ON SILVER'S AND RELATED PRINCIPLES

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INTRODUCTION

Silver's Principle W_κ has been around for quite some time. It fits well into the remarkable program initiated by Jensen of formulating useful combinatorial principles which hold in the constructible universe, and which moreover can be appended to any model of set theory by straightforward forcing. Such principles have found wide application not only in the continuing investigation of set theory itself, but to problems in more general mathematics which implicitly involve the transfinite. W_κ is a useful extraction from the full structure of a morass, not so deeply embedded in the definability considerations of the constructible hierarchy, and more akin to $\Diamond_\kappa$ and $\Box_\kappa$ in comparable combinatorial complexity. This paper is perhaps the first systematic discussion of W_κ, and one objective is to establish W_κ to a wider audience as an independent and useful principle of construction, alongside $\Diamond_\kappa$ and $\Box_\kappa$.

Silver initially isolated his principle from a morass in order to effect constructions in set-theoretic topology. To backtrack a bit, Prikry had introduced the method of forcing with side conditions in order to establish the relative consistency of a combinatorial principle which in turn had an application in the partition calculus of Erdős, Hajnal, and Rado. Combinatorialists in Eastern Europe, inspired by Prikry, used this new approach to solve further problems in set-theoretic topology and in the partition calculus. In somewhat ad hoc fashion, many of these new principles were then shown to follow from the full structure of a morass, by the morass experts. After the introduction to W_κ in the first section of this paper, the second section brings together results--some known, others not --which show how these principles actually form a hierarchy of implications emanating from just the relatively simple proposition $\underline{W}_\kappa$. It should be noted that recent work of Shelah has considerably extended the method of forcing with side conditions, to establish the relative consistency of propositions for which it is no longer clear that analogues in the constructible universe exist.

The third section of this paper discusses a form of W_κ available at limit cardinals which is endowed with the requisite strength. Its relative consistency is established through a forcing technique involving a new kind of density argument. Then, the recent work of Shelah and Stanley on a Martin's Axiom-type characterization of morasses, in the particular case of canonical limit partial orderings at non-weakly compact cardinals, is called upon to establish the relative consistency in L, and moreover to provide some new characterizations of weak compactness there. An evident precursor to this section is the author's paper [Ka2].

The set theoretical notation is standard, and the following short litany should take care of any variations: The first Greek letters $\alpha,\beta,\gamma,\ldots$ denote ordinals, whereas the middle Greek letters $\kappa,\lambda,\mu,\ldots$ are reserved for infinite cardinals. If x is a set, $|x|$ denotes its cardinality, $[x]^{\kappa}$ denotes the collection of subsets of x of cardinality κ, and if f is a function, $f''x = \{f(y) \mid y \in x\}$. Finally, ${}^{y}x$ denotes the set of functions from y into x.

§1. THE PRINCIPLE

In order to formulate Silver's Principle, it will be helpful to establish once and for all some conventions regarding trees. If T is a tree, T_α will denote the members of T at its αth level; if $x \in T_\beta$ and $\alpha \leq \beta$, then $\pi_\alpha(x)$ is the tree predecessor of x at level α. Let us assume that trees are normalized at limits, i.e. if δ is a limit ordinal and $x \neq y$ are both in T_δ, then there is an $\alpha < \delta$ such that $\pi_\alpha(x) \neq \pi_\alpha(y)$. A <u>$\kappa$-Kurepa tree</u> is a tree with height $\kappa+1$ such that $|T_\kappa| > \kappa$, yet $|T_\alpha| \leq \alpha$ for every $\alpha < \kappa$. (This is clearly congruous with the usual definition; it will be convenient to identify cofinal branches with a top level.) As usual, a <u>Kurepa tree</u> is a ω_1-Kurepa tree. This settled, we can state Silver's Principle W_κ for κ a successor cardinal with κ^- its predecessor:

(W_κ) There is a κ-Kurepa tree T and a function W with domain κ such that:
- (a) for each $\alpha < \kappa$, we have $W(\alpha) \subseteq [T_\alpha]^{\kappa^-}$ with $|W(\alpha)| \leq \kappa^-$.
- (b) for any $s \in [T_\kappa]^{\kappa^-}$, there is a $\gamma < \kappa$ such that whenever $\gamma \leq \alpha < \kappa$, we have $\pi_\alpha''s \in W(\alpha)$.

Like $\Diamond_\kappa$, W_κ can provide constructions in κ stages which are universal in some sense. But whereas $\Diamond_\kappa$ and its variants anticipate subsets of κ and manage to meet requirements at cofinally many stages, W_κ takes care of subsets of κ^+ and manages to meet requirements at all sufficiently large stages. Mind you, $\Diamond_\kappa$ is an enumeration of potentialities for all arbitrary subsets of κ, whilst W_κ is only able to handle all the actual size κ^- subsets of κ^+. The two principles, in any case, are quite disparate.

The potency of W_κ comes from the function W. Suppose that T is a tree with height $\kappa+1$. Even if T were not necessarily a κ-Kurepa tree, as long as $|T_\kappa| \geq \kappa^-$ and there is a function W satisfying (a) and (b) above for this T, it is not difficult to see that $2^{\kappa^-} = \kappa$ must be satisfied. On the other hand, merely having a Kurepa tree is consistent with the negation of the Continuum Hypothesis; such trees exist in L, so just add many reals to L generically whilst preserving cardinals.

Already, it is convenient to focus on the concrete case $\kappa = \omega_1$, even though the following remarks hold generally. Interestingly enough, an amplification of an old remark of Silver and Rowbottom about Kurepa trees provides a simple plausibility argument for W_{ω_1} :

Suppose that λ is a strongly inaccessible cardinal. Let T be the complete binary tree $\bigcup\{{}^{\alpha}2 \mid \alpha \leq \lambda\}$ in V, and let $V[G]$ be a generic extension via the usual Lévy collapse of λ to ω_1. Silver and Rowbottom's comment was that in $V[G]$ T is a Kurepa tree since its top level has $(2^{\lambda})^V > \omega_1$ members yet its αth level $({}^{\alpha}2)^V$ is now countable for $\alpha < \lambda$. Next, consider G as $\langle G_{\alpha} \mid \alpha < \lambda \rangle$, where G_{α} codes generic bijections of all ordinals $\leq \alpha$ with ω. Let $W(\alpha)$ be $[{}^{\alpha}2]^{\omega} \cap V[G_{\alpha}]$, which by standard arguments is countable in $V[G]$. Then W_{ω_1} holds in $V[G]$ with these definitions, since if s is a countable subset of $({}^{\lambda}2)^V$, then by the λ-c.c. of the forcing, there is a $\gamma < \lambda$ such that $s \in V[G_{\alpha}]$ for $\gamma \leq \alpha < \lambda$. Clearly, $\pi_{\alpha}{}''s \in W(\alpha)$ for such α, if we take γ large enough so that $\pi_{\gamma} \restriction s$ is injective.

This simple argument makes W_{ω_1} seem quite natural. The official relative consistency result through forcing has more details, but does not need the consistency strength of a strongly inaccessible cardinal. To the usual notion of forcing for adjoining a Kurepa tree one appends further side conditions which renders the strengthened W_{ω_1} in the generic extension (see Burgess [Bu1] for an exposition, and 3.1 below). This notion of forcing is countably closed and moreover has the ω_2-c.c. if the CH holds, and so preserves all cardinals in this case. Also, it is worthwhile mentioning that this forcing meets the requirements for Baumgartner's [Ba] generalization of Martin's Axiom, as is outlined in Tall [T], and hence W_{ω_1} holds in any model of Baumgartner's Axiom, CH, and $2^{\omega_1} > \omega_2$.

More to the point is that if $V = L$, then W_{κ} holds for every successor cardinal κ. In fact, if there is a $(\kappa,1)$-morass (a gap-1 morass at κ) for κ a successor cardinal with $2^{\kappa^-} = \kappa$, then W_{κ} holds. Indeed, Silver had originally extracted W_{κ} as a useful combinatorial residue from a $(\kappa,1)$-morass. That so many of the long-winded combinatorial emanations from morasses actually follow from W_{κ}, as we shall see, is a testimonial to Silver's insight.

More recently, Shelah and Stanley [SS1] and Velleman [V] have made the formidable apparatus of a $(\kappa,1)$-morass more tractable by providing a Martin's Axiom-type characterization. That is, certain partial orders and collections of dense sets are described, and the existence of a morass is shown to be equivalent to the proposition that for every such partial order and every such collection F

of dense sets, there is an F-generic filter in the usual sense. This provides an alternate way of seeing that W_κ holds in L. It is noteworthy that Shelah-Stanley and Velleman came up with quite distinctive formulations, with Velleman's more compact. On the other hand, Shelah and Stanley [SS2] have an extension when κ is a regular but not weakly compact cardinal which will be crucial in §3.

It was already observed that W_κ implies $2^{\kappa^-} = \kappa$. The consistency strength of CH plus $\neg W_{\omega_1}$ is not difficult to determine. Silver [Si] long ago established that in any generic extension via the Lévy collapse of a strongly inaccessible cardinal to ω_2, there are no Kurepa trees at all. Conversely, it is known that if $A \subseteq \omega_1$, then inside $L[A]$ there is a $(\omega_1,1)$-morass. Thus, a standard argument shows that if the CH holds yet W_{ω_1} is false, then ω_2 must be inaccessible in L : Otherwise, one can find an $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$, $\omega_2^{L[A]} = \omega_2$, and (by CH) $[\omega_2]^\omega \subseteq L[A]$. Now inside $L[A]$, there is an $(\omega_1,1)$-morass, so W_{ω_1} holds. But then W_{ω_1} must hold in the universe by absoluteness, by the conditions on A. Thus, we have the following equiconsistency:

Con(ZFC + There is an inaccessible cardinal) <u>iff</u> Con(ZFC + CH + $\neg W_{\omega_1}$).

Concerning Jensen's principle $\square_{\omega_1}$, notice that if λ is an inaccessible non-Mahlo cardinal in L, then any generic extension of L via the Lévy collapse of λ to ω_2^L is a model of the theory ZFC + CH + $\square_{\omega_1}$ + $\neg W_{\omega_1}$, since Jensen has shown ([Jen]p.286) that if $\square_\kappa$ fails, then κ^+ must be Mahlo in L. An argument due to Baumgartner yields ZFC + W_{ω_1} + $\neg\square_{\omega_1}$: Let μ be Mahlo and first adjoin a witness for W_{ω_1} with μ cofinal branches, i.e. do the usual forcing (see [Bul]) but provide for the labeling of μ branches. Then the collapse of μ to ω_2 yields $\neg\square_{\omega_1}$ by an argument due to Solovay, and W_{ω_1} is still retained.

<u>§2. THE RELATED PRINCIPLES</u>

This section describes how several higher combinatorial principles first devised in set theoretical praxis, particularly in the partition calculus and in set-theoretic topology, actually form a hierarchy emanating from W_κ. It is significant that the principles were each formulated by combinatorialists to isolate salient features of particular constructions, and shown by them to be consistent by forcing. Then, the specialists in L established how they hold there, using morasses. The realization that they are all derivable from the relatively simple W_κ is a more recent, synthetic phenomenon.

Pondering the existence of special topological spaces of large cardinality, Hajnal and Juhász realized in the early 1970's that concrete constructions readily follow from certain existential principles concerning matrices of sets. The following proposition is a conglomeration of these principles, and can be appropriately dubbed the <u>Hajnal-Juhász Principle</u>. It is somewhat of a long-winded generalization in the Shelavian manner, but a convenient unification.

(HJ_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $\rho < \kappa^-$ and $s: \kappa^- \times \rho \to \kappa^+$ is injective, there is a $\gamma < \kappa$ such that: if $x \in [\kappa-\gamma]^{<\omega}$ and $\{\varepsilon_\tau \mid \tau < \rho\} \subseteq {}^x 2$, there is a $\sigma < \kappa^-$ with $\varepsilon_\tau \subseteq f_{s(\sigma,\tau)}$ for every $\tau < \rho$.

In set-theoretic topology, HL and HS are acronymic for hereditarily Lindelöf and hereditarily separable, respectively, and an L(S) space is an HL(HS) space which is not HS(HL). Finally, X is a strong L(S) space <u>iff</u> for every $n \in \omega$, X^n is a L(S) space. There is quite a literature on the study of these spaces nowadays, particularly in connection with Martin's Axiom, and a good but older reference is M.E.Rudin [R] Chapter 5. An initial version of HJ_κ was considered by Hajnal and Juhász with the restriction to just $\rho = 1$. Taking again the concrete case $\kappa = \omega_1$ (otherwise, we would have to frame the discussion in general terms around κ^--Lindelof and κ^--separable), they show [HJ1] that this restricted principle implies the existence of normal S spaces of large cardinality, the so-called HFD spaces, and establish its consistency by forcing. Then Devlin [D] established this restricted principle in L, directly using morasses. As Hajnal and Juhász later realized, the full principle HJ_κ implies the existence of normal, <u>strong</u> S spaces of large cardinality. Kunen [Ku] has shown that under MA + ¬CH, there are no strong S spaces.

Concerning L spaces, Hajnal and Juhász early on [HJ2] formulated the following principle to construct L spaces of large cardinality:

(HJ_κ^-) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $\rho < \kappa^-$ and $s: \kappa^- \times \rho \to \kappa^+$ is injective and $\phi: \kappa^- \times \rho \to 2$, we have $|\{\xi < \kappa \mid \forall\sigma<\kappa^- \exists\tau<\rho (f_{s(\sigma,\tau)}(\xi) \neq \phi(\sigma,\tau))\}| < \kappa$.

(Actually, they had a further condition on $\{f_\alpha \mid \alpha < \kappa^+\}$ to insure good separation properties for the space constructed, but this is the crux of the matter.) That HJ_κ^- follows from HJ_κ is not unexpected; in fact, it follows from the simplified version of HJ_κ where $|x| = 1$:

<u>Lemma 2.1</u>: $HJ_\kappa \to HJ_\kappa^-$.

$\vdash$ Let $\Gamma: \kappa^+ \times 2 \leftrightarrow \kappa^+$ be a bijection, and given $\{f_\alpha \mid \alpha < \kappa^+\}$ as provided by HJ_κ, set

$$g_\alpha(\xi) = \begin{cases} i & \text{if } f_{\Gamma(\alpha,i)}(\xi) = 0 \text{ and } f_{\Gamma(\alpha,1-i)}(\xi) = 1 \\ 0 & \text{if there is no such } i\,. \end{cases}$$

Then $\{g_\alpha \mid \alpha < \kappa^+\}$ satisfies HJ_κ^-: Suppose that $\rho < \kappa^-$ and $s: \kappa^- \times \rho \to \kappa^+$ is injective and $\phi: \kappa^- \times \rho \to 2$. Define $\bar{s}: \kappa^- \times \rho 2 \to \kappa^+$ by: $\bar{s}(\sigma,\tau) = \Gamma(s(\sigma,\tau),\phi(\sigma,\tau))$ and $\bar{s}(\sigma,\rho+\tau) = \Gamma(s(\sigma,\tau), 1-\phi(\sigma,\tau))$ for every $\tau < \rho$. Then $\bar{s}$ is also injective, so let γ be as in HJ_κ for $\bar{s}$. Hence, if $\gamma \leq \xi < \kappa$, taking $x = \{\xi\}$ and $\varepsilon_\tau(\xi) = 0$ if $\tau < \rho$, and $= 1$ if $\rho \leq \tau < \rho 2$, there is a $\sigma < \kappa^-$ such that

$f_{\bar{s}(\sigma,\tau)}(\xi) = \varepsilon_\tau(\xi)$ for every $\tau < \rho 2$. But then, $g_{s(\sigma,\tau)}(\xi) = \phi(\sigma,\tau)$ for every $\tau < \rho$. Thus, $\{\xi < \kappa \mid \forall\sigma<\kappa^- \exists\tau<\rho(g_{s(\sigma,\tau)}(\xi) \neq \phi(\sigma,\tau))\} \subseteq \gamma$ and hence has cardinality $< \kappa$.

$\dashv$

The following theorem makes the connection with Silver's Principle. The proof can be culled from [HJ1], [HJ21, and [Ju], but let me provide the details for the interested reader. If this first derivation from W_κ proves too daunting see Theorem 2.4 below for a notationally simpler use of W_κ.

Theorem 2.2: $2^{<\kappa^-} = \kappa^-$ & $W_\kappa \rightarrow HJ_\kappa$.

$\vdash$ In applying W_κ one can actually assume that there is a κ-Kurepa tree T with a function W such that:

(i) $T_\kappa = \kappa^+$,

(ii) for any $\xi < \kappa$, $W(\xi) \subseteq \{s \mid \exists\rho<\kappa^-(s\colon \kappa^-\times\rho \rightarrow T_\xi$ is injective$)\}$ and $|W(\xi)| \leq \kappa^-$, and

(iii) for any injective $s\colon \kappa^-\times\rho \rightarrow \kappa^+$ where $\rho < \kappa^-$, there is a $\gamma < \kappa$ such that whenever $\gamma \leq \xi < \kappa$, we have $\pi_\xi \circ s \in W(\xi)$.

This can be seen as follows: Let $\bar{T}$ and $\bar{W}$ be as provided by W_κ, and $\xi_0 < \kappa$ be sufficiently large so that $|\bar{T}_{\xi_0}| = \kappa^-$. By an inductive relabelling which involves some minor pruning and grafting, we can assume that for $\xi_0 \leq \xi \leq \kappa$, each level $\bar{T}_\xi$ is the set $\kappa^-\times\kappa^-\times T_\xi$ for some T_ξ in such a way that: (a) $T_\kappa = \kappa^+$ and $|T_\xi| = \kappa^-$ for $\xi < \kappa$; (b) $\xi \neq \zeta$ implies $T_\xi \cap T_\zeta = \emptyset$; (c) $\langle\sigma_1,\tau_1,u\rangle < \langle\sigma_2,\tau_2,v\rangle$ in $\bar{T}$ implies $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$; and (d) $\langle\sigma_1,\tau_1,u\rangle < \langle\sigma_1,\tau_1,w\rangle$ and $\langle\sigma_2,\tau_2,v\rangle < \langle\sigma_2,\tau_2,w\rangle$ in $\bar{T}$ implies $u = v$. Now $T = \bigcup_{\xi<\xi_0}\bar{T}_\xi \cup \bigcup_{\xi_0\leq\xi\leq\kappa} T_\xi$, which by (d) has a naturally inherited tree ordering, is also a κ-Kurepa tree. Next, for $\xi < \kappa$ set $W(\xi) = \bar{W}(\xi) \cap \{s \mid \exists\rho<\kappa^-(s\colon \kappa^-\times\rho \rightarrow T_\xi$ is injective$)\}$, i.e. take only those members of $\bar{W}(\xi)$ which are functions of the first two variables of this form. Then T and W satisfy (i), (ii), and (iii) above, regarding functions s as in (iii) as sets of ordered triples.

Now, we shall inductively define functions $g_\xi\colon T_\xi \rightarrow 2$ for $\xi < \kappa$, satisfying the following inductive hypothesis:

(*) Suppose that $s \in W(\xi)$, say with $s\colon \kappa^-\times\rho \rightarrow T_\xi$, $x \in [\xi+1]^{<\omega}$ is such that $\delta \in x$ implies $\pi_\delta \circ s \in W(\delta)$, and $\{\varepsilon_\tau \mid \tau < \rho\} \subseteq {}^x2$. Then the set defined by $Z(s,\{\varepsilon_\tau \mid \tau < \rho\}) = \{\sigma < \kappa^- \mid \forall\tau<\rho\forall\delta\in x(g_\delta(\pi_\delta(s(\sigma,\tau))) = \varepsilon_\tau(\delta))\}$ has cardinality κ^-.

Once this is done, the proof can be completed by defining $f_\alpha\colon \kappa \rightarrow 2$ for $\alpha < \kappa^+$ as follows. Set

$$f_\alpha(\xi) = g_\xi(\pi_\xi(\alpha)) .$$

To ascertain HJ_κ, let $\rho < \kappa^-$ and $s\colon \kappa^-\times\rho \rightarrow \kappa^+$ be injective. By hypothesis, let $\gamma < \kappa$ be such that $\gamma \leq \xi < \kappa$ implies $\pi_\xi \cdot s \in W(\xi)$. Then using (*) and tracing through the definitions, it is easy to see that the conclusion of HJ_κ holds for s.

All that remains is to define the functions g_ξ. Suppose that this has already been done so that (*) is inductively satisfied for every $\delta < \xi$. Let K_ξ be the collection of all the κ^- size sets of form $Z(s,\{\varepsilon_\tau | \tau < \rho\})$ as defined in (*), with $s \in W(\xi)$, say with $s\colon \kappa^- \times \rho \to T_\xi$, $x \in [\xi]^{<\omega}$ is such that $\delta \in x$ implies $\pi_\delta \cdot s \in W(\delta)$, and $\{\varepsilon_\tau | \tau < \rho\} \subseteq {}^{x_\xi}2$. To perpetuate (*) at ξ, we must essentially add a layer of values with g_ξ to take care of all $x \in [\xi+1]^{<\omega}$.

Since $W(\xi)$, ξ, and $2^{<\kappa^-}$ all have cardinality $\leq \kappa^-$, there is an enumeration $\langle\langle s_\eta,\{\varepsilon^\eta_\tau | \tau < \rho_\eta\},k_\eta\rangle | \eta < \kappa^-\rangle$ of all possible triples $\langle s,\{\varepsilon_\tau | \tau < \rho\},k\rangle$ where $Z(s,\{\varepsilon_\tau | \tau < \rho\}) \in K_\xi$ and $k\colon \rho \to 2$. Moreover, we can arrange that: (a) each triple occurs κ^- many times in the enumeration, and (b) $\sum_{\zeta<\eta} |\rho_\zeta| < \kappa^-$ for every $\eta < \kappa^-$, whether κ^- is regular or not, by a dovetailing process akin to Gödel's pairing function for ordinals.

Finally, define $g_\xi\colon T_\xi \to 2$ inductively in κ^- stages, so that at the ηth stage only ρ_η many values of g_ξ are determined. Thus, before the ηth stage, only $\sum_{\zeta<\eta} |\rho_\zeta| < \kappa^-$ values have been determined. Since $Z(s_\eta,\{\varepsilon^\eta_\tau | \tau < \rho_\eta\})$ has cardinality κ^-, it must have a member σ such that g_ξ has not been defined for $s_\eta(\sigma,\tau)$ for any $\tau < \rho_\eta$. Thus, we can set $g_\xi(s_\eta(\sigma,\tau)) = k_\eta(\tau)$ at the ηth stage. Having carried out the construction through all κ^- stages, extend g_ξ arbitrarily to the whole of T_ξ.

It is now clear that (*) is satisfied at ξ: Suppose that $s \in W(\xi)$, say with $s\colon \kappa^- \times \rho \to T_\xi$, $x \in [\xi+1]^{<\omega}$ is such that $\delta \in x$ implies $\pi_\delta \cdot s \in W(\delta)$, and $\{\varepsilon_\tau | \tau < \rho\} \subseteq {}^{x}2$. If $x \subseteq \xi$, by induction there is nothing to prove. Otherwise $x = \bar{x} \cup \{\xi\}$ where $\bar{x} \subseteq \xi$, and inductively $Z(s,\{\varepsilon_\tau \restriction \xi | \tau < \rho\})$ has cardinality κ^-, i.e. is a member of K_ξ. But $\langle\varepsilon_\tau(\xi) | \tau < \rho\rangle$ is k_η for κ^- many η's, and thus our construction insures that $Z(s,\{\varepsilon_\tau | \tau < \rho\})$ has cardinality κ^-. This completes the proof of 2.2. ⊣

The next principle can be appropriately dubbed <u>Prikry's Principle</u>, and is denoted $\Delta^+(\kappa,\kappa^+)$ in Kanamori [Ka2].

(P_κ) There is a collection $\{f_\alpha | \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s\kappa$, we have $|\{\xi < \kappa | \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

Historically, P_κ was the first of these higher combinatorial principles to be formulated. In the ground breaking paper which inspired the work of Hajnal and Juhász, Prikry [P] devised his principle and established its consistency with the GCH by forcing. Prikry was answering a question of Erdős, Hajnal, and Rado [EHR] in the partition calculus, and this was the first of several examples of the phenomenon of relative consistency results, rather than outright demonstrations, in this area of set theoretical investigation. To recall the relevant case of the polarized partition symbol of [EHR],

$$\begin{pmatrix}\lambda\\ \kappa\end{pmatrix} \rightarrow \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{\gamma}$$

means that whenever $F\colon \lambda\times\kappa \rightarrow \gamma$, there are $X \in [\lambda]^{\mu}$ and $Y \in [\kappa]^{\nu}$ such that $F''(X\times Y) \neq \gamma$. To denote the negation of this proposition, $\rightarrow$ is replaced by $\not\rightarrow$. Besides [EHR], see the secondary source Williams [Wi] Chapter 4 for background.

Prikry educed P_{κ} from the negative partition relation that he wanted to establish, which is equivalent to the ostensibly weaker version where the $\phi \in {}^{s}\kappa$ just range over the constant functions:

<u>Lemma 2.3</u>: $P_{\kappa} \rightarrow \begin{pmatrix}\kappa^{+}\\ \kappa\end{pmatrix} \not\rightarrow \begin{bmatrix}\kappa^{-}\\ \kappa\end{bmatrix}_{\kappa}$.

$\vdash$ If $\{f_{\alpha} \mid \alpha < \kappa^{+}\}$ is as provided by P_{κ}, set $F(\alpha,\beta) = f_{\alpha}(\beta)$. $\dashv$

Other applications of P_{κ} are discussed in Kanamori [Ka2], where it is denoted $\Delta^{+}(\kappa,\kappa^{+})$. The first result about L in this whole context was Jensen's; he showed that P_{ω_1} follows from the morass structure that he developed in L. The following derivation from W_{κ} needs less of an edifice than Theorem 2.2.

<u>Theorem 2.4</u>: $2^{\kappa^{-}} = \kappa$ & $W_{\kappa} \rightarrow P_{\kappa}$.

$\vdash$ Let T and W be as provided by W_{κ}; as before, we might as well assume that $T_{\kappa} = \kappa^{+}$ by renaming. For each $\delta < \kappa$ and $s \in W(\delta)$, by $2^{\kappa^{-}} = \kappa$ we can enumerate ${}^{s}\kappa$ as $\{h_{\xi}^{s} \mid \delta \leq \xi < \kappa\}$. To take care of more and more of the h_{ξ}^{s}'s, for each $\xi < \kappa$ we shall define functions $g_{\xi}\colon T_{\xi} \rightarrow \kappa$ such that:

(†) Whenever $\delta \leq \eta \leq \xi$, $s \in W(\xi)$, and $\pi_{\delta}''s \in W(\delta)$, there is an $x \in s$ such that $g_{\xi}(x) = h_{\eta}^{\pi_{\delta}''s}(\pi_{\delta}(x))$.

Once this is done, the proof can be completed by defining $f_{\alpha}\colon \kappa \rightarrow \kappa$ for $\alpha < \kappa^{+}$ by:

$$f_{\alpha}(\xi) = g_{\xi}(\pi_{\xi}(\alpha)).$$

To verify P_{κ}, let $s \in [\kappa^{+}]^{\kappa^{-}}$ and $\phi \in {}^{s}\kappa$. By hypothesis, there is a $\delta < \kappa$ such that $\delta \leq \xi < \kappa$ implies $\pi_{\xi}''s \in W(\xi)$, where we can take δ sufficiently large so that π_{δ} is injective on s. For some $\eta \geq \delta$, we have $h_{\eta}^{\pi_{\delta}''s} = \phi\circ\pi_{\delta}^{-1}$. Then for any ξ such that $\eta \leq \xi < \kappa$, there is an $x \in \pi_{\xi}''s$ such that $g_{\xi}(x) = h_{\eta}^{\pi_{\delta}''s}(\pi_{\delta}(x))$. But if $\alpha = \pi_{\xi}^{-1}(x)$, then $f_{\alpha}(\xi) = g_{\xi}(x) = \phi(\alpha)$.

All that remains is to define the functions g_{ξ} so as to satisfy (†). But doing this is easy: Fix $\xi < \kappa$, and let $\{\langle s_{\zeta},\delta_{\zeta},\eta_{\zeta}\rangle \mid \zeta < \kappa^{-}\}$ enumerate all triples $\langle s,\delta,\eta\rangle$ where $\delta \leq \eta \leq \xi$, $s \in W(\xi)$, and $\pi_\delta{}''s \in W(\delta)$. Now define exactly one value of g_ξ in each of κ^- stages inductively: If $\zeta < \kappa^-$, since only ζ values have been determined before the ζth stage and s_ζ has cardinality κ^-, there is an $x \in s_\zeta$ such that $g_\xi(x)$ has not yet been defined. Set $g_\xi(x) = h_\eta^{\pi_\delta{}''s}(\pi_\delta(x))$, where $\delta = \delta_\zeta$ and $\eta = \eta_\zeta$. Finally, after κ^- stages extend g_ξ arbitrarily to all of T_ξ. This completes the construction of g_ξ, and the proof is thus complete. $\dashv$

To correlate Prikry's Principle with the work of Hajnal and Juhász, the range of the functions must be delimited to 2. Consider

(P_κ^-) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s 2$, we have $|\{\xi < \kappa \mid \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

The following is immediate.

Lemma 2.5: (i) $HJ_\kappa^- \rightarrow P_\kappa^-$.
(ii) $P_\kappa \rightarrow P_\kappa^-$.

$\vdash$ For (ii), take $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ as in P_κ and $k\colon \kappa \rightarrow 2$ any surjective function. Then $\{k \cdot f_\alpha \mid \alpha < \kappa^+\}$ satisfies P_κ^-. $\dashv$

The final principle to be discussed here is the Hajnal-Máté Principle, formulated in the course of the study of set mappings in combinatorial set theory, in Hajnal-Máté [HM].

(HM_κ) There is a collection $\{h_\xi \mid \xi < \kappa\} \subseteq {}^{\kappa^+}\kappa^+$ with $h_\xi(\alpha) \neq \alpha$ for every $\xi < \kappa$ and $\alpha < \kappa^+$ such that: whenever $s \in [\kappa^+]^{\kappa^-}$, we have $|\{\xi < \kappa \mid \forall\alpha\in s(h_\xi(\alpha) \notin s)\}| < \kappa$.

In the most general setting, a set mapping on a set X is a function f from a subset of $P(X)$ into $P(X)$ such that $x \cap f(x) = \emptyset$ for every x in the domain. A subset $H \subseteq X$ is free with respect to f iff $H \cap f(x) = \emptyset$ for every $x \subseteq H$ in the domain. The general problem of when large free subsets for set mappings exist was extensively investigated through the fifties by classical combinatorial means in Eastern Europe. (See the secondary source Williams [Wi] Chapter 3 for background; a timely application of this theory is found in Galvin-Hajnal [GH].) Forcing and in particular Prikry's method of forcing with side conditions extended the realm of possibility, and Hajnal and Máté distilled their principle with the following implication in mind: If HM_κ and there is a κ-Kurepa tree, then there is an $f\colon [\kappa^+]^3 \rightarrow \kappa^+$ such that no set of cardinality κ^- is free with respect to f. Note that HM_κ itself is a proposition about set mappings and free sets. Fitting into the pattern, Hajnal and Máté established the consistency by forcing, and then it was shown later to be true in L, this time by Burgess

[Bu2], who incidentally establishes a stronger principle in L from which W_κ is also derivable.

Theorem 2.6: $P_\kappa \to HM_\kappa$.

$\vdash$ Suppose that $\{f_\alpha \mid \alpha < \kappa^+\}$ is as provided by P_κ. For each $\alpha < \kappa^+$, let $\psi_\alpha : \alpha \to \kappa$ be injective. Finally, define $h_\xi : \kappa^+ \to \kappa^+$ for $\xi < \kappa$ by:

$$h_\xi(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0, \text{ else} \\ \psi_\alpha^{-1}(f_\alpha(\xi)) & \text{if this is defined, and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we took care that $h_\xi(\alpha) \neq \alpha$ for every $\alpha < \kappa^+$.

To verify HM_κ, suppose that $s \in [\kappa^+]^{\kappa^-}$. Let β be the least element in $s - \{0,1\}$, and set $t = s - (\beta+1)$. Define $\phi: t \to \kappa$ by $\phi(\alpha) = \psi_\alpha(\beta)$. Then $\{\xi < \kappa \mid \forall \alpha \varepsilon t (f_\alpha(\xi) \neq \phi(\alpha))\} \supseteq \{\xi < \kappa \mid \forall \alpha \varepsilon s (h_\xi(\alpha) \notin s)\}$ and hence by P_κ, this last set has cardinality $< \kappa$. $\dashv$

The following diagram summarizes the implications in this section, assuming the GCH :

$$\begin{tikzcd} W_\kappa \arrow[d] \arrow[dr] & \\ HJ_\kappa \arrow[d] & P_\kappa \arrow[d] \arrow[dl] \\ HJ_\kappa^- \arrow[d] & HM_\kappa \\ P_\kappa^- & \end{tikzcd}$$

I do not know whether any converses are possible. Particularly desirable would be the implication $P_\kappa^- \to P_\kappa$, for then the hierarchy of principles would become linear.

§3. THE GENERALIZATION

This final section is devoted to a version of W_κ available at limit cardinals. We still want a κ-Kurepa tree with a companion function W, but without assuming that there is a cardinal predecessor κ^-, we can only expect to take care of all $s \in [T_\kappa]^\kappa$ in clause (b). Since this way we will no longer have the capability of carrying out the inductive constructions of the previous section, we must enhance the principle in some other direction to get the requisite strength. There seems to be several ways of doing this; one strengthening, which presages the application to a Prikry-type principle and is natural in the context of 2.4 and [Ka2], is the following, which will be dubbed the weak Silver's Principle. We first need a definition: If $S \subseteq \{\langle s,\phi\rangle \mid s \text{ is a set } \& \ \phi \in {}^s\kappa\}$, say that an $h: \bigcup\{s \mid \exists\phi(\langle s,\phi\rangle \in S)\} \to \kappa$ is a consistent map for S iff for every $\langle s,\phi\rangle \in S$, there are infinitely many elements $x \in s$ so that $h(x) = \phi(x)$. Now the principle:

(wW_κ) There is a κ-Kurepa tree T and a function W with domain κ such that:

(a) for each $\alpha < \kappa$, we have $W(\alpha) \subseteq \{\langle s,\phi\rangle \mid s \subseteq T_\alpha \;\&\; \phi \in {}^s\kappa\}$ with $|W(\alpha)| \leq |T(\alpha)| \leq \alpha$ such that $W(\alpha)$ possesses a consistent map.

(b) for any $t \in [T_\kappa]^\kappa$ and $\phi \in {}^t\kappa$, there is an $s \in [t]^{<\kappa}$ and a $\gamma < \kappa$ such that $\pi_\gamma \restriction s$ is injective and whenever $\gamma \leq \alpha < \kappa$, we have $\langle \pi_\alpha{}''s,\ \phi \circ \pi_\alpha^{-1} \restriction \pi_\alpha{}''s\rangle \in W(\alpha)$.

Somewhat cumbersome, but one can no longer trust to chance when not assuming that there is a κ^-. It is clear from the proof of Theorem 2.4 that $2^{\kappa^-} = \kappa$ and W_κ together imply wW_κ. Let me mention that for S to possess a consistent map is a natural generalization of $X = \{s \mid \exists\phi(\langle s,\phi\rangle \in S)\}$ having Property B in the sense of Erdös and Hajnal [EH]: there is a set B such that if $s \in X$, then $s \cap B \neq \emptyset$ and $s - B \neq \emptyset$. Possessing a consistent map is a stronger transversal principle.

The consistency of wW_κ via forcing entails a new and rather elegant kind of density argument discovered independently (and much earlier) by Shelah; the argument is similar to Theorem 3.1 of [Ka2], but is presented here for the convenience of the reader. The initial Shelah argument established directly the negative partition relation which is a consequence of the forthcoming 3.2 and 3.3.

Theorem 3.1: If the ground model V satisfies $\kappa^{<\kappa} = \kappa$, then there is a κ^+-c.c. $<\kappa$-distributive forcing extension in which wW_κ holds. (Furthermore, properties like the Mahloness of κ are preserved.)

$\vdash$ The notion of forcing is an adjustment of the one for W_κ (see e.g. Burgess [Bu1]), which in turn is an augmentation of the standard Stewart [Ste] conditions for adjoining a κ-Kurepa tree.

Let Q_κ consist of quadruples $p = \langle T_p, l_p, W_p, S_p\rangle$ such that:

(i) T_p is a tree consisting of nodes which are ordinals $< \kappa$, with $|(T_p)_\alpha| \leq \alpha$ for every α. Moreover, T_p has height a successor ordinal $\alpha_p+1 < \kappa$, i.e. T_p has a top level, the α_pth.

(ii) W_p is a function with domain α_p such that each $W_p(\alpha) \subseteq \{\langle s,\phi\rangle \mid s \subseteq (T_p)_\alpha \;\&\; \phi \in {}^s\kappa\}$, possesses a consistent map, and $|W_p(\alpha)| \leq |(T_p)_\alpha| \leq \alpha$.

(iii) l_p is a bijection from a subset of $\kappa^+ - \kappa$ onto the top level of T_p.

(iv) S_p is a subset of $\{\langle s,\phi\rangle \mid s \subseteq \mathrm{domain}(l_p) \;\&\; \phi \in {}^s\kappa\}$ of cardinality $< \kappa$ possessing a consistent map.

Partially order Q_κ by setting $p \leq q$ iff

(a) T_p is an end extension of T_q.

(b) $\mathrm{domain}(l_q) \subseteq \mathrm{domain}(l_p)$, and for every $\rho \in \mathrm{domain}(l_q)$, we have $l_q(\rho) \leq l_p(\rho)$ in the tree ordering of T_p.

(c) W_p extends W_q.

(d) $S_q \subseteq S_p$, and for every α with $\alpha_q < \alpha < \alpha_p$ and every $<s,\phi> \in S_q$, we have $<(\pi_\alpha \cdot l_p)"s,\ \phi \cdot (\pi_\alpha \cdot l_p)^{-1} \restriction (\pi_\alpha \cdot l_p)"s> \ \in \ W_p(\alpha)$.

(e) Any consistent map for S_q can be extended to a consistent map for S_p.

Thus, each $p \in Q_\kappa$ is a $< \kappa$ size approximation to a witness for wW_κ, with l_p a labelling of the κ^+ eventual cofinal branches through the κ-Kurepa tree and S_p a set of requirements which must thence be met by any extension of the condition. Clause (e) is the key new idea needed to establish the important preservation properties of the forcing. With it, one can define a notion of continuity for a strictly decreasing sequence $<p_\xi | \ \xi < \eta>$, η a limit ordinal $< \kappa$, and an operation Lim on such continuous sequences such that:

(I) if $<p_\xi | \ \xi < \eta>$ is continuous, then $\mathrm{Lim}(<p_\xi | \ \xi < \eta>) \in Q_\kappa$, and then

(II) if $p = \mathrm{Lim}(<p_\xi | \ \xi < \eta>)$, then $p \leq p_\xi$ for every $\xi < \eta$ and $S_p = \bigcup_{\xi<\eta} S_{p_\xi}$.

This is done by induction on the length $\eta < \kappa$ of sequences as follows. Suppose that these notions have already been defined for limit ordinals $\delta < \eta$. Then say that a strictly decreasing sequence $<p_\xi | \ \xi < \eta>$ is <u>continuous</u> <u>iff</u> for every limit ordinal $\delta < \eta$, $p_\delta = \mathrm{Lim}(<p_\xi | \ \xi < \delta>)$. In order to define Lim for such a sequence, first set $T = \bigcup_{\xi<\eta} T_{p_\xi}$, $d = \bigcup_{\xi<\eta} \mathrm{domain}(l_{p_\xi})$, $W = \bigcup_{\xi<\eta} W_{p_\xi}$, $S = \bigcup_{\xi<\eta} S_{p_\xi}$, and $\beta = \sup_{\xi<\eta} \alpha_{p_\xi}$. Since the T_{p_ξ}'s are end extensions, T inherits a tree ordering with height β. In order to have a condition however, we must adjoin a top level--but there is canonical way of doing this.

For each $\rho \in d$, notice that $b(\rho) = \{x \in T | \ \exists\xi<\eta(\rho \in \mathrm{domain}(l_{p_\xi}) \ \& \ x \leq l_{p_\xi}(\rho)\}$ is a cofinal branch through T. We can top each branch in a canonical fashion: if ρ_0 is the least element of d, let x_{ρ_0} be the least ordinal in $\kappa - T$ and specify $x_{\rho_0} \geq$ every $x \in b(\rho_0)$ on the tree; if ρ_1 is the second least element of d, let x_{ρ_1} be the second least ordinal in $\kappa - T$ and specify $x_{\rho_1} \geq$ every $x \in b(\rho_1)$ on the tree; and so forth. Define $\bar{T}$ to be the extended tree $T \cup \{x_\rho | \rho \in d\}$, and define a map $\bar{l}$ with domain d by setting $\bar{l}(\rho) = x_\rho$. Finally, set $p = <\bar{T},\bar{l},W,S>$. The important clause (e) in the definition of the partial order combined with the inductive assumption (II) about Lim insures that $p \in Q_\kappa$ since S has a consistent map, as well as $p \leq p_\xi$ for every $\xi < \eta$. Set $p = \mathrm{Lim}(<p_\xi | \ \xi < \eta>)$. This completes the inductive definition of continuity and Lim.

$\mathbb{Q}_\kappa$ becomes a <u>κ-canonical limit ordering</u> in the sense of Shelah and Stanley [SS1][SS2] with these formulations of continuity and Lim , and this fact will figure prominently in the discussion of L below. Using these notions, it is easy to see that $\mathbb{Q}_\kappa$ is $<\kappa$-distributive, i.e. adds no new η sequences of ordinals for any $\eta < \kappa$, and for example preserves the Mahloness of κ by standard arguments. Also, for any $\alpha < \kappa$ and $\rho \in \kappa^+ - \kappa$, one can check that $D_{\alpha\rho} = \{p \in \mathbb{Q}_\kappa \mid \alpha_p \geq \alpha \ \ \& \ \ \rho \in \text{domain}(1_p)\}$ is dense in $\mathbb{Q}_\kappa$. (To see this, note that given any $p \in \mathbb{Q}_\kappa$, a one-height extension is possible by topping every $x \in T_{\alpha_p}$ with a t_x to get a new tree $\bar{T}$, defining $\bar{1}$ with the same domain as 1_p by $\bar{1}(\rho) = t_{1_p(\rho)}$, and extending W_p one step to $\bar{W}$ by setting $\bar{W}(\alpha_p) = \{<1_p"s, \ \phi \cdot 1_p^{-1} \restriction 1_p"s> \mid \ <s,\phi> \in S_p\}$. $\bar{W}(\alpha_p)$ has a consistent map since S_p does, and so $<\bar{T},\bar{1},\bar{W},S_p>$ is a condition $\leq p$. Thus, one-height extensions are always possible, and the rest follows from the use of Lim .)

With all this in mind, if G is $\mathbb{Q}_\kappa$-generic over V , set $W^G = \bigcup\{W_p \mid p \in G\}$ and $T^G = \bigcup\{T_p \mid p \in G\} \cup (\kappa^+ - \kappa)$ with the inherited tree ordering for ordinals $< \kappa$, and if $\rho \in \kappa^+ - \kappa$, specify: $x \leq \rho$ in T^G <u>iff</u> $\exists p \in G(x \leq 1_p(\rho) \text{ in } T_p)$. Then T^G is a κ-Kurepa tree (once the κ^+-c.c. is verified; see below), and the function W^G satisfies clause (a) of wW_κ .

The next task is to verify clause (b) of wW_κ in V[G] . So, suppose that $p \Vdash \dot{t} \in [\kappa^+ - \kappa]^\kappa \ \ \& \ \ \dot{\psi} \in {}^{\dot{t}}\kappa$. By induction on $n \in \omega$, construct conditions p_n , ordinals α_n , sets $t_n \in [\kappa^+]^{<\kappa}$ and functions $\psi_n : t_n \to \kappa$ as follows. Set $p_0 = p$. Given p_n , since $\dot{t}$ is forced to have cardinality κ and $\mathbb{Q}_\kappa$ is sufficiently closed, produce a condition $p_{n+1} \leq p_n$, an ordinal α_n , a set $t_n \in [\kappa^+]^{<\kappa}$, and a function $\psi_n : t_n \to \kappa$ such that

$$p_{n+1} \Vdash \dot{t} \cap \alpha_n = t_n \ \ \& \ \ \dot{\psi} \restriction t_n = \psi_n \ \ \& \ \ \alpha_n \in \dot{t} - \bigcup\{s \mid \exists\phi(<s,\phi> \in S_{p_n})\} .$$

By a trivial extension, we can further assume that $t_n \cup \{\alpha_n\} \subseteq \bigcup\{s \mid \exists\phi(<s,\phi> \in S_{p_{n+1}})\}$. Next, set $q = \text{Lim}(<p_n \mid n \in \omega>)$, $\bar{s} = \bigcup t_n$, $\bar{\psi} = \bigcup \psi_n$, $\bar{\alpha} = \bigcup(\alpha_n + 1)$, $\bar{S} = S_q \cup \{<\bar{s},\bar{\phi}>\}$. Finally, set $\bar{q} = <T_q, 1_q, W_q, \bar{S}>$.

There is now a <u>CLAIM</u>: $\bar{q}$ is a condition extending every p_n (but certainly not q !). Since it would then be the case that $\bar{q} \Vdash \dot{t} \cap \bar{\alpha} = \bar{s} \ \ \& \ \ \dot{\psi} \restriction \bar{s} = \bar{\psi}$, this would certainly establish clause (b) of wW_κ , since if by density $\bar{q} \in G$, then $<\pi_\alpha"s, \ \bar{\phi} \cdot \pi_\alpha^{-1} \restriction \pi_\alpha"\bar{s}> \ \in \ W^G(\alpha)$ for any α such that $\alpha_q < \alpha < \kappa$.

To establish the claim, it is necessary to show that for any n and consistent map h for S_{p_n} , h can be extended to a consistent map for $\bar{S}$. So, fix such an n and h , and proceed by induction to define consistent maps h_i for $S_{p_{n+i}}$ for every $i \in \omega$ as follows: Set $h_0 = h$. Given h_i , since

$p_{n+i+1} \leq p_{n+i}$, let $g_{i+1} \supseteq h_i$ be a consistent map for $S_{p_{n+i+1}}$. Remembering that $\alpha_{n+i} \in \bigcup\{s \mid \exists\phi(\langle s,\phi\rangle \in S_{p_{n+i+1}})\} - \bigcup\{s \mid \exists\phi(\langle s,\phi\rangle \in S_{p_{n+i}})\}$, define h_{i+1} by:

$$h_{i+1}(\rho) = \begin{cases} g_{i+1}(\rho) & \text{if } \rho \neq \alpha_{n+i} \\ \bar{\phi}(\alpha_{n+i}) & \text{if } \rho = \alpha_{n+i} . \end{cases}$$

Clearly, $h_{i+1} \supseteq h_i$ is again a consistent map for $S_{p_{n+i+1}}$ since only one value was changed. Finally, set $\bar{h} = \bigcup h_i$, so that $\bar{h}$ is consistent map for S_q . Moreover, for each $i \in \omega$, we have $\bar{h}(\alpha_{n+i}) = \bar{\phi}(\alpha_{n+i})$, so that $\bar{h}$ is actually a consistent map for $S_q \cup \{\langle\bar{s},\bar{\phi}\rangle\} = \bar{S}$. This establishes the Claim.

All that remains is to establish the κ^+-c.c. for Q_{κ} . So, suppose that $\{p_\xi \mid \xi < \kappa^+\} \subseteq Q_\kappa$. Standard Δ-system arguments using $\kappa^{<\kappa} = \kappa$ (see Jech[Jec] p.248 for an example) show that there is an $A \in [\kappa^+]^{\kappa^+}$, a tree T of height $\alpha+1$ for some α , a function W , and a $z \in [\kappa^+ - \kappa]^{<\kappa}$ such that:

(1) $\xi \in A$ implies $T_{p_\xi} = T$ and $W_{p_\xi} = W$,

(2) $\xi \neq \zeta \in A$ implies $\mathrm{domain}(l_{p_\xi}) \cap \mathrm{domain}(l_{p_\zeta}) = z$, and

(3) $\xi,\zeta \in A$ implies $l_{p_\xi} \restriction z = l_{p_\zeta} \restriction z$.

To take care of the S_{p_ξ}'s, first find $B \in [A]^{\kappa^+}$ and a $\mu < \kappa$ such that $\xi \in B$ implies $|S_{p_\xi}| = \mu$. For such ξ , write $S_{p_\xi} = \{\langle s^\xi_\delta, \phi^\xi_\delta\rangle \mid \delta < \mu\}$. By a further Δ-system argument using $\kappa^{<\kappa} = \kappa$, one can find $C \in [B]^{\kappa^+}$ and a y such that:

(4) $\xi \neq \zeta \in C$ implies $\bigcup\{s^\xi_\delta \mid \delta < \mu\} \cap \bigcup\{s^\zeta_\delta \mid \delta < \mu\} = y$, and

(5) $\xi,\zeta \in C$ implies $\langle\langle s^\xi_\delta \cap y,\ \phi^\xi_\delta \restriction y\rangle \mid \delta < \mu\rangle = \langle\langle s^\zeta_\delta \cap y,\ \phi^\zeta_\delta \restriction y\rangle \mid \delta < \mu\rangle$.

For $\xi \in C$, write $y_\xi = \bigcup\{s^\xi_\delta \mid \delta < \mu\} - y$. By $\kappa^{<\kappa} = \kappa$, there are at most κ structures $\langle\sigma,<,U_\delta\rangle_{\delta<\mu}$ where $\sigma < \kappa$ and the U_δ's are unary predicates. Each $M_\xi = \langle y_\xi, <, s^\xi_\delta \cap y_\xi\rangle_{\delta<\mu}$ is isomorphic to one of these, so by cardinality considerations there is a $D \in [C]^{\kappa^+}$ such that:

(6) $\xi,\zeta \in D$ implies there is an isomorphism $\pi_{\xi\zeta}: M_\xi \to M_\zeta$, and

(7) $\phi^\xi_\delta(\rho) = \phi^\zeta_\delta(\pi_{\xi\zeta}(\rho))$ for $\rho \in s^\xi_\delta \cap y_\xi$.

It is now claimed that if $\xi,\zeta \in D$, then p_ξ and p_ζ are compatible, thereby completing the proof of the theorem. To show this, first add one new level to the top of T by specifying that each $l_{p_\xi}(\rho)$ for $\rho \in z$ is to have exactly one immediate successor, and that each $l_{p_\xi}(\rho)$ for $\rho \in \mathrm{domain}(l_{p_\xi}) - z$

is to have exactly two immediate successors. This forms a new tree $\bar{T}$ of height $\alpha+2$. Define a function $\bar{l}$ with domain equal to domain $(l_{p_\xi}) \cup \mathrm{domain}(l_{p_\zeta})$ and range the top level of $\bar{T}$ by setting

$$\bar{l}(\rho) = \begin{cases} \text{the node above } l_{p_\xi}(\rho), & \text{if } \rho \in z \\ \text{the first node above } l_{p_\xi}(\rho), & \text{if } \rho \in \mathrm{domain}(l_{p_\xi}) - z \\ \text{the second node above } l_{p_\zeta}(\rho), & \text{if } \rho \in \mathrm{domain}(l_{p_\zeta}) - z \,. \end{cases}$$

Next, set $\bar{S} = S_{p_\xi} \cup S_{p_\zeta}$, and finally, extend W to a function $\bar{W}$ with domain $\alpha+1$ by specifying $\bar{W}(\alpha) = \{<\bar{l}"s, \phi\cdot\bar{l}^{-1}\restriction\bar{l}"s>\mid <s,\phi> \in \bar{S} >$.

Now, set $q = <\bar{T},\bar{l},\bar{W},\bar{S}>$. To establish that q is a condition extending both p_ξ and p_ζ, it suffices to show mutatis mutandis that if h is a consistent map for S_{p_ξ}, then h can be extended to a consistent map for $\bar{S}$. (This will incidentally show that $\bar{W}(\alpha)$ has a consistent map.) Let $\pi_{\xi\zeta}$ be as in (6) and (7) above. Then it is straightforward using (4) and (5) to show that if $\bar{h}$ is defined by:

$$\bar{h}(\rho) = \begin{cases} h(\rho) & \text{if } \rho \in y \cup y_\xi \\ h(\pi_{\xi\zeta}^{-1}(\rho)) & \text{if } \rho \in y_\zeta \,, \end{cases}$$

then $\bar{h}$ is a consistent map for $\bar{S}$. This completes the proof of the theorem. $\dashv$

The particular formulation of wW_κ was designed for application to the following direct analogue of Prikry's Principle available at limit cardinals:

(wP_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ so that whenever $s \in [\kappa^+]^\kappa$ and $\phi \in {}^s\kappa$, we have $|\{\xi < \kappa \mid \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

This principle is denoted $\Delta(\kappa,\kappa^+)$ in [Ka2], and the following is just as immediate as Lemma 2.3:

<u>Lemma 3.2</u>: $wP_\kappa \rightarrow \binom{\kappa^+}{\kappa} \not\rightarrow \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}_\kappa$.

Now the presaged application:

<u>Theorem 3.3</u>: $wW_\kappa \rightarrow wP_\kappa$.

$\vdash$ If T and W are as provided by wW_κ, with everything so prearranged, we can let $g_\xi\colon T_\xi \rightarrow \kappa$ for each $\xi < \kappa$ be any extension of a consistent map for $W(\xi)$. As usual, assume $T_\kappa = \kappa^+$ and define $f_\alpha\colon \kappa \rightarrow \kappa$ for $\alpha < \kappa^+$ by:

$$f_\alpha(\xi) = g_\xi(\pi_\xi(\alpha)) \,.$$

To verify wP_κ, let $t \in [\kappa^+]^\kappa$ and $\phi \in {}^t\kappa$. By hypothesis, there is a $\gamma < \kappa$

and an $s \in [t]^{<\kappa}$ such that $\pi_\gamma \restriction s$ is injective, and whenever $\gamma \leq \xi < \kappa$, we have $<\pi_\xi "s, \phi \cdot \pi_\xi^{-1} \restriction \pi_\xi "s> \in W(\xi)$. Thus, for such ξ there is an $x \in \pi_\xi "s$ such that $g_\xi(x) = \phi \cdot \pi_\xi^{-1}(x)$ since g_ξ is consistent for $W(\xi)$, i.e. if $\alpha = \pi_\xi^{-1}(x)$, then $f_\alpha(\xi) = \phi(\alpha)$. This establishes wP_κ. $\dashv$

Actually, wP_κ is equivalent to the weak version of wW_κ where the thinness requirements on the tree are dropped, i.e. the reformulation of wW_κ with T just a tree of height κ with κ^+ branches, and the condition $|W(\alpha)| \leq |T(\alpha)| \leq \alpha$ eliminated. There does not seem to be a good analogue for HJ_κ, but HM_κ can be weakened by replacing $s \in [\kappa^+]^{\kappa^-}$ by $s \in [\kappa^+]^\kappa$ to yield a principle wHM_κ, which follows from wP_κ with the same proof as for Theorem 2.6.

It is appropriate here to digress somewhat with a discussion of the polarized partition relation denied by wP_κ, before the final synthesis which characterizes wW_κ in the constructible universe. In general, the proposition

$$(*) \qquad \begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{bmatrix} \kappa \\ \kappa \end{bmatrix}_2$$

seems to hold but rarely. The earliest result along these lines was due to Erdös and Rado[ER]Theorem 48, who established (*) for $\kappa = \omega$. Hajnal[H2] then established that (*) is true for κ a measurable cardinal; see also Chudnovsky[C] and Kanamori[Ka1] for some refinements. Chudnovsky claims without proof in his paper that (*) holds for κ weakly compact, and proofs have since been provided by Wolfsdorf[Wo] and Shelah. We shall soon see that, at least in L, this is as far as one can go. The following is yet another proof of the implication from weak compactness, which is more compact than the published proof in [Wo]; it uses an idea from [Ka1].

Theorem 3.4: If κ is weakly compact, then $\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{bmatrix} \eta \\ \kappa \end{bmatrix}_2$ for every $\eta < \kappa^+$.

$\vdash$ Suppose that $F\colon \kappa^+ \times \kappa \to 2$ is given, and set $X^i_\alpha = \{\delta < \kappa \mid F(\alpha, \delta) = i\}$ for $\alpha < \kappa^+$ and $i < 2$. Say that an interval I of ordinals $< \kappa^+$ is <u>full iff</u> $|I| = \kappa$ and whenever $\beta > \sup I$, for every $\xi < \kappa$ we have

$$|\{\alpha \in I \mid X^0_\alpha \cap \xi = X^0_\beta \cap \xi\}| = \kappa .$$

Of course, the 0 could have been replaced by 1 here, since $X^0_\alpha \cup X^1_\alpha = \kappa$. Notice that any $\bar\alpha < \kappa^+$ is the start of a full interval:

First, let $\{s_\rho \mid \rho < \kappa\}$ enumerate $[\kappa]^{<\kappa}$ so that each set appears κ times. Then define ordinals α_ρ for $\rho < \kappa$ by induction starting with $\alpha_0 = \bar\alpha$ and in general letting α_ρ be the least $\alpha > \sup\{\alpha_\delta \mid \delta < \rho\}$ so that s_ρ is an initial segment of X^0_α, if there is such an α (and $\alpha = \sup\{\alpha_\rho \mid \delta < \rho\} + 1$ otherwise). Finally, set $\bar\beta = \sup\{\alpha_\rho \mid \rho < \kappa\}$. To verify that $I = [\bar\alpha, \bar\beta)$ is a full interval, assume to the contrary that $\beta \geq \bar{\beta}$ yet for some $\xi < \kappa$ we have $|\{\alpha \in I \mid X^0_\alpha \cap \xi = X^0_\beta \cap \xi\}| < \kappa$. Let $\rho_1 < \kappa$ be such that α_{ρ_1} is greater than any α in this set, and let $\rho_2 > \rho_1$ be such that $s_{\rho_2} = X^0_\beta \cap \xi$. Then α_{ρ_2} must have been defined via the first, non-trivial clause, since β was an available candidate. This contradicts the choice of ρ_1.

To establish the theorem, let $\eta < \kappa^+$ be given. The goal is to find a $j < 2$, a set D of ordertype η, and a set $Z \in [\kappa]^\kappa$ such that $Z \subseteq \bigcap\{X^j_\alpha \mid \alpha \in D\}$. By increasing η if necessary, we might as well assume for convenience that η is an indecomposable limit ordinal $\geq \kappa$. Now let $\langle\sigma_\zeta \mid \zeta \leq \eta\rangle$ be any closed, increasing sequence of ordinals $< \kappa^+$ such that each $[\sigma_\zeta, \sigma_{\zeta+1})$ is a full interval; such a sequence exists by the previous paragraph. Next, invoking the weak compactness of κ, let U be any non-principal κ-complete ultrafilter on the κ-field of sets generated by $\{X^i_\alpha \mid i < 2 \;\&\; \alpha < \sigma_\eta\}$ and the bounded subsets of κ. Surely, there is a fixed $j < 2$ and a set $A \in [\kappa^+]^{\kappa^+}$ such that for any $\beta \in A$, there are order-type η many full intervals $[\sigma_\zeta, \sigma_{\zeta+1})$ in the natural ordering on which: for every $\xi < \kappa$,

$$|\{X^j_\alpha \in U \mid X^j_\alpha \cap \xi = X^j_\beta \cap \xi \quad \& \quad \sigma_\zeta \leq \alpha < \sigma_{\zeta+1}\}| = \kappa .$$

Call these $[\sigma_\zeta, \sigma_{\zeta+1})$ the <u>full intervals for β</u>.

Now, we might as well assume that there is a $\beta_0 \in A$ such that $|X^j_{\beta_0} \cap X| = \kappa$ for every $X \in U$. If this were not so, there would be a $B \in [A]^{\kappa^+}$ and a fixed $X \in U$ such that for every $\beta \in B$, $|X^j_\beta \cap X| < \kappa$; in fact, we can further assume that $y = X^j_\beta \cap X$ is also fixed. But then, $X - y \subseteq X^{1-j}_\beta$, and hence we would have $F''(B \times (X - y)) = \{1-j\}$, which is more than adequate to establish the theorem.

Next, since $|U| = \kappa$ and U is κ-complete, we can easily produce a $Y \in [X^j_{\beta_0}]^\kappa$ such that $|Y - X| < \kappa$ for every $X \in U$. Finally, fix a bijection $\psi\colon \kappa \leftrightarrow \eta$.

We now proceed by induction to choose a closed unbounded set $\{\xi_\gamma \mid \gamma < \kappa\} \subseteq \kappa$ and a set $\{\alpha_\delta \mid \alpha < \kappa\} \subseteq \kappa^+$ as follows: At the γth step, if γ is a limit ordinal $< \kappa$, set $\xi_\gamma = \sup_{\delta<\gamma} \xi_\delta$. Otherwise, with ξ_γ already given, first choose α_γ in the $\psi(\gamma)$th full interval for β_0 such that $X^j_{\alpha_\gamma} \in U$ and $X^j_{\alpha_\gamma} \cap \xi_\gamma = X^j_{\beta_0} \cap \xi_\gamma$. (This is possible since $\beta_0 \in A$.) Then let $\xi_{\gamma+1}$ be the least $\xi > \xi_\gamma$ such that $Y - \xi \subseteq X^j_{\alpha_\gamma}$.

With this construction, $\{\alpha_\gamma \mid \alpha < \kappa\}$ clearly has ordertype η. So, by splitting it into two parts each of ordertype η, we can see that there must be a $D \subseteq \{\alpha_\gamma \mid \gamma < \kappa\}$ of ordertype η such that if $T = \bigcup_{\alpha_\gamma \in D} [\xi_\gamma, \xi_{\gamma+1})$, then $Z = Y - T$ still has cardinality κ.

The proof can now be concluded by verifying that $Z \subseteq \bigcap\{X^j_\alpha \mid \alpha \in D\}$. Indeed, if $\alpha_\gamma \in D$ and $\delta \in Z$, then either $\delta < \xi_\gamma$ or $\xi_{\gamma+1} \leq \delta$. If $\delta < \xi_\gamma$, then $\delta \in Z \subseteq Y \subseteq X^j_{\beta_0}$ implies that $\delta \in X^j_{\alpha_\gamma}$ by choice of α_γ. If $\xi_{\gamma+1} \leq \delta$, then $\delta \in Y - \xi_{\gamma+1} \subseteq X^j_{\alpha_\gamma}$. Hence, in either case $\delta \in X^j_{\alpha_\gamma}$, and the proof is complete. ⊣

To bring this section and this paper to a close, let us first take another look at the proof of Theorem 3.1. The key feature of the notion of forcing Q_κ was the capability of taking lower bounds which are not the natural ones provided by taking unions of the coordinates; in general, Q_κ does not have greatest lower bounds. Recall that a partial order P is <u>$<\kappa$-directed closed</u> <u>iff</u> whenever $D \subseteq P$ is directed (i.e. $p,q \in D$ implies $r \leq p,q$ for some $r \in D$) and $|D| < \kappa$, then D has a lower bound. Q_κ is not in general $<\kappa$-directed closed, as the following argument attests:

If κ is weakly compact and Q_κ were $<\kappa$-directed closed, then one could call upon the Silver technique of upward Easton extensions (see [Jec]§36 or [KM] §25) to show that it is consistent (via a preliminary extension) to have a weakly compact κ such that if one forces with Q_κ, then κ remains weakly compact in the extension. But, this would contradict 3.1-3.4.

Q_κ, in having a natural operation Lim for procuring lower bounds but also other possibilities for lower bounds to the side, is a paradigm case of a <u>κ-canonical limit partial ordering</u>, in the sense of Shelah and Stanley[SS1]. It was to handle such orderings that led Shelah and Stanley[SS2] to extend their characterization of morasses. They show that a Martin's Axiom-type characterization, with strong properties attributable to the corresponding generic filters, which also accomodates canonical limit partial orderings is equivalent to the existence of $(\kappa,1)$-morasses with "built-in $\Diamond$-principle", when there is a non-reflecting stationary subset of κ, i.e. an $S \subseteq \kappa$ which is stationary in κ yet $S \cap \alpha$ is not stationary in α for any $\alpha < \kappa$. They establish that such morasses with built-in $\Diamond$-principle exist in L, and of course, it is a well-known result of Jensen that in L, κ is not weakly compact <u>iff</u> there is a non-reflecting stationary subset of κ. That Q_κ does satisfy the Shelah-Stanley formulation is just a matter of checking, and there is enough provision for the generic object to satisfy the needed conclusions. Thus, we have the following confluence of 3.1-3.4.

<u>Theorem 3.5</u>: If $V = L$, then the following are equivalent for regular $\kappa > \omega$:

(i) κ is not weakly compact.

(ii) wW_κ.

(iii) wP_κ.

(iv) $\binom{\kappa^+}{\kappa} \not\to \begin{bmatrix} \kappa \\ \kappa \end{bmatrix}_\kappa$

(v) $\binom{\kappa^+}{\kappa} \not\to \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}_2$.

This characterization in L of weak compactness in terms of wW_κ is a pleasing complement to Jensen's characterization in L that a regular cardinal κ is ineffable just in case there are no κ-Kurepa trees at all.

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