to \kappa$ for some $c \supseteq b$. Thus, one step extensions are always possible, and the rest follows from $<\kappa$-continuous closure.)

It now follows that if G is any generic filter over V, then one can define $\{f_\alpha^G \mid \alpha < \lambda\} \subseteq {}^\kappa\kappa$ in $V[G]$ by: $f_\alpha^G(\xi) = \beta$ iff $\exists \langle F,S\rangle \varepsilon G(F(\alpha,\xi) = \beta)$. The next task is to verify that this collection of functions satisfies $\Delta(\kappa,\lambda)$. In fact, I shall establish the following:

(†) Whenever $t \in [\lambda]^\kappa \cap V[G]$ and $\psi \in {}^t\kappa \cap V[G]$, there is an initial segment s of t of cardinality $<\kappa$ such that $|\{\xi < \kappa \mid \forall \alpha \varepsilon s(f_\alpha^G(\xi) \neq \psi(\alpha))\}| < \kappa$.

This would more than suffice.

So, suppose $\langle F,S\rangle \Vdash \dot{t} \in [\lambda]^\kappa \ \&\ \dot{\psi} \in {}^{\dot{t}}\kappa$. By induction, construct conditions $\langle F_n,S_n\rangle$, ordinals α_n, sets $t_n \in [\lambda]^{<\kappa}$ and functions $\psi_n\colon t_n \to \kappa$ as follows. Set $\langle F_0,S_0\rangle = \langle F,S\rangle$. Given $\langle F_n,S_n\rangle$, since $\dot{t}$ is forced to have cardinality κ, and Q_λ^κ is sufficiently closed, produce a condition $\langle F_{n+1},S_{n+1}\rangle \leq \langle F_n,S_n\rangle$, an ordinal α_n, a set $t_n \in [\lambda]^{<\kappa}$, and a function $\psi_n\colon t_n \to \kappa$ such that $\langle F_{n+1},S_{n+1}\rangle \Vdash \dot{t} \cap \alpha_n = t_n \ \&\ \dot{\psi}\restriction t_n = \psi_n \ \&\ \alpha_n \in \dot{t} - \bigcup\{s \mid \langle s,\phi\rangle \in S_n\}$. By a trivial extension, we can assume that $\alpha_n \in \bigcup\{s \mid \langle s,\phi\rangle \in S_{n+1}\}$. Finally, set $\langle G,T\rangle = \langle \bigcup F_n, \bigcup S_n\rangle$, $\beta = \sup \alpha_n$, $s = \bigcup t_n$, and $\phi = \bigcup \psi_n$. There is now a Claim: $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle$ is a condition extending $\langle F_n,S_n\rangle$ for every n (but not necessarily extending $\langle G,T\rangle$!). Since it would then be the case that $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle \Vdash \dot{t} \cap \beta = s \ \&\ \dot{\psi}\restriction s = \phi$, this would certainly establish (†), since if by density $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle \in G$, and $G\colon b\times\delta \to \kappa$, then $\{\xi < \kappa \mid \forall \alpha \varepsilon s\ (f_\alpha^G(\xi) \neq \phi(\alpha))\} \subseteq \delta$.

To establish the Claim, it is necessary to show that for any n and any consistent map h for $\langle F_n,S_n\rangle$, h can be extended to a consistent map for $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle$. So, fix such an n and h, and proceed by induction to define consistent maps h_i for S_{n+i} for every $i \in \omega$ as follows: Set $h_0 = h$. Given h_i, since $\langle F_{n+i+1}, S_{n+i+1}\rangle \leq \langle F_{n+i},S_{n+i}\rangle$, let $g_{i+1} \supseteq h_i$ be a consistent map for S_{n+i+1}. Remembering that $\alpha_{n+i} \in \bigcup\{s \mid \langle s,\phi\rangle \in S_{n+i+1}\} - \bigcup\{s \mid \langle s,\phi\rangle \in S_{n+i}\}$, define h_{i+1} by

$$h_{i+1}(\xi) = \begin{cases} g_{i+1}(\xi) & \text{if } \xi \neq \alpha_{n+i} \\ \phi(\alpha_{n+i}) & \text{if } \xi = \alpha_{n+i} \end{cases}$$

Clearly, $h_{i+1} \supseteq h_i$ is again a consistent map for S_{n+i+1} since only one value was changed. Finally, set $\bar{h} = \bigcup_i h_i$, so that $\bar{h}$ is a consistent map for $T = \bigcup_{j\in\omega} S_j$. Moreover, for each $i \in \omega$, we have $\bar{h}(\alpha_{n+i}) = \phi(\alpha_{n+i})$, so that $\bar{h}$ is actually a consistent map for $T \cup \{<s,\phi>\}$. This establishes the Claim.

All that remains is to establish the κ^+-c.c. for Q^κ_λ. So, suppose that $\{<F_\alpha,S_\alpha>|\ \alpha < \kappa^+\} \subseteq Q^\kappa_\lambda$, where $F_\alpha\colon a_\alpha \times \gamma_\alpha \to \kappa$. Standard Δ-system arguments using $\kappa^{<\kappa} = \kappa$ (see Jech[J1], p.248, for an example) show that there is a $W \in [\kappa^+]^{\kappa^+}$, a $\gamma < \kappa$, and a $z \in [\lambda]^{<\kappa}$ such that:

(1) $\alpha \in W$ implies $\gamma_\alpha = \gamma$.

(2) $\alpha \neq \beta \in W$ implies $a_\alpha \cap a_\beta = z$.

(3) $\alpha,\beta \in W$ implies $F_\alpha \restriction z\times\gamma = F_\beta \restriction z\times\gamma$.

Thus, for any $\alpha,\beta \in W$, $F_\alpha \cup F_\beta$ is still a function. To take care of the S_α's, first find $X \in [W]^{\kappa^+}$ and a $\mu < \kappa$ such that $\alpha \in X$ implies $|S_\alpha| = \mu$. For such α, write $S_\alpha = \{<s^\alpha_\xi,\phi^\alpha_\xi>|\ \xi < \mu\}$. By a further Δ-system argument using $\kappa^{<\kappa} = \kappa$, one can find $Y \in [X]^{\kappa^+}$ and a T such that:

(4) $\alpha \neq \beta \in Y$ implies $\bigcup\{s|<s,\phi> \in S_\alpha\} \cap \bigcup\{s|<s,\phi> \in S_\beta\} = T$.

(5) $\alpha,\beta \in Y$ implies $<<s^\alpha_\xi \cap T, \phi^\gamma_\xi \restriction T>|\ \xi < \mu> = <<s^\beta_\xi \cap T, \phi^\beta_\xi \restriction T>|\ \xi < \mu>$.

For $\alpha \in Y$, write $T_\alpha = \bigcup\{s|<s,\phi> \in S_\alpha\} - T$. By $\kappa^{<\kappa} = \kappa$, there are at most κ structures $<\rho,<,A_\xi>_{\xi<\mu}$ where $\rho < \kappa$ and the A_ξ's are unary predicates. Each $M_\alpha = <T_\alpha,<,s^\alpha_\xi \cap T_\alpha>_{\xi<\mu}$ when transitized is isomorphic to one of these, so by cardinality considerations there is a $Z \in [Y]^{\kappa^+}$ such that:

(6) $\alpha,\beta \in Z$ implies there is an isomorphism $\pi_{\alpha\beta}\colon M_\alpha \to M_\beta$, and

(7) $\phi^\alpha_\xi(\delta) = \phi^\beta_\xi(\pi_{\alpha\beta}(\delta))$ for $\delta \in s^\alpha_\xi \cap T_\alpha$.

It is now claimed that if $\alpha,\beta \in Z$, then $<F_\alpha \cup F_\beta, S_\alpha \cup S_\beta>$ is a condition extending both $<F_\alpha,S_\alpha>$ and $<F_\beta,S_\beta>$, thereby completing the proof. It suffices mutatis mutandis to show that if h is a consistent map for S_α, then h can be extended to a consistent map for $S_\alpha \cup S_\beta$. Let $\pi_{\alpha\beta}: M_\alpha \to M_\beta$ be as in (6) and (7). Then it is straightforward using (4) and (5) that if $\bar{h}$ is defined by:

$$\bar{h}(\xi) = \begin{cases} h(\xi) & \text{if } \xi \in \bigcup\{s \mid \langle s,\phi\rangle \in S_\alpha\} \\ h(\pi_{\alpha\beta}^{-1}(\xi)) & \text{if } \xi \in T_\beta, \end{cases}$$

then $\bar{h}$ is a consistent map for $S_\alpha \cup S_\beta$. This completes the proof of the theorem.

In this consistency proof, it was essential that one actually establishes (†), a $\Diamond$-like proposition which says that any $X \in [\lambda]^\kappa$ has a proper initial segment which serves as a sufficiently good guess. To make this more precise, one can impose a further constraint on the forcing conditions to read off more information from the second coordinates. Consider:

(b)' S is a function with domain a $<\kappa$ size subset of $[\lambda]^{<\kappa}$, where $S(t) = \langle s,\phi\rangle$ for some $s \subseteq t$ and $\phi \in {}^s\kappa$, and the range of S possesses a consistent map.

Condition (b)' can be used instead of (b) for better bookkeeping, with provisions (ii) and (iii) slightly rewritten to accomodate the fact that S is now to be a function whose range should have the corresponding properties. Jech[J2] first formulated the notions of closed unbounded and stationary subsets of $[\lambda]^{<\kappa}$, and the following natural generalization of $\Diamond$:

($\Diamond^\kappa_\lambda$) There is a sequence $\langle A_t \mid t \in [\lambda]^{<\kappa}\rangle$ with $A_t \subseteq t$ such that for every $A \subseteq \lambda$, $\{t \mid A \cap t = A_t\}$ is stationary in $[\lambda]^{<\kappa}$.

($\Diamond^\kappa_\lambda$ was not Jech's notation.) The following is quite straightforward.

PROPOSITION 3.2: If G is generic over V for the notion of forcing Q^κ_λ modified with (b)' replacing (b), then $V[G]$ satisfies $\Diamond^\kappa_\lambda$. Also, $V[G]$ satisfies the usual $\Diamond_\kappa$.

PROOF: With (b)', we can define from G sets $A^G_t \subseteq t$ for every $t \in [\lambda]^{<\kappa}$ by:

$A^G_t = s$ iff $\exists \langle F,S\rangle \in G(t \in \text{domain}(S)\ \&\ S(t) = \langle s,\phi\rangle)$.

The usual argument for satisfying $\diamondsuit$ via forcing can be incorporated into the density argument for (†) in the proof of 3.1 to establish: If in $V[G]$, $A \subseteq \lambda$ with $|A| \geq \kappa$ and C is a closed unbounded subset of $[\lambda]^{<\kappa}$, then there is a $\langle t,\langle s,\phi\rangle\rangle \in S$ for a $\langle F,S\rangle \in G$ such that $t \in C$, and $s \subseteq t$ is an initial segment of A. Thus, $\langle A_t^G \mid t \in [\lambda]^{<\kappa}\rangle$ is a $\diamondsuit_\lambda^\kappa$ sequence at least for $A \subseteq \lambda$ with $|A| \geq \kappa$. But then, it is not difficult to see that it actually <u>codes</u> a $\diamondsuit_\lambda^\kappa$ sequence for all $A \subseteq \lambda$. (For instance, by using a master bijection: $\lambda \to \lambda\times 2$, first transform $\langle A_t^G \mid t \in [\lambda]^{<\kappa}\rangle$ into functional form: $\langle f_t^G \mid t \in [\lambda]^{<\kappa}\rangle$ with $f_t^G\colon t \to 2$ such that for any $f\colon \lambda \to 2$ we have that $\{t \mid f \restriction t = f_t^G\}$ is stationary in $[\lambda]^{<\kappa}$. Then convert back to set form again with $\bar{A}_t^G = \{\alpha \in t \mid f_t(\alpha) = 1\}$; by then, $\langle \bar{A}_t^G \mid t \in [\lambda]^{<\kappa}\rangle$ is a <u>bona fide</u> $\diamondsuit_\lambda^\kappa$ sequence.)

An analogous argument establishes that $\langle A_\alpha^G \mid \alpha < \kappa\rangle$ codes in similar fashion a $\diamondsuit_\kappa$ sequence.

Still another twist can be added. Erdös and Hajnal[EH] formulated the following: a set X has <u>property B</u> <u>iff</u> there is a $B \subseteq \bigcup X$ such that if $s \in X$, then $s \cap B \neq \emptyset$ and $s - B \neq \emptyset$. Define a stronger property: a set X has a <u>division</u> <u>iff</u> there is a $B \subseteq \bigcup X$ such that if $s \in X$, both $s \cap B$ and $s - B$ are infinite. Consider the further amendment:

(b)" in addition to (b)', $\{s \mid \exists t,\phi(S(t) = \langle s,\phi\rangle)\}$ is to possess a division;

and add to the definition of the partial order:

(iv) any division of $\{s \mid \exists t,\phi(T(t) = \langle s,\phi\rangle)\}$ can be extended to one for $\{s \mid \exists t,\phi(S(t) = \langle s,\phi\rangle)\}$.

If G is generic over V for this further version of the forcing, then the density argument for (†) in 3.1 still works, and $\{s \mid \exists\langle F,S\rangle \in G\,\exists t,\phi(S(t) = \langle s,\phi\rangle)\}$ is such that: (a) every subset of cardinality less than κ has property B, yet (b) the whole collection cannot, as any $A \in [\lambda]^\kappa$ has an initial segment in the collection. For the first consistency results about property B, see Shelah[Sh].

In all these guises, the key feature of the forcing is the capability of taking a lower bound (like the $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle$ in 3.1) for a sequence of conditions which is not the natural one (i.e. $\langle G,T\rangle$) provided by taking unions of the coordinates. In general, $\langle G, T \cup \{\langle s,\phi\rangle\}\rangle$ is not $\leq \langle G,T\rangle$, and so Q_λ^κ is not

$<\kappa$-complete as P^{κ}_{λ} was. Recall that a partial order P is <u>$<\kappa$-directed closed iff</u> whenever $D \subseteq P$ is directed (i.e. $p,q \in D$ implies $r \leq p,q$ for some $r \in D$) and $|D| < \kappa$, then D has a lower bound. The following argument shows that Q^{κ}_{λ} is not generally $<\kappa$-directed closed:

Hajnal ([H1],Theorem 9) once established that if $2^{\mu} = \mu^+$ and μ is regular, then there is a family $F \subseteq [\mu^+]^{\mu}$ with $|F| = \mu^+$ consisting of pairwise μ-almost disjoint sets (i.e. $s \neq t \in F$ implies $|s \cap t| < \mu$), such that for every $S \in [\mu^+]^{\mu^+}$ there is an $s \in F$ with $s \subseteq X$. For any set s, let ψ^i_s be the constant function with domain s and range $\{i\}$. Given a family F as above and assuming that $\mu^+ < \kappa$, consider $D = \{<\emptyset,\{<s,\psi^i_s>\}>|\ s \in F \ \&\ i \in \{0,1\}\}$. Then $D \subseteq Q^{\kappa}_{\lambda}$ is directed; in fact, any μ members of D have a common lower bound. However, D itself cannot have a lower bound, for suppose $f\colon \mu^+ \to 2$ were a consistent map for $\{<s,\psi^i_s>|\ s \in F \ \&\ i \in \{0,1\}\}$. Either $f^{-1}(\{0\})$ or $f^{-1}(\{1\}))$ has cardinality μ^+, say the former. By hypothesis, there is an $s \in F$ with $s \subseteq f^{-1}(\{0\})$. But this is a contradiction, as there should be infinitely many $\alpha \in s$ so that $f(\alpha) = 1$.

There is one situation where it is crucial that Q^{κ}_{λ} not be $<\kappa$-directed closed. As mentioned in §1, Chudnovsky had stated that if κ is weakly compact, then

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_2 .$$

If $Q^{\kappa}_{\kappa^+}$ for such a κ were $<\kappa$-directed closed, then we can call upon the standard Silver technique of upward Easton extensions (see [J1]§36 or [KM]§25) to show that it is consistent (via a preliminary extension) to have a weakly compact κ such that if one forces with $Q^{\kappa}_{\kappa^+}$, then κ remains weakly compact. However, the $\Delta(\kappa,\kappa^+)$ family added clearly contradicts Shelah's result.

Q^{κ}_{λ}, in having a natural limit operation for procuring lower bounds but also other possibilities for lower bounds to the side, is a paradigm case of a <u>canonical limit partial order</u>, as formulated by Shelah and Stanley[SS1]. It was to handle such orderings that led Shelah and Stanley[SS2] to extend their characterization of morasses. They show that a Martin's Axiom-type characterization, with strong properties attributable to the corresponding generic filters, which also accomodates canonical limit partial orders is equivalent to the existence of $(\kappa,1)$-morasses with a "built-in $\Diamond$-principle", when there is a non-reflecting stationary subset of κ, i.e. an $S \subseteq \kappa$ which is stationary in κ yet $S \cap \alpha$ is not stationary in α for any $\alpha < \kappa$. They establish that such morasses with built-in $\Diamond$-principle hold in L, and of course, it is a well-known result of Jensen that in L, κ is not weakly compact <u>iff</u> there is a non-reflecting stationary subset of κ. That $Q^{\kappa}_{\kappa^+}$ does satisfy the Shelah-Stanley formulation is just a matter of checking, and there is enough provision in their characterization for the generic object to take care of the requirement $(\dagger)$ in 3.1 for all $t \in [\kappa^+]^{\kappa}$. Thus, we have

<u>THEOREM 3.3</u>: If $V = L$, then the following are equivalent for regular uncountable κ:

(i) κ is not weakly compact.

(ii) $\Delta(\kappa,\kappa^+)$.

(iii) $\binom{\kappa^+}{\kappa} \not\to \binom{\kappa}{\kappa}_2$

By previous remarks, the $\Delta(\kappa,\kappa^+)$ family here can also be made to satisfy $(\dagger)$ of 3.1, code a $\Diamond^{\kappa}_{\kappa^+}$ family, and exhibit the "non-compactness" of property B. As a side remark, let me mention that Jech[J2] had shown that if $V = L$, then $\Diamond^{\kappa}_{\lambda}$ holds for every $\lambda \geq \kappa$ as long as κ is a successor cardinal. The present result extends this to $\Diamond^{\kappa}_{\kappa^+}$ for any uncountable regular κ (but note the limitation to $\lambda = \kappa^+$). However, it is not clear that all this morass structure is necessary just to establish in L that $\Diamond^{\kappa}_{\kappa^+}$ holds for every uncountable regular κ. Theorem 3.3, mostly due to Shelah and Stanley, elegantly highlights the close connection that really exists between forcing and definability.

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