SEPARATING ULTRAFILTERS
ON UNCOUNTABLE CARDINALS

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ABSTRACT
A uniform ultrafilter U on \kappa is said to be \lambda-separating if distinct elements of the ultrapower never project U to the same uniform ultrafilter V on \lambda. It is shown that, in the presence of CH, an \omega-separating ultrafilter U on \kappa > \omega is non-(\omega, \omega)\text{-regular and, in fact, if } \kappa < \aleph_\omega \text{ then } U \text{ is } \lambda\text{-separating for all } \lambda. Several large cardinal consequences of the existence of such an ultrafilter U are derived.

§1. Introduction

We begin by establishing our notation and terminology. Throughout this paper \kappa, \lambda, \mu etc. will denote infinite (but not necessarily regular) cardinals, and \mathfrak{A} will denote the set of all functions mapping \kappa to \lambda. Suppose now that U is an ultrafilter on \kappa. U is said to be uniform if every set in U has cardinality \kappa.

The usual equivalence relation \sim_U on \mathfrak{A} is given by \( f \sim_U g \) iff \( \{ \alpha < \kappa : f(\alpha) = g(\alpha) \} \subseteq U \), and we let the equivalence class of f be denoted by \([f]_U\). The set of such equivalence classes can be linearly ordered by setting \([f]_U \leq [g]_U\), iff \( \{ \alpha < \kappa : f(\alpha) \leq g(\alpha) \} \subseteq U \); the resulting structure is referred to as the ultrapower of \lambda with respect to U. If \( f \in \mathfrak{A} \) then \( f \) projects U to an ultrafilter \( f^*(U) \) on \lambda where \( X \in f^*(U) \) iff \( f^{-1}(X) \subseteq U \). The ordering given by declaring \( f^*(U) \leq_{RR} U \) is called the Rudin–Keisler ordering. The property of ultrafilters that we will consider here is given by the following.

DEFINITION 1.1. Suppose that U is a uniform ultrafilter on \kappa and \lambda \leq \kappa. Then U will be called \lambda-separating iff whenever \( f^*(U) \) is a uniform ultrafilter on \lambda, the following implication holds:

\[ \forall g \in \mathfrak{A} \left( [f]_U \neq [g]_U \Rightarrow f^*(U) \neq g^*(U) \right) . \]

U is said to be separating if U is \lambda-separating for every \lambda \leq \kappa.

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The notion of a separating ideal was introduced in [10]; it is an easy exercise to show that an ultrafilter \( U \) is separating in the sense of Definition 1.1 iff the ideal on \( \kappa \) dual to \( U \) is separating in the sense of [10].

In Section 2, we consider non-regularity properties of separating ultrafilters and obtain some companion results to those of Pelletier [11]. In particular, we show that if \( U \) is an \( \omega \)-separating ultrafilter on \( \kappa \) and CH holds, then \( U \) is non-(\( \omega, \omega_1 \))-regular, and if \( \kappa < \aleph_\omega \) then \( U \) is non-(\( \lambda, \lambda^+ \))-regular for every \( \lambda \leq \kappa \). Several large cardinal consequences of the existence of a separating ultrafilter are discussed in Section 3. In Section 4, we show that if \( U \) is \( \lambda \)-separating and non-(\( \lambda, \lambda^+ \))-regular, then \( U \) is \( \lambda^+ \)-separating; this result is reminiscent of the well-known analogous result for \( \lambda \)-descendingly incomplete ultrafilters [3], [4], [9].

\[ \frac{1}{2} \]

\section{Non-regularity properties of separating ultrafilters}

Recall that a uniform ultrafilter \( U \) on \( \kappa \) is said to be \((\lambda, \mu)\)-regular iff there are \( \mu \) sets in \( U \) any \( \lambda \) of which have empty intersection. Such a collection is called a \((\lambda, \mu)\)-regularizing family for \( U \). If \( U \) fails to be \((\omega, \kappa)\)-regular, then \( U \) is said to be non-regular. Pelletier was the first to point out that separating ultrafilters possess a degree of non-regularity; his method of proof yields the following (although only a special case is explicitly stated in [11]).

\begin{theorem}[Pelletier [11]]
Suppose that \( U \) is a separating ultrafilter on \( \kappa \) and that \( \gamma \) is a cardinal satisfying:
\[ 2^{2^{<\gamma}} < 2^\gamma. \]
Then \( U \) is non-(\( \gamma, \kappa \))-regular.
\end{theorem}

The above result, however, yields no information for the case \( \kappa = \omega_1 \). Thus, we take another approach to irregularity. This approach requires the following three lemmas, the first of which combines ideas of Blass [2] p. 34, Benda–Ketonen [1], and Jorgensen [6].

\begin{lemma}
Suppose that \( U \) is an \((\omega, 2^\kappa)\)-regular uniform ultrafilter on \( \kappa \) and \( V \) is an arbitrary uniform ultrafilter on \( \lambda \). Then there are \((2^\kappa)^+\) distinct elements of the ultrapower, all of which project \( U \) onto \( V \).
\end{lemma}

\begin{proof}
Let \( \{ A_\alpha : \alpha < 2^\kappa \} \) be an \((\omega, 2^\kappa)\)-regularizing family for \( U \) and let \( \{ X_\alpha : \alpha < 2^\kappa \} \) be an enumeration of the sets in \( V \). It clearly suffices to show that for any collection \( \{ f_\alpha : \alpha < 2^\kappa \} \) of functions mapping \( \kappa \) to \( \lambda \), we can find a function \( f : \kappa \rightarrow \lambda \) so that

\[ \text{.} \]

(a) \([f_\alpha] \neq [f]\) for every \(\alpha < 2^\lambda\), and
(b) \(f_\alpha(U) = V\).

We will accomplish this by constructing \(f : \kappa \to \lambda\) so that
\((a') f_\alpha(\xi) < f(\xi)\) for every \(\xi \in A_\alpha\) and \(\alpha < 2^\lambda\), and
\((b') f^{-1}(X_\alpha) \supseteq A_\alpha\) for every \(\alpha < 2^\lambda\).

For each \(\xi < \kappa\), let \(\mathcal{O}(\xi) = \{\alpha < 2^\lambda : \xi \in A_\alpha\}\). Since infinite intersections of the \(A_\alpha\)'s are empty, we know that \(\mathcal{O}(\xi)\) is finite. Hence, if we let \(X = \bigcap\{X_\alpha : \alpha \in \mathcal{O}(\xi)\}\), then \(|X| = \lambda\) and so we can choose \(f(\xi) \in X\) so that for every \(\alpha \in \mathcal{O}(\xi)\) we have \(f_\alpha(\xi) < f(\xi)\). Notice that
\((a'')\) if \(\alpha < 2^\lambda\) and \(\xi \in A_\alpha\) then \(\alpha \in \mathcal{O}(\xi)\) so \(f_\alpha(\xi) < f(\xi)\), and
\((b'')\) if \(\xi \in A_\alpha\) then \(\alpha \in \mathcal{O}(\xi)\) so \(f(\xi) \in X_\alpha\).

Since \((a'') \rightarrow (a') \rightarrow (a)\) and \((b'') \rightarrow (b') \rightarrow (b)\), the proof is complete.

The next lemma is again heavily based on ideas of Benda–Ketonen [1]; its statement is aided by the following bit of terminology.

DEFINITION 2.3. If \(U\) is a uniform ultrafilter on \(\kappa\), then \(\mathcal{F}\) will be called a \(\lambda\)-family for \(U\) iff \(\mathcal{F}\) consists of functions each mapping a set in \(U\) to \(\lambda\) so that if \(f, g \in \mathcal{F}\) and \(f \neq g\) then
\[|\{\xi \in \text{domain}(f) \cap \text{domain}(g) : f(\xi) = g(\xi)\}| < \kappa.\]

LEMMA 2.4. Suppose that \(U\) is a uniform \((\lambda^+, \lambda^{++})\)-regular ultrafilter on \(\kappa\), and assume that there is a \(\lambda^+\)-family for \(U\) of size \(\lambda^{++}\). Then \(U\) is \((\lambda, \lambda^+)\)-regular.

PROOF. Let \(\{A_\alpha : \alpha < \lambda^{++}\}\) show that \(U\) is \((\lambda^+, \lambda^{++})\)-regular and let \(\{f_\alpha : \alpha < \lambda^{++}\}\) be a \(\lambda^+\)-family for \(U\) where \(f_\alpha : X_\alpha \to \lambda^+\). Define \(g : \kappa \to \lambda^+\) so that if \(\xi \in A_\alpha\) then \(f_\alpha(\xi) < g(\xi)\). This is possible since \(\xi\) occurs in only \(\lambda\) many \(A_\alpha\)'s. For each \(\gamma < \lambda^+\) let \(h_\gamma : \gamma \to \lambda\) be one to one and for each \(\alpha < \lambda^{++}\) let \(f'_\alpha : A_\alpha \to \lambda\) be given by \(f'_\alpha(\xi) = h_{\mathcal{O}(\xi)}(f_\alpha(\xi))\). Notice that \(\{f'_\alpha : \alpha < \lambda^{++}\}\) is a \(\lambda\)-family for \(U\). Without loss of generality, assume that for each \(\alpha < \lambda^+\) there is a set \(B_\alpha \subseteq U\) so that \(f'_\alpha(\xi) < f_\alpha(\xi)\) for every \(\xi \in B_\alpha\). Finally, let \(C_\alpha \subseteq U\) be given by \(C_\alpha = B_\alpha - \{\xi < \kappa : \exists \beta < \alpha : (f'_\beta(\xi) = f'_\alpha(\xi))\}\). It is easy to see that \(\{C_\alpha : \alpha < \lambda^+\}\) is a \((\lambda, \lambda^+)\)-regularizing family for \(U\).

The non-regularity results for separating ultrafilters that follow from Lemmas 2.2 and 2.4 are summarized in the following.

THEOREM 2.5. Suppose that \(U\) is a uniform ultrafilter on \(\kappa\).
(a) If \(U\) is \(\lambda\)-separating, then \(U\) is non-\((\omega, 2^\lambda)\)-regular.
(b) (CH). If \(U\) is \(\omega\)-separating then \(U\) is non-\((\omega, \omega_1)\)-regular; in particular, \(U\) is non-regular.
(c) (CH). If $\kappa < \aleph_\omega$ and $U$ is $\omega$-separating then $U$ is non-$(\lambda, \lambda^*)$-regular for every $\lambda$.

**Proof.** Parts (a) and (b) are immediate from Lemma 2.2. Part (c) follows from part (b), Lemma 2.4, and the observation that if $\kappa < \aleph_\omega$ and $\lambda < \kappa$ then there is a $\lambda^*$-family for $U$ of size $\lambda^{++}$. (One starts with a family of $\kappa^+$ eventually different functions from $\kappa$ to $\kappa$, i.e. the case $\lambda = \kappa^+$, and then works one's way down to $\lambda$ using the same argument that occurred in the proof of Lemma 2.4.)

§3. Large cardinal consequences

An ultrafilter $U$ on $\kappa$ is said to be weakly normal iff whenever $\{\alpha < \kappa : f(\alpha) < \alpha\} \subseteq U$, there is a $\beta < \kappa$ so that $\{\alpha < \kappa : f(\alpha) \leq \beta\} \subseteq U$. $U$ is said to be $\lambda$-indecomposable iff there is no uniform ultrafilter $V$ on $\lambda$ such that $V \leq_{RK} U$. Notice that if $U$ is $\lambda$-indecomposable then $U$ is $\lambda$-separating. The large cardinal consequences of the existence of a separating ultrafilter on $\kappa$ that we obtain in this section are derived from the following well-known results.

**Theorem 3.1** (a) (Kanamori [7]). If there is a uniform non-$((\kappa, \kappa^+)\$)-regular ultrafilter $U$ on $\kappa^+$, then there is such an ultrafilter $V$ on $\kappa^+$ which is also weakly normal and less than or equal to $U$ in the Rudin–Keisler ordering.

(b), (Kanamori [7] and Ketonen [8] independently). If there is a uniform ultrafilter $U$ on a regular cardinal $\kappa$ which in non-$(\omega, \lambda)$-regular for some $\lambda < \kappa$, then there is such an ultrafilter $V$ on $\kappa$ which is also weakly normal.

(c) (Jensen [5]). Suppose that $\kappa^{< \kappa} = \kappa$ and there is a uniform weakly normal ultrafilter on $\kappa$. Then there is an inner model with a measurable cardinal.

(d) (Koppelberg for regular $\kappa$ [5]; Donder for singular $\kappa$). Suppose that there is a uniform ultrafilter on $\kappa$ which is $\lambda$-indecomposable for some regular $\lambda^\prime < \kappa$. Then there is an inner model with a measurable cardinal.

The following is now straightforward.

**Theorem 3.2.** Suppose that $U$ is an $\omega$-separating ultrafilter on $\kappa > \omega$, and either

(i) CH holds, or

(ii) $\kappa > 2^{\omega_1}$ and $\kappa^{< \kappa} = \kappa$.

Then there is an inner model with a measurable cardinal.

**Proof.** Suppose first that (i) holds. Then either $U$ is $\omega_1$-indecomposable, in which case we are done by Theorem 3.1(d), or there is a uniform ultrafilter $V$ on $\omega_1$ with $V \leq_{RK} U$. It is an easy exercise to show that in this case $V$ is also
ω-separating and, hence, non-regular by Theorem 2.5(c). But now we are done by Theorem 3.1(a) and (c).
If (ii) holds, then \( \overline{U} \) is non-(ω, \( \lambda \))-regular for \( \lambda = 2^\omega < \kappa \) by Theorem 2.5(a).
The desired result now follows from Theorem 3.1(b) and (c).

This is the best possible result on the consistency strength of the existence of a separating ultrafilter on some \( \kappa > \omega_1 \) except in cases like \( \kappa \leq 2^\omega \). When \( \kappa \) is strongly inaccessible, the following result shows that \( \kappa \) itself has substantial large cardinal properties.

\textbf{THEOREM 3.4.} Suppose that \( U \) is a separating ultrafilter on the strongly inaccessible cardinal \( \kappa \). Then:

(a) \( \kappa \) is in the \( \omega \)th strong Mahlo class.
(b) If the GCH holds below \( \kappa \), then \( 2^\alpha = \kappa^+ \).
(c) Kunen's Hypothesis for \( \kappa \) fails.

\textbf{PROOF.} The proofs amount to a recasting of results in [12]. For (a), note first that by 2.5(a) and 3.1(b) we can assume that \( U \) is weakly normal. Moreover, it is easy to see that \( |\gamma/\mu| < \kappa \) for every \( \gamma < \kappa \); i.e., if \( f, g \in \gamma \) and \( [f]_\mu \neq [g]_\mu \) then \( f_*(U) \neq g_*(U) \), and there are fewer than \( \kappa \) many ultrafilters on \( \gamma \). By straightforward arguments (see proposition 8 of [12]) this is enough to verify that \( \{\alpha < \kappa : \alpha \text{ is strongly inaccessible}\} \in U \). We can now proceed by induction to establish that for each \( n \in \omega \), \( \{\alpha < \kappa : \alpha \text{ is } n\text{-th strongly Mahlo}\} \in U \). This is achieved by following the proof of theorem 6 of [12], using for the 1st case in that proof the fact that if \( V \equiv_k U \), then \( V \) is also separating.

For (b), we again assume that \( U \) is weakly normal and \( |\gamma/\mu| < \kappa \) for every \( \gamma < \kappa \) and call upon the proof of theorem 16 of [12]; this argument is essentially Scott's proof that if \( V \) is a normal ultrafilter on a measurable cardinal \( \mu \) and \( \{\alpha < \mu : 2^\alpha = \alpha^+\} \in V \), then \( 2^\alpha = \mu^+ \).

Finally, (c) follows in analogous fashion from theorem 7 of [12].

Whilst on the topic of large cardinals, let us mention a result of Sureson (unpublished). A normal ultrafilter on a measurable cardinal is separating, so it is natural to ask whether being a \( p \)-point, a well-known property of ultrafilters weaker than normality, is also a sufficient condition. Sureson established that this is not so. Specifically, she established that if \( \kappa \) is \( 2^\omega \)-supercompact (sic), then there is a \( p \)-point on \( \kappa \) which is not separating. Sureson has also shown that the consistency of the existence of a measurable cardinal is enough to obtain the consistency of the existence of a measurable cardinal which carries a non-separating \( p \)-point ultrafilter.
§4. A stepping up theorem

It is well-known that if $\lambda$ is regular and $U$ is a $\lambda$-indecomposable ultrafilter, then $U$ is also $\lambda^+$-indecomposable. (This was first proved by Chang [3] assuming $2^\lambda = \lambda^+$ and in general by Chudnovsky and Chudnovsky [4] and Kunen and Prikry [9].) The following result provides a partial analogue of this property for $\lambda$-separating ultrafilters.

**Theorem 4.1.** Suppose that $\lambda$ is regular and that $U$ is $\lambda$-separating and non-$(\lambda, \lambda^+)$-regular. Then $U$ is $\lambda^+$-separating.

**Proof.** Assume that $U$ is a uniform ultrafilter on $\kappa$ and that $f, g : \kappa \to \lambda^+$ show that $U$ is not $\lambda^+$-separating. We want to show that $U$ is either $(\lambda, \lambda^+)$-regular or not $\lambda$-separating. For this, we will need the following lemmas.

**Lemma 4.2.** There exists a collection $\{f_\alpha : \alpha < \lambda^+\}$ of functions satisfying the following:

(i) for each $\alpha < \lambda^+$, $f_\alpha : |\alpha| \to \alpha$ is a bijection, and

(ii) if $\beta < \alpha < \lambda^+$ then $|\{\xi < \lambda : f_\alpha(\xi) = f_\beta(\xi)\}| < \lambda$.

**Proof.** For $\alpha < \lambda$, choose any $f_\alpha$ satisfying (i). Suppose now that $\lambda \leq \alpha < \lambda^+$ and that $f_\beta$ has been defined for each $\beta < \alpha$. Let $\{g_\xi : \xi < \lambda\}$ enumerate $\{f_\beta : \beta < \alpha\}$ in order-type $\lambda$ and let $\{\gamma_\xi : \xi < \lambda\}$ enumerate $\alpha$ in order-type $\lambda$. We will define a bijection $f_\alpha : \lambda \to \alpha$ by a back and forth induction involving $\lambda$ steps, where at step $\xi < \lambda$ we specify values for $f_\alpha(\xi)$ and $f_\alpha^{-1}(\gamma_\xi)$. In order to ensure that (i) and (ii) hold we need only do this so that $f_\alpha$ remains one to one and the following are satisfied:

(iii) if $\eta \leq \xi$ and $f_\alpha(\xi)$ has not yet been defined then $f_\alpha(\xi) \neq g_\eta(\xi)$;

(iv) if $\eta \leq \xi$ and $f_\alpha^{-1}(\gamma_\eta)$ has not yet been defined then $f_\alpha^{-1}(\gamma_\eta) \neq g_\xi^{-1}(\gamma_\eta)$.

It is easy to see that this is possible. To see that (ii) holds notice that if $\eta < \lambda$ and $f_\alpha(\xi) = g_\eta(\xi) = \gamma_\eta$, then $\xi < \max\{\eta, \eta'\}$; i.e. if $f_\alpha(\xi)$ was defined at stage $\xi$ and $\xi \equiv \eta$ then $f_\alpha(\xi) \neq g_\eta(\xi)$ by (iii) and if $f_\alpha(\xi)$ was defined at stage $\eta' < \xi$ then $f_\alpha^{-1}(\gamma_\eta) \neq g^{-1}(\gamma_\eta)$ by (iv).

Now, to complete the proof of Theorem 4.1 we define, for each $\alpha < \lambda^+$, a function $h_\alpha : \lambda^+ - (\alpha + 1) \to \lambda$ by

$$h_\alpha(\beta) = f_\beta^{-1}(\alpha).$$

Recall that $f, g : \kappa \to \lambda^+$ were chosen so that $[f]_U \neq [g]_U$ but $f_*(U)$ and $g_*(U)$ are the same uniform ultrafilter on $\lambda^+$. Without loss of generality, assume that $f(\xi) < g(\xi)$ for every $\xi < \kappa$. We consider 3 cases.
Case 1. \( \{ \alpha < \lambda^+: (h_\alpha \circ f)_*(U) \text{ is not uniform on } \lambda \} \) has cardinality \( \lambda^+ \).

In this case we get a cardinal \( \mu < \lambda \), a set \( Z \subseteq \lambda^+ \) and for each \( \alpha \in Z \) a set \( X_\alpha \subseteq U \) so that \( |Z| = \lambda^+ \) and \( h_\mu(f(X_\alpha)) \subseteq \mu \). Let \( Y_\alpha = X_\alpha - \{ \gamma < \kappa : f(\gamma) \leq \alpha \} \).

Notice that \( Y_\alpha \subseteq U \) since \( f_\mu(U) \) is a uniform ultrafilter on \( \lambda^+ \). We claim that \( \{ Y_\alpha : \alpha < \lambda^+ \} \) shows that \( U \) is \( (\lambda, \lambda^+)-\text{regular} \). To see this, suppose not and choose \( \gamma \) occurring in \( \lambda \) many \( Y_\alpha \)'s. Let \( \beta = f(\gamma) \). Since \( h_\alpha(\beta) < \mu \) we get a set \( A \subseteq \lambda^+ \) so that \( |A| = \lambda \) and for each \( \alpha, \alpha' \in A \) we have \( h_\alpha(\beta) = h_{\alpha'}(\beta) \). (Notice that for each such \( \alpha \) we have \( h_\alpha(\beta) \) defined since \( \gamma \in Y_\alpha \rightarrow f(\gamma) > \alpha \rightarrow \beta > \alpha \). Thus \( \alpha < \beta \) so \( \beta \in \text{domain}(h_\alpha) \).) But now we have \( f_\beta^{-1}(\alpha) = f_\beta^{-1}(\alpha') \), contradicting the fact that \( f_\beta \) is one to one.

Case 2. \( \{ \alpha < \lambda^+: [h_\alpha \circ f]_U = [h_\alpha \circ g]_U \} \) has cardinality \( \lambda^+ \).

Let \( Z \) be the set of such \( \alpha \) and choose \( X_\alpha \subseteq U \) for each \( \alpha \in Z \) so that \( h_\mu \circ f(\gamma) = h_\alpha \circ g(\gamma) \) for every \( \gamma \in X_\alpha \). We claim that the collection \( \{ X_\alpha : \alpha \in Z \} \) shows that \( U \) is \( (\lambda, \lambda^+)-\text{regular} \). To see this, suppose not and choose \( \gamma \) occurring in \( \lambda \) many \( X_\alpha \)'s. Then for each such \( \alpha \) we have \( f_\gamma^{-1} \circ (\alpha) = f_\gamma^{-1} \circ (\alpha') \) and so \( f_\gamma^{-1} \) and \( f_\gamma^{-1} \) agree on a set of size \( \lambda \). Thus \( f(\gamma) = g(\gamma) \), contradiction.

Case 3. Otherwise.

In this case we have at least one \( h_\alpha \) so that

\[
[h_\alpha \circ f]_U \neq [h_\alpha \circ g]_U
\]

and \( (h_\alpha \circ f)_*(U) \) is a uniform ultrafilter on \( \lambda \). Since \( f_\alpha(U) = g_\alpha(U) \) it follows that \( (h_\alpha \circ f)_*(U) = (h_\alpha \circ g)_*(U) \) and so \( U \) is not \( \lambda \)-separating in this case.

Combining Theorem 4.1 with the non-regularity results in Theorem 2.5(b) and (c), we obtain the following.

**Theorem 4.3 ([3]).** Assume that \( U \) is an \( \omega \)-separating ultrafilter on \( \kappa \). Then

(a) \( U \) is \( \omega_1 \)-separating, and

(b) if \( \kappa < \aleph_\omega \), then \( U \) is a separating ultrafilter (i.e., \( \lambda \)-separating for all \( \lambda \)).

It is worth noting that the converse of Theorem 4.3(a) is not provable. In fact, the existence of an \( \omega_1 \)-separating ultrafilter on \( \omega_1 \) has no large cardinal consequences. For example, if \( 2^{\omega_1} = \omega_2 \), then a straightforward inductive construction yields a uniform ultrafilter \( U \) on \( \omega_1 \) having the property that any \( f : \omega_1 \rightarrow \omega_1 \) is either bounded (mod \( U \)) or one to one (mod \( U \)). (This was pointed out to us several years ago by Prikry.) But, as shown in [10], every ideal (in particular: \( U^* \)) is separating with respect to one-one functions, and so \( U \) is \( \omega_1 \)-separating.
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