MORASSES IN COMBINATORIAL SET THEORY

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Jensen invented the morass in order to establish strong model-theoretic transfer principles in the constructible universe. Morasses are structures of considerable complexity, a culminating edifice in Jensen's remarkable program of formulating useful combinatorial principles which obtain in the constructible universe, and which moreover can be appended to any model of set theory by straightforward forcing. Gödel's Axiom of Constructibility V = L is surely the ultimate combinatorial principle in ZFC, and the morass codifies a substantial portion of the structure of L. As set theorists looked beyond the well-known $\Diamond$ and $\Box$ for applicable combinatorial principles, it was natural to consider extractions from the full structure of a morass.

This paper is an expository survey that schematizes the higher combinatorial principles derivable from morasses which have emerged in set theoretical praxis. It is notable that most of these principles were formulated by combinatorial set theorists to isolate salient features of particular constructions, and shown by them to be consistent first by forcing. Then, the specialists in L established how they hold there, in ad hoc fashion using the full structure of the morass. The sections of this paper deal successively with Prikry's Principle, Silver's Principle, Burgess' Principle, and finally limit cardinal versions. This sequence reflects the historical development of ideas, the progression toward further complexity, and coincidentally the author's series of papers [Ka2][Ka3] [Ka4]. The cumulative layers of sophistication provide an illuminating approach to the full morass structure, whilst at the same time providing a hierarchy of principles which, seen in this scheme, will hopefully find wider application in the future. The emphasis will be on shorter, illustrative proofs for the casual but interested reader, with adequate references for the more persistent researcher.

The set theoretical notation is standard, and here is a short litany: The first Greek letters

$\alpha,\beta,\gamma,\ldots$ denote ordinals, whereas the middle Greek letters $\kappa,\lambda,\mu,\ldots$ are reserved for infinite cardinals. If x is a set, $|x|$ denotes its cardinality, $[x]^\kappa$ denotes the collection of subsets of x of cardinality κ, and if f is a function, $f"x = \{f(y)\,|\,y \in x\}$. Finally, ${}^y x$ denotes the set of functions from y into x.

§1. PRIKRY'S PRINCIPLE

In the first three sections, κ will always denote a successor cardinal, with κ^- its predecessor. The general situation will be considered in the last section. Prikry's Principle is the following proposition:

(P_κ) There is a collection $\{f_\alpha | \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s\kappa$, we have $|\{\xi < \kappa | \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

This first approach to the system of approximations which comprises a morass says roughly that there are κ^+ functions: $\kappa \to \kappa$ such that: if guesses are made at possible values for any κ^- many of them, then for sufficiently large $\xi < \kappa$, at least one guess is rendered correct at ξ. Although it is not made explicit, notice that we can assume that the f_α's are pairwise distinct and have range $= \kappa$, since easy applications of P_κ show that only $< \kappa^-$ of these functions do not have these properties. Historically, P_κ was the first of the higher combinatorial principles to be formulated. In a ground-breaking paper, Prikry [P] devised his principle and established its consistency with the GCH by a method of forcing with side conditions. Assuming $2^{\kappa^-} = \kappa$, a simple diagonal argument provides κ functions satisfying the conclusion of P_κ; Prikry's

argument yields κ^+ many, and in fact can provide arbitrarily many in a cardinal-preserving forcing extension. There was no particular emphasis laid on possibilities in L, but at any rate, the first result about L in this whole context was due to Jensen, who showed that if V = L, then P_κ holds for every successor cardinal κ, using the morass structure that he invented.

Prikry was answering a question of Erdős, Hajnal and Rado [EHR] in the partition calculus, and this was the first example of the phenomenon of relative consistency results, rather than outright demonstrations, in this area of set-theoretical research. To recall the relevant case of the polarized partition symbol of [EHR],

$$\begin{pmatrix}\lambda\\ \kappa\end{pmatrix} \longrightarrow \begin{bmatrix}\mu\\ \nu\end{bmatrix}_\gamma$$

means that whenever $F: \lambda\times\kappa \to \gamma$, there are $X \in [\lambda]^\mu$ and $Y \in [\kappa]^\nu$ such that $F''(X\times Y) \neq \gamma$. To denote the negation of this proposition, $\to$ is replaced by $\not\to$. Besides [EHR], see the secondary source Williams [Wi] Chapter 4 for background. Prikry educed P_κ from the negative partition relation that he wanted to show consistent, which is equivalent to the ostensibly weaker version where the $\phi \in {}^s\kappa$ just range over the constant functions:

$$P_\kappa \to \begin{pmatrix}\kappa^+\\ \kappa\end{pmatrix} \not\to \begin{bmatrix}\kappa^-\\ \kappa\end{bmatrix}_\kappa$$

(If $\{f_\alpha | \alpha < \kappa^+\}$ is as provided by P_κ, set $F(\alpha,\beta) = f_\alpha(\beta)$ to get a counter-example.)

P_κ has several other applications. Prikry himself in [P] provided a consequence about an old problem of Ulam's on measurability with respect to a sequence of measures. In a related development, Szymański [Sz] formulated the following concept in order to establish some Baire Category-type theorems for $U(\omega_1)$, the space of uniform ultrafilters over ω_1. For any infinite cardinal λ, a matrix $\{A^n_\alpha | n < \omega, \alpha < \lambda\}$ is a <u>λ-matrix</u> <u>iff</u>

(a) if $m < n$ and $\alpha < \lambda$, then $A^m_\alpha \subseteq A^n_\alpha$,

(b) $\bigcup\{A^n_\alpha | n < \omega\} = \omega_1$ for each $\alpha < \lambda$, and

(c) for every infinite $s \subseteq \lambda$ and $\phi \in {}^s\omega$,
$|\bigcap\{A^{\phi(\alpha)}_\alpha | \alpha \in s\}| < \omega_1$.

A basic clopen set for $U(\omega_1)$ is a set of form $\{u \in U(\omega_1) | A \in u\}$ for some $A \subseteq \omega_1$; and a G_δ closed set is a countable intersection of basic clopen sets. Szymański established the following equivalence: A λ-matrix exists <u>iff</u> there is a family of λ G_δ closed and nowhere dense subsets of $U(\omega_1)$ such that the union of any infinite subfamily is dense in $U(\omega_1)$. The connection with P_{ω_1} is clear:

If P_{ω_1}, then there is an ω_2-matrix.

(Let $g: \omega_1 \to \omega$ be any surjection, and given $\{f_\alpha | \alpha < \omega_2\}$ as provided by P_{ω_1} set $A^n_\alpha = \{\xi | g \cdot f_\alpha(\xi) \le n\}$.) Incidentally, Baumgartner has established that the following are equivalent (see [Ka2] for a proof): (a) an ω_1-matrix exists; (b) an ω-matrix exists; and (c) there is a subset of ${}^\omega\omega$ of cardinality ω_1 without an upper bound in ${}^\omega\omega$ under the ordering of eventual dominance.

Delimiting the ranges of the functions in P_κ even further by composing with a fixed surjection $g: \kappa \to 2$ yields the following:

(P^-_κ) There is a collection $\{f_\alpha | \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s 2$, we have $|\{\xi < \kappa | \forall\alpha\in s(f_\alpha(\xi) \ne \phi(\alpha))\}| < \kappa$.

Even this weakened principle has its uses: Balcar, Simon and Vojtáš asked([BSV] Problem 20b) whether the following is consistent: whenever λ is regular and uncountable and U is a uniform ultrafilter over λ, then there are λ^+ sets in U such that the intersection of any infinitely many of them has cardinality $< \lambda$. Probably, this is true in L, and the proof will depend heavily on the structure of ultrafilters. But at least, one can affirm the case $\lambda = \omega_1$: If P^-_κ, then for any uniform ultrafilter over κ there are κ^+ sets in U such that any κ^- of them has intersection of cardinality $< \kappa$. (The proof is immediate.)

Perhaps a more substantial application of P_κ is to the Hajnal-Máté Principle, formulated in the study of set mappings in combinatorial set theory, in Hajnal-Máté [HM]:

(HM_κ) There is a collection $\{h_\xi \mid \xi < \kappa\} \subseteq {}^{\kappa^+}\kappa^+$ with $h_\xi(\alpha) \neq \alpha$ for every $\xi < \kappa$ and $\alpha < \kappa^+$ such that: whenever $s \in [\kappa^+]^{\kappa^-}$, we have $|\{\xi < \kappa \mid \forall \alpha \in s (h_\xi(\alpha) \notin s)\}| < \kappa$.

In the most general setting, a set mapping on a set X is a function f from a subset of P(X) into P(X) such that $x \cap f(x) = \emptyset$ for every x in the domain. A subset $H \subseteq X$ is free with respect to f iff $H \cap f(x) = \emptyset$ for every $x \subseteq H$ in the domain. The general problem of when large free subsets for set mappings exist was extensively investigated through the fifties by classical combinatorial means in Eastern Europe. (See the secondary source Williams [Wi] Chapter 3 for background; a timely application of this theory is found in Galvin-Hajnal [GH].) Forcing and in particular Prikry's method of forcing with side conditions extended the realms of possibility, and Hajnal and Máté distilled their principle with the following implication in mind: If HM_κ and there is a κ-Kurepa tree (see the next section), then there is an $f: [\kappa^+]^3 \to \kappa^+$ such that no set of cardinality κ^- is free with respect to f. Note that HM_κ itself is a proposition about set mappings and free sets. Fitting into the pattern, Hajnal and Máté established the consistency of HM_κ by forcing, and then it was shown later to be true in L, this time by Burgess [Bu2], who first established a stronger principle in L to be discussed in §3.

Theorem 1 (Kanamori): $P_\kappa \to HM_\kappa$.

$\vdash$ Suppose that $\{f_\alpha \mid \alpha < \kappa^+\}$ is as provided by P_κ. For each $\alpha < \kappa^+$, let $\psi_\alpha: \alpha \to \kappa$ be injective. Finally, define $h_\xi: \kappa^+ \to \kappa^+$ for $\xi < \kappa$ by:

$$h_\xi(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0\text{, else} \\ \psi_\alpha^{-1}(f_\alpha(\xi)) & \text{if this is defined, and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we took care that $h_\xi(\alpha) \neq \alpha$ for every $\alpha < \kappa^+$.

To verify HM_κ, suppose that $s \in [\kappa^+]^{\kappa^-}$. Let β be the least element in $s - \{0,1\}$, and set $t = s - (\beta+1)$. Define $\phi: t \to \kappa$ by $\phi(\alpha) = \psi_\alpha(\beta)$. Then $\{\xi < \kappa \mid \forall \alpha \in t (f_\alpha(\xi) \neq \phi(\alpha))\} \supseteq \{\xi < \kappa \mid \forall \alpha \in s (h_\xi(\alpha) \notin s)\}$ and hence by P_κ, this last set has cardinality $< \kappa$. $\dashv$

§2. SILVER'S PRINCIPLE

In order to formulate Silver's Principle (and also Burgess' Principle in the next section), it will be helpful to establish once and for all some conventions regarding trees. If T is a tree, T_ξ will denote the members of T at its ξth level; if $x \in T_\zeta$ and $\xi \leq \zeta$, then $\pi_\xi(x)$ is the tree predecessor of x at level ξ. Let us assume that trees are normalized at limits, i.e. if δ is a limit ordinal and $x \neq y$ are both in the δth level, then there is a $\xi < \delta$ such that $\pi_\xi(x) \neq \pi_\xi(y)$. A <u>$\kappa$-Kurepa tree</u> is a tree with height $\kappa+1$ such that $|T_\kappa| > \kappa$, yet $|T_\xi| \leq \xi$ for every $\xi < \kappa$. (This is congruous with the usual definition; it will be convenient to identify cofinal branches with a top level.) This settled, here is Silver's Principle:

(W_κ) There is a κ-Kurepa tree T and a function W with domain κ such that:

(a) for each $\xi < \kappa$, we have $W(\xi) \subseteq [T_\xi]^{\kappa^-}$ with $|W(\xi)| \leq \kappa^-$.

(b) for any $s \in [T_\kappa]^{\kappa^-}$, there is a $\gamma < \kappa$ such that whenever $\gamma \leq \xi < \kappa$, we have $\pi_\xi{}''s \in W(\xi)$.

Like P_κ, W_κ meets requirements mandated by κ^- size subsets of κ^+ (in essence, as $|T_\kappa| \geq \kappa^+$) at all

sufficiently large stages. The new feature in the ascent towards the morass is the κ-Kurepa tree structure: the system of approximations has small initial stages. This combines with the potency provided by the function W, which plays a role somewhat akin to the sequence of distinguished subsets in $\diamondsuit^+$. Notice that if T is any tree of height $\kappa+1$, even if T were not necessarily a κ-Kurepa tree, as long as $|T_\kappa| \geq \kappa^-$ and there is a function W satisfying (a) and (b) above for this T, it is not difficult to see that $2^{\kappa^-} = \kappa$ must be satisfied.

The formulation of W_κ evinced an evolution from P_κ in the focus of attention as well. Upon seeing some consistency results constructed by Hajnal and Juhász in set-theoretic topology, Silver extracted W_κ from the morass in order to effect these constructions in L. So, unlike P_κ, W_κ was formulated with ramifications in L in mind. That many, long-winded combinatorial emanations from a morass actually follow from W_κ is a testimonial to Silver's insight. Incidentally, the consistency argument for W_κ through forcing is not difficult. To the usual notion of forcing for adjoining a κ-Kurepa tree one appends further clauses reminiscent of Prikry's side conditions (see Burgess [Bul] for an exposition). The following proof illustrates the constructions possible with W_κ:

Theorem 2 (Silver): $W_\kappa \to P_\kappa$.

$\vdash$ Let T and W be as provided by W_κ; we might as well assume that $T_\kappa = \kappa^+$ by renaming. Since W_κ implies $2^{\kappa^-} = \kappa$, for each $\delta < \kappa$ and $s \in W(\delta)$, we can enumerate ${}^s\kappa$ as $\{h^s_\xi \mid \delta \le \xi < \kappa\}$. To take care of more and more of the h^s_ξ's, for each $\xi < \kappa$ we shall define functions $g_\xi: T_\xi \to \kappa$ such that:

(†) Whenever $\delta \le \eta \le \xi$, $s \in W(\xi)$, and $\pi_\delta"s \in W(\delta)$, there is an $x \in s$ such that $g_\xi(x) = h^{\pi_\delta"s}_\eta(\pi_\delta(x))$.

Once this is done, the proof can be completed by defining $f_\alpha: \kappa \to \kappa$ for $\alpha < \kappa^+$ by:

$$f_\alpha(\xi) = g_\xi(\pi_\xi(\alpha)).$$

To verify P_κ, let $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s\kappa$. By hypothesis, there is a $\delta < \kappa$ such that $\delta \le \xi < \kappa$ implies $\pi_\xi"s \in W(\xi)$, where we can take δ sufficiently large so that π_δ is injective on s. For some $\eta \ge \delta$, we have $h^{\pi_\delta"s}_\eta = \phi\cdot\pi_\delta^{-1}$. (Here, π_δ^{-1} is clear from the context as that inverse of π_δ whose range is the top level.) Then for any ξ such that $\eta \le \xi < \kappa$, there is an $x \in \pi_\xi"s$ such that $g_\xi(x) = h^{\pi_\delta"s}_\eta(\pi_\delta(x))$. But if $\alpha = \pi_\xi^{-1}(x)$, then $f_\alpha(\xi) = g_\xi(x) = \phi(\alpha)$.

All that remains is to define the functions g_ε so as to satisfy (†). But doing this is easy: Fix $\xi < \kappa$, and let $\{\langle s_\zeta, \delta_\zeta, \eta_\zeta\rangle \mid \zeta < \kappa^-\}$ enumerate all triples $\langle s, \delta, \eta\rangle$ where $\delta \le \eta \le \xi$, $s \in W(\xi)$, and $\pi_\delta"s \in W(\delta)$. Now define exactly one value for g_ξ in each of κ^- stages inductively: If $\zeta < \kappa^-$, since only ζ values have been determined before the ζth stage and s_ζ has cardinality κ^-, there is an $x \in s_\zeta$ such that $g_\xi(x)$ has not yet been defined. Set $g_\xi(x) = h^{\pi_\delta"s}_\eta(\pi_\delta(x))$, where $\delta = \delta_\zeta$ and $\eta = \eta_\zeta$. Finally, after κ^- stages extend g_ξ arbitrarily to all of T_ξ. This completes the construction of g_ξ, and the proof is thus complete. $\dashv$

Let us now turn to the work of Hajnal and Juhász alluded to earlier. Pondering the existence of special topological spaces of large cardinality, Hajnal and Juhász realized in the early 1970's that concrete constructions readily follow from certain existential principles concerning matrices of sets. The following proposition is the strongest form of these principles, and can be appropriately dubbed the <u>Hajnal-Juhász Principle</u>:

(HJ_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $\rho < \kappa^-$ and $s: \kappa^- \times \rho \to \kappa^+$ is injective, there is a $\gamma < \kappa$ such that: if $x \in [\kappa - \gamma]^{<\omega}$ and $\{\varepsilon_\tau \mid \tau < \rho\} \subseteq {}^x 2$, there is a $\sigma < \kappa^-$ with $\varepsilon_\tau \subseteq f_{s(\sigma,\tau)}$ for every $\tau < \rho$.

In set-theoretic topology, HL and HS are acronymic for hereditarily Lindelöf and hereditarily separable, respectively, and an L space is an HL space which is not HS, whilst an S space is an HS space which is not HL. There is quite a literature on the study of these spaces nowadays, particularly in connection with Martin's Axiom, and a good but older reference is M.E. Rudin [Ru] Chapter 5. An initial version of HJ_κ was considered by Hajnal and Juhász with the restriction to just $\rho = 1$. Taking the concrete case $\kappa = \omega_1$ (otherwise, we would have to frame the discussion in general terms around κ^--Lindelöf and κ^--separable), they show [HJ1] that this restricted principle implies the existence of normal S spaces of large cardinality, the so-called HFD spaces, and establish its consistency by forcing. Then Devlin [D] established this restricted principle in L, directly using morasses. As Hajnal and Juhász later realized, the full principle HJ_{ω_1} implies the existence of normal, <u>strong</u> S spaces of large cardinality. (A strong S space is a space X such that X^n is an S space for every $n \in \omega$.) Kunen [Ku2] has shown that under MA + $\neg$CH, there are no strong S spaces.

Concerning L spaces, Hajnal and Juhász early on [HJ2] formulated the following principle to construct

L spaces of large cardinality:

(HJ_κ^-) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa 2$ so that whenever $\rho < \kappa^-$ and $s: \kappa^- \times \rho \to \kappa^+$ is injective and $\phi: \kappa^- \times \rho \to 2$, we have $|\{\xi < \kappa \mid \forall\sigma<\kappa^- \exists\tau<\rho (f_{s(\sigma,\tau)}(\xi) \neq \phi(\sigma,\tau))\}| < \kappa$.

(Actually, they had a further condition on $\{f_\alpha \mid \alpha < \kappa^+\}$ to insure good separation properties for the space constructed, but this is the crux of the matter.) HJ_κ^- immediately implies P_κ^-; that HJ_κ^- follows from HJ_κ is not unexpected, and details are provided in [Ka3]. The proof of the following theorem is also given in full in [Ka3]; it can be culled [HJ1][HJ2] and especially [Ju] where a simpler conclusion is derived.

Theorem 3 (Silver): $2^{<\kappa^-} = \kappa^-$ and $W_\kappa \to HJ_\kappa$.

§3. BURGESS' PRINCIPLE

Although we saw in §1 that the Hajnal-Máté Principle follows from Prikry's Principle, Burgess [Bu2] originally established the Hajnal-Máté Principle in L from a more complicated principle. The following, asserting the existence of what he calls quagmires, can be dubbed <u>Burgess' Principle</u>. Here, the notation for κ-Kurepa trees developed in §2 is still in effect.

(B_κ) There is a quagmire, i.e. a κ-Kurepa tree T with tree ordering $<$, equipped with a binary relation $\lhd$ and a ternary function Q such that:

(1) $y \lhd x$ implies that x and y are distinct elements on the same level, and $\lhd$ linearly orders every level.

(2) Q is defined on triples $\langle \bar{y}, \bar{x}, x\rangle$ just in case $\bar{y} \lhd \bar{x} < x$, and for any such, $y < Q(\bar{y}, \bar{x}, x) \lhd x$.

(3) (Commutativity) If $\bar{y} \lhd \bar{x} < x' < x$, then $Q(Q(\bar{y}, \bar{x}, x'), x', x) = Q(\bar{y}, \bar{x}, x)$.

(4) (Coherence) If $\bar{z} \lhd \bar{y} \lhd \bar{x} < x$, then $Q(\bar{z}, \bar{y}, Q(\bar{y}, \bar{x}, x)) = Q(\bar{z}, \bar{x}, x)$.

(5) (Completeness) If $y \lhd x \in T_\kappa$, then for some $\xi < \kappa$, $\pi_\xi(y) \lhd \pi_\xi(x)$ and $Q(\pi_\xi(y), \pi_\xi(x), x) = y$.

The reader familiar with morasses will already see a growing resemblance, and as with morasses, he or she is advised to draw pictures to get the picture. B_κ may seem a bit ad hoc, but it is really the next natural rung in the evolutionary ladder toward a morass. Whereas W_κ merely hypothesized κ^+ cofinal branches and a system of approximations by κ^- size subsets, B_κ endows a linear order on these branches which is moreover reflected in the $\lhd$ orderings through the previous levels. Thus, B_κ incorporates an important feature of morasses; the main ingredient which must still be added to get the full structure of a morass is the limit continuity across levels. Burgess [Bu2] established the following result:

Theorem 4 (Burgess): If $2^{\kappa^-} = \kappa$ and B_κ, then W_κ.

$\vdash$ The idea here is first to enumerate the powerset $P(T_\xi)$ as $\{X_{\xi\rho} \mid \rho < \kappa\}$ for each $\xi < \kappa$, using $2^{\kappa^-} = \kappa$. For $x \in T_\gamma$ and $\xi, \rho < \gamma$ let $S(\xi,\rho,x) = \{Q(\bar{y}, \pi_\xi(x), x) \mid \bar{y} \lhd \pi_\xi(x) \text{ and } \bar{y} \in X_{\xi\rho}\}$. Finally, for $\gamma < \kappa$ set $W(\gamma) = \{S(\xi,\rho,x) \mid \xi,\rho < \gamma \text{ and } x \in T_\gamma\}$. Then this function W works, capturing more and more of the images of the κ^- size subsets of the top level as we move to the right along $\lhd$ and upwards along $<$:

The Completeness condition (5) implies that any $x \in T_\kappa$ has at most κ $\lhd$-predecessors. Hence, given any $s \in [T_\kappa]^{\kappa^-}$, there is an $x \in T_\kappa$ such that $y \lhd x$ for every $y \in s$. Again by Completeness, for each $y \in s$ there is a $\xi(y) < \kappa$ such that $Q(\pi_{\xi(y)}(y), \pi_{\xi(y)}(x), x) = y$. Set $\delta = \sup\{\xi(y) \mid y \in s\}$. Then an easy application of Commutativity shows that for any ξ with $\delta \le \xi < \kappa$ and $y \in s$, we have $\pi_\xi(y) \lhd \pi_\xi(x)$ and $Q(\pi_\xi(y), \pi_\xi(x), x) = y$. Finally, let ρ be such that $X_{\delta\rho} = \pi_\delta``s$. Then for any $\xi \ge \max(\delta,\rho)$, the above arguments confirm that $\pi_\xi``s = S(\delta,\rho,\pi_\xi(x)) \in W(\xi)$, completing the proof. $\dashv$

Just as Silver's Principle establishes Prikry's Principle, Burgess' Principle establishes an extended

Prikry's Principle, first formulated by Rebholz [Re] soon after morasses first saw the light of day.

Rebholz' Principle is the following:

(R_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\}$ of functions with $f_\alpha: \alpha \to \alpha$, so that whenever $s \in [\kappa^+]^{\kappa^-}$ and ϕ is a regressive function with domain $= s$ (i.e. $\phi(\alpha) < \alpha$ for $\alpha \in s$), then $|\{\xi < \bigcap s \mid \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

Using morasses (and $\diamondsuit_\kappa$, although $2^{\kappa^-} = \kappa$ seems to be sufficient by a more complicated proof), Rebholz established that if $V = L$, then R_κ holds for every successor cardinal κ. Clearly, R_κ is equivalent to the following principle if we compose each f_α with a bijection $\alpha \leftrightarrow \kappa$ for $\kappa \leq \alpha < \kappa^+$:

(R'_κ) There is a collection $\{g_\alpha \mid \alpha < \kappa^+\}$ of functions with $g_\alpha: \alpha \to \kappa$, so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi: s \to \kappa$, then $|\{\xi < \bigcap s \mid \forall\alpha\in s(g_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

Also, it is easy to see, by considering $\{g_\alpha \restriction \kappa \mid \alpha < \kappa^+\}$, that:

$$R_\kappa \to P_\kappa.$$

Finally, like P_κ, R_κ has a consequence in the partition calculus. To be precise, $\lambda \to [\mu:\nu]^2_\gamma$ means that whenever $f: [\lambda]^2 \to \gamma$, there is an $X \in [\lambda]^\mu$ and a $Y \in [\lambda]^\nu$ with $\bigcup X < \bigcap Y$, and $f''\{\langle\xi,\zeta\rangle \mid \xi \in X \text{ and } \zeta \in Y\} \neq \gamma$.

Note that the negation $\lambda \not\to [\mu:\nu]^2_\gamma$ implies $\lambda \not\to [\mu\dot{+}\nu]^2_\gamma$, and so provides a strong counterexample to the ordinary partition symbol. Rebholz formulated his principle with the following immediate consequence in mind:

$$R_\kappa \to \kappa^+ \not\to [\kappa:\kappa^-]^2_\kappa.$$

(If $\{g_\alpha \mid \alpha > \kappa^+\}$ is as provided by R'_κ, set $F(\beta,\alpha) = g_\alpha(\beta)$ for $\beta > \alpha$ to get a counterexample.) The conclusion here is stronger than the negative polarized partition relation entailed by P_κ, and the difference is revealing: there, each f_α need only be: $\kappa \to \kappa$, and here, we must have an elongated $f_\alpha: \alpha \to \alpha$. Incidentally, this is the best possible limitative result, since Shelah [S] established in ZFC + GCH that: If $\kappa > \omega$ is regular and $\gamma^+ < \kappa$, then $\kappa^+ \to (\kappa+\gamma)^2_2$. Rebholz [Re] also provides an application of R_κ to the theory of free subsets for set mappings, answering a question of Máté.

The following derivation highlights the lateral approximations provided by $\triangleleft$ in B_κ.

Theorem 5 (Kanamori): $(2^{\kappa^-} = \kappa \text{ and } B_\kappa) \to R_\kappa$.

$\vdash$ As mentioned in the proof of Theorem 4, by the Completeness condition, any $x \in T_\kappa$ has at most κ $\triangleleft$-predecessors. Hence, as T_κ has cardinality $> \kappa$, it must have a subset well-ordered by $\triangleleft$ in order-type κ^+.

So, by renaming and trimming, we might as well assume further that:

(6) $T_\kappa = \kappa^+$, and for $\alpha, \beta < \kappa^+$, we have $\alpha \lhd \beta$ <u>iff</u> $\alpha < \beta$.

To prove the theorem, it suffices to establish the more tractable R_κ'. By Theorem 4, there is a function W satisfying the clauses of Silver's Principle for our tree T. Thus, by $2^{\kappa^-} = \kappa$, for each $\delta < \kappa$ and $s \in W(\delta)$, we can enumerate ${}^s\kappa$ as $\{h_\xi^s \mid \delta \le \xi < \kappa\}$, and define functions $g_\xi\colon T_\xi \to \kappa$ satisfying the condition (†), just as in the proof of Theorem 2. Finally, let us define $f_\alpha\colon \alpha \to \kappa$ for $\alpha < \kappa^+$ as follows:

If $\zeta < \alpha$, by Completeness, there is a ρ such that $\pi_\rho(\zeta) \lhd \pi_\rho(\alpha)$. Let ρ_ζ^α be the least such ρ, and set

$$f_\alpha(\zeta) = g_{\rho_\zeta^\alpha}(\pi_{\rho_\zeta^\alpha}(\alpha)).$$

This definition underscores the importance of $\lhd$; once an ordering of the top nodes is established, $\lhd$ reflects and completely approximates this ordering through the lower levels.

To verify R_κ', let $s \in [\kappa^+]^{\kappa^-}$ and $\phi \in {}^s\kappa$. There is a $\delta < \kappa$ such that:

(a) $\delta \le \xi < \kappa$ implies π_ξ"$s \in W(\xi)$, and

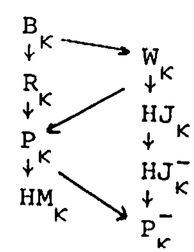
(b) $\delta \le \xi < \kappa$ and $\alpha < \beta \in s$ implies $\pi_\xi(\alpha) \lhd \pi_\xi(\beta)$ and $Q(\pi_\xi(\alpha), \pi_\xi(\beta), \beta) = \alpha$.

Here, we can accomplish (b) using Completeness and Commutativity much the same as in the proof of Theorem 4. Notice the following FACT: If $\zeta < \bigcap s$, $\delta \le \xi < \kappa$, and there is an $\alpha \in s$ such that $\pi_\xi(\zeta) \lhd \pi_\xi(\alpha)$, then for <u>every</u> $\beta \in s$ we also have $\pi_\xi(\zeta) \lhd \pi_\xi(\beta)$. This is so by transitivity of $\lhd$ if $\alpha < \beta$, and by the Coherence condition (4) if $\beta < \alpha$.

Now for some $\eta \ge \delta$, we have $h_\eta^{\pi_\delta"s} = \phi \cdot \pi_\delta^{-1}$. Set $E = \{\zeta < \bigcap s \mid \exists \alpha \in s \exists \xi < \eta\ (\pi_\xi(\zeta) \lhd \pi_\xi(\alpha)$ and $Q(\pi_\xi(\zeta), \pi_\xi(\alpha), \alpha) = \zeta)\}$. Clearly $|E| < \kappa$; E consists of the exceptional ordinals:

Suppose that $\zeta \in (\bigcap s - E)$. By Completeness and the FACT, there is a fixed ρ with $\eta \le \rho < \kappa$ such that $\rho = \rho_\zeta^\alpha$ for every $\alpha \in s$ (where ρ_ζ^α was defined in the course of the definition of f_α). Now we can complete the proof as in Theorem 2. By condition (†) on the g_ξ's, there is an $x \in \pi_\rho$"s such that $g_\rho(x) = h_\eta^{\pi_\delta"s}(\pi_\delta(x))$. But if $\alpha = \pi_\rho^{-1}(x) \in s$, then $f_\alpha(\zeta) = g_\rho(x) = \phi(\alpha)$. This establishes R_κ'. ⊣

The following diagram summarizes the implications in the first three sections assuming the GCH:



I do not know whether any converses are true.

§4. GENERALIZATIONS

This section considers versions of the various combinatorial principles also available at limit cardinals; perhaps the main interest in these generalizations lies in the consequent limitative results in the partition calculus which counterpoint the positive results available from large cardinals. So, let me provide the backdrop of historical context, first of all for the polarized partition relation.

In general, the proposition

$$(*) \qquad \begin{pmatrix}\kappa^+\\ \kappa\end{pmatrix} \to \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}^2$$

seem to hold but rarely. The earliest result along these lines was due to Erdős and Rado [ER] Theorem 48, who established (*) for $\kappa = \omega$. Hajnal [H] then established (*) for κ a measurable cardinal; see also Chudnovsky [C] and Kanamori [Ka1] for some refinements. Chudnovsky claims without proof in his paper that (*) holds for κ a weakly compact cardinal, and proofs have since been provided by Wolfsdorf [Wo], Shelah, and Kanamori [Ka2].

For successor cardinals κ, we saw in §1 that P_κ denies (*) in strong fashion. Unpublished work of

Laver [L] provides a positive consistency result: Say that a non-trivial ideal over a regular cardinal $\kappa > \omega$ is a <u>Laver ideal</u> <u>iff</u> whenever $X \subseteq P(\kappa) - I$ with $|X| \geq \kappa^+$, there is a $Y \in [X]^{\kappa^+}$ so that: whenever $Z \in [Y]^{<\kappa}$, then $\bigcap Z \notin I$. Notice that any measure over a measurable cardinal κ is dual to a Laver ideal over κ. Laver noted that the existence of a Laver ideal implies (*) (where the 2 can be replaced by any ordinal $< \kappa$). Refining an argument of Kunen [Ku1], he then established the relative consistency of ω_1 carrying a Laver ideal, by forcing over a ground model satisfying ZFC and a strong large cardinal hypothesis, the existence of a huge cardinal.

The study of the even rarer

$$(**) \qquad \kappa^+ \to (\alpha)^2_2 \quad \text{for every } \alpha < \kappa^+$$

also has a rambling history. After years of partial results and conjectures, Baumgartner and Hajnal [BH] established (**) for $\kappa = \omega$, as a consequence of a more general result which they established in elegant fashion by using Martin's Axiom and an absoluteness argument. Avoiding these tricks of the trade, Galvin [G] provided a direct proof which is a combinatorial tour de force. More recently, Tedorcevic has announced further refinements. It is not known whether (**) holds for κ a measurable cardinal; perhaps the best partial result is due to Laver. He was first to observe that if there is a Laver ideal over κ, then $\kappa^+ \to (\kappa+\kappa+1,\alpha)^2_2$ for every $\alpha < \kappa^+$. I established Laver's result without knowing of it, and a full proof is provided in [Ka4]. Also in [Ka4] is the result that if κ is a weakly compact cardinal, then $\kappa^+ \to (\kappa+\kappa+1,[\kappa:\kappa^+])^2_2$, a technical statement somewhat stronger than $\kappa^+ \to [\kappa:\kappa]^2_2$, which already follows from the known relation (*) for weakly compact cardinals. For successor cardinals, we saw in §3 that R_κ denies (**) in strong fashion. In the positive direction, there is again Laver's result about a Laver ideal over ω_1, starting with the consistency strength of a huge cardinal. Gray also has some partial positive results.

Turning to the subject at hand, just as P_κ and R_κ deny partition relations for successor cardinals, there are weaker versions which delimit the situation for possibly limit cardinals.

(wP_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ so that whenever $s \in [\kappa^+]^\kappa$ and $\phi \in {}^s\kappa$, we have $|\{\xi < \kappa \mid \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

(wR_κ) There is a collection $\{f_\alpha \mid \alpha < \kappa^+\}$ of functions $f_\alpha : \alpha \to \alpha$ so that whenever $s \in [\kappa^+]^\kappa$ and ϕ is a regressive function with domain $= s$, then $|\{\xi < \bigcap s \mid \forall\alpha\in s(f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$.

There are versions of the other combinatorial principles with the requisite strength (one is stated for W_κ in [Ka3]), but to discuss them would take us somewhat afield. In direct analogy to previous results, we have:

$$wP_\kappa \;\to\; \begin{pmatrix}\kappa^+\\ \kappa\end{pmatrix} \not\to \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}_2$$

$$wR_\kappa \;\to\; \kappa^+ \not\to [\kappa:\kappa]^2_\kappa$$

I established [Ka2][Ka4] the consistency of these propositions via a forcing which does not (too much) disturb the universe; e.g. for the stronger wR_κ,

<u>Theorem 6 (Kanamori)</u>: If the ground model satisfies $\kappa^{<\kappa} = \kappa$, then there is a κ^+-c.c., $<\kappa$-distributive forcing extension in which wR_κ holds. (Furthermore, properties like the Mahloness of κ are preserved.)

The proof involves a new and elegant kind of density argument, first seen in the work of Shelah. By itself, this is a piecemeal result, and to genuinely contrast the positive partition relations from large cardinals, the actual situation in L must be ascertained. Recent and continuing work of Shelah and Stanley [SS1][SS2] and Velleman [V] have made the formidable apparatus of the $(\kappa,1)$-morass (a gap-1 morass at κ) more tractable (at least for some) by providing a Martin's Axiom-type characterization. That is, certain partial orders and collections of dense sets are described, and the existence of a morass is shown to be equivalent to the proposition that for every such partial order and every such collection F of dense sets, there is an F-generic filter in the usual sense. The partial order used in the proof of Theorem 6 is a paradigm case of a canonical limit partial order, in the sense of Shelah and Stanley. It was to handle such orders that led Shelah and Stanley to extend their characterization of morasses. They show how canonical limit partial orders can be accommodated in a Martin's Axiom-type characterization for $(\kappa,1)$-morasses "with built-in $\Diamond$ principle", when there is a non-reflecting stationary subset of κ, i.e. an $S \subseteq \kappa$ which is stationary in κ yet $S \cap \alpha$ is not stationary in α for any $\alpha < \kappa$. They establish that such morasses with built-in $\Diamond$ principle exist in L, and, of course, it is a well-known result of Jensen [Je] that in L, a regular $\kappa > \omega$ is not weakly compact iff there is a non-reflecting stationary subset of κ. Velleman is also developing a scheme along similar lines, but with a more concise formulation. Assuming that the partial order used in Theorem 6 fits into either the Shelah-Stanley or Velleman scheme in its final form, we have the following characterization of

weak compactness in L:

Theorem 7: If V = L, then the following are equivalent for regular $\kappa > \omega$:

(i) κ is not weakly compact

(ii) wR_κ

(iii) wP_κ

(iv) $\begin{pmatrix}\kappa^+\\ \kappa\end{pmatrix} \not\to \begin{bmatrix}\kappa\\ \kappa\end{bmatrix}_\kappa$

(v) $\kappa^+ \not\to [\kappa:\kappa]^2_\kappa$

This is the heralded counterpoint to large cardinals.

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