ADVANCES IN MATHEMATICS 59, 152–169 (1986)

Partition Relations for Successor Cardinals

AKIHIRO KANAMORI

Department of Mathematics, Boston University, Boston, Massachusetts 02215

This paper investigates the relations $\kappa^+ \to (\alpha)^2_2$ and its variants for uncountable cardinals κ. First of all, the extensive literature in this area is reviewed. Then, some possibilities afforded by large cardinal hypotheses are derived, for example, if κ is measurable, then $\kappa^+ \to (\kappa+\kappa+1, \alpha)^2_2$ for every $\alpha < \kappa^+$. Finally, the limitations imposed on provability in ZFC by L and by relative consistency via forcing are considered, primarily the consistency of: if κ is not weakly compact, then $\kappa^+ \not\to [\kappa : \kappa]^2_\kappa$. © 1986 Academic Press, Inc.

This paper discusses a basic property of infinite cardinals, and thereby presents an episode in combinatorial set theory. Although abstract and concise, the property became the focus of attention for the development of ideas and methods of considerable sophistication. Beyond the direct consequences of the ZFC axioms for set theory, there is a contemporary typicality: large cardinal hypotheses extend the limits of possibility, and combinatorial propositions true in the constructible universe delimit provability in ZFC. The three sections of this paper take up the several aspects: the first section reviews the background which frames the entire discussion; the second section provides the further consequences available through the addition of large cardinal assumptions; and the third section discusses the limitations imposed by forcing and the categoricity of the constructible universe. Aspects of this paper have been considerably enhanced by conversations with Hans-Dieter Donder and Richard Laver, and by the fruits of their research.

1. BACKGROUND

Let us reaffirm some notation. For X a set of ordinals and α an ordinal, $[X]^\alpha$ denotes the set of subsets of X with ordertype α.

(i) The *ordinary partition relation* of Erdös and Rado for ordinals $\alpha \to (\beta)^n_\delta$ asserts that whenever $f\colon [\alpha]^n \to \delta$, there are an $X \in [\alpha]^\beta$ and a $\rho < \delta$ such that $f''[X]^n = \{\rho\}$.

0001-8708/86 $7.50

(ii) The *disjunctive form*, for the principal case of the subscript 2, is $\alpha \to (\beta, \gamma)^n$, which asserts that whenever $f\colon [\alpha]^n \to 2$, either there is an $X \in [\alpha]^\beta$ such that $f''[X]^n = \{0\}$, or else there is a $Y \in [\alpha]^\gamma$ such that $f''[Y]^n = \{1\}$.

(iii) The *square-brackets partition relation* $\alpha \to [\beta]^n_\delta$ asserts that whenever $f\colon [\alpha]^n \to \delta$, there is an $X \in [\alpha]^\beta$ such that $f''[X]^n \neq \delta$, i.e., f omits a value on $[X]^n$.

(iv) There is one important variant: If the β in say the ordinary partition relation is replaced by $\eta : \zeta$, the assertion is then that there are a $\rho < \delta$, an $A \in [\alpha]^\eta$, and a $B \in [\alpha - \sup A]^\zeta$ such that whenever $\sigma \in A$ and $\tau \in B, f(\{\sigma, \tau\}) = \rho$. Note that $\alpha \to (\eta + \zeta)^n_\delta$ implies $\alpha \to (\eta : \zeta)^n_\delta$. The β or γ in the disjunctive or square-bracket forms may be similarly replaced.

(v) We will also have occasion to refer to related *polarized partition relations*.

$$\begin{pmatrix} \lambda \\ \kappa \end{pmatrix} \to \begin{bmatrix} \mu \\ \nu \end{bmatrix}_\delta$$

asserts that whenever $f\colon \lambda \times \kappa \to \delta$, there are $X \in [\lambda]^\mu$ and $Y \in [\kappa]^\nu$ such that $f''(X \times Y) \neq \delta$.

(vi) Finally, the negation of any of these assertions is denoted by a corresponding $\not\to$.

For more on the whole subject of the partition calculus, see the good secondary source Williams [30].

The main question is to what extent we have

$$\kappa^+ \to (\alpha)^2_\delta$$

for regular cardinals κ. We restrict ourselves to regular κ, as well as the superscript 2, merely because we already encounter substantial difficulty in this case. These propositions have the feel of basic set theoretical assertions generalizing the pigeon-hole principle, and their study has been a recurring theme since the 1950s. We take some time to chronicle that study, in order to establish the context for this paper.

Casting ideas of Sierpinski into a general setting, the 1953 paper of Erdös and Rado [9] first noted that $\lambda \to (\lambda)^2_2$ never holds for successor cardinals λ. Of course, we know now that λ's satisfying this relation, the weakly compact cardinals, must be extremely inaccessible from below. If the subscript is raised, we encounter another classical Sierpinski restriction, $2^\kappa \not\to (3)^2_\kappa$.

The first significant positive result occurs in the 1956 paper of Erdös and Rado [10]: If $\kappa^{<\kappa} = \kappa$, then $\kappa^+ \to (\kappa^+, \kappa + 1)^2$. A few years later, Hajnal

[12] limited this approach by establishing that if $2^\kappa = \kappa^+$, then $\kappa^+ \not\rightarrow (\kappa^+, \kappa : 2)^2$. Taylor recently noted that this conclusion already follows from an enumeration principle strictly weaker than $2^\kappa = \kappa^+$; see Carlson [3] for a proof. Also, Laver [17] established the consistency of $2^\omega \not\rightarrow (2^\omega, \omega : 2)^2$ for values of 2^ω other than ω_1. Recently, Todorcevic [26] established the consistency of $\omega_1 \rightarrow (\omega_1, \alpha)_2^2$ for every $\alpha < \omega_1$. Actually, a standard proof of the classical Erdös–Rado result using Fodor's regressive function lemma can be amplified to provide a known extension which just skirts the Sierpinski restriction: if $\kappa^{<\kappa} = \kappa$ and $\delta < \kappa$, then $\kappa^+ \rightarrow (\kappa^+, (\kappa+1)_\delta)^2$, which means that whenever f: $[\kappa^+]^2 \rightarrow \delta$, either there is an $X \in [\kappa^+]^{\kappa^+}$ such that $f''[X]^2 = \{0\}$, or else there are a $0 < \rho < \delta$ and a $Y \in [\kappa^+]^{\kappa+1}$ such that $f''[Y]^2 = \{\rho\}$. We shall observe in passing (Theorem 3.4) that this is the best possible for successor cardinals: If $V = L$ and κ is a successor cardinal with κ^- its predecessor, then $\kappa^+ \not\rightarrow (\kappa : 2)^2_{\kappa^-}$.

Tackling the ordinary partition relation directly, the 1973 paper of Shelah established the following result in ZFC + GCH: If $\gamma^+ < \kappa$, then $\kappa^+ \rightarrow (\kappa + \gamma)_2^2$. See Todorcevic [27] for a more general result, and a well-rendered proof. Here, the subscript 2 is essential; it is not known whether the GCH implies $\omega_2 \rightarrow (\omega_1 + 2)_3^2$. Soon after, Rebholz established in his 1974 paper [23] that the Shelah result is the best possible for successor cardinals, by showing that if $V = L$ and κ is a successor cardinal, then $\kappa^+ \not\rightarrow [\kappa : \kappa^-]^2_\kappa$. Very recently, Donder [8] provided the following improvement based on the Jensen Covering Theorem: If κ is a successor cardinal $> \omega_2$, $2^{\kappa^-} = \kappa$, and $0^\#$ does not exist, then $\kappa^+ \not\rightarrow [\kappa : \kappa^-]^2_\kappa$. Here, $\kappa > \omega_2$ is an annoying but essential restriction of the proof. Nonetheless, the Rebholz paper is noteworthy for its early appearance; soon after Jensen's morasses saw the light of day, Rebholz grasped their applicability to propositions of combinatorial set theory.

Jensen invented the morass in the early 1970s in order to establish strong model-theoretic transfer principles in L. Morasses are structures of considerable complexity, a culminating edifice in Jensen's remarkable program of formulating useful combinatorial principles which obtain in L, and which moreover can be appended to any model of set theory by straightforward forcing. The axiom $V = L$ is surely the ultimate combinatorial principle in ZFC, and the morass codifies a substantial portion of the structure of L. As set theorists looked beyond the well-known $\diamond$ and $\square$ for applicable combinatorial principles, it was natural to consider extractions from the full structure of a morass.

The main result of Section 3 extends Rebholz' work to regular limit cardinals. His isolation of the salient combinatorial structure is put into a contemporary context and generalized. The approach is to develop a forcing scheme, and to extend it to limit cardinals with the requisite strength by using a new kind of density argument first discovered by Shelah. Recent work of Velleman and Donder, building on efforts by Shelah and Stanley, can then be cited to apply this construction with a morass with particularly strong properties to provide a new characterization of weakly compact cardinals in L, as those cardinals κ satisfying $\kappa^+ \to [\kappa : \kappa]^2_\kappa$.

In counterpoint to these various limitative results, there are the possibilities afforded by cardinals endowed with special closure properties of large cardinal character. For $\kappa = \omega$, a long-standing conjecture was that $\omega_1 \to (\alpha)^2_n$ holds for every $\alpha < \omega_1$ and $n < \omega$. After various partial results (e.g., Hajnal [12] and Prikry [21]), Baumgartner and Hajnal affirmed this conjecture, as an immediate consequence of an even more general result which they established in elegant fashion by using Martin's axiom and an absoluteness argument (the notation has the obvious interpretation):

> If ψ is an ordertype such that $\psi \to (\omega)^1_\omega$, then $\psi \to (\alpha)^2_n$ for every $\alpha < \omega_1$ and $n < \omega$.

Avoiding these tricks of the trade, Galvin [11] provided a direct proof of the Baumgartner–Hajnal theorem which is a combinatorial tour de force. There have since been further developments. In his paper, Galvin asked whether the hypothesis of the Baumgartner–Hajnal theorem can be weakened to : ψ is a partially ordered set such that $\psi \to (\omega)^1_\omega$. Todorcevic [27] confirmed this with an attractive proof, streamlining Galvin's combinatorics with a forcing and absoluteness argument.

It is not known whether $\kappa^+ \to (\alpha)^2_2$ for every $\alpha < \kappa^+$ holds for any uncountable κ. While being the least infinite ordinal is a strong property of ω which, of course, does not generalize, large cardinal properties that espouse other structural properties of ω lead to partial positive results. Directly applying a related polarized partition relation, the following weak positive result is noted in Section 2: If κ is weakly compact, then $\kappa^+ \to (\kappa : \eta)^2_2$ for every $\eta < \kappa^+$. This already complements the limitative results for non-weakly compact cardinals in Section 3. Having a measure-theoretic overlay leads to stronger results: If κ is a measurable cardinal, then $\kappa^+ \to (\kappa + \kappa + 1, \alpha)^2_2$ for every $\alpha < \kappa^+$. Actually, the conclusion already follows from the existence of a Laver ideal over κ, and this was first proved by Laver [19]. The author rediscovered this proof and it is given in Section 2 with Laver's permission.

Finally, it should be mentioned that various speculations concerning uncountable cardinals seem to encounter a recurring difficulty. This is the well-known Milner–Rado "paradox" [20]: For any κ, any $\alpha < \kappa^+$ can be written as a disjoint union $\alpha = \bigcup_{n \in \omega} A^\alpha_n$, where each A^α_n has ordertype $< \kappa^n$. For example, these decompositions immediately impose the following restriction on infinite subscripts: $\kappa^+ \nrightarrow (\kappa^\omega : 1)^2_\omega$. To verify this, just let f:

$[\kappa^+]^2 \to \omega$ be defined by $f(\{\beta, \alpha\}) =$ that n such that $\beta \in A_n^\alpha$. Various attempts to generalize the techniques of Galvin [11] are also thwarted by the paradox.

2. The Possibilities

This section deals with the possibilities afforded by large cardinal hypotheses. The results are weak partial results if one were to conjecture the full analogue from the ω case: If κ is weakly compact, then $\kappa^+ \to (\alpha)_n^2$ for every $\alpha < \kappa^+$ and $n < \omega$. Remember that the aforementioned Milner–Rado paradox restricts the subscript somewhat, but one can still conjecture: If κ is weakly compact, then $\kappa^+ \to (\kappa^\omega)_\omega^2$. There is no reason to believe that these assertions cannot be established in ZFC.

The first result is an immediate consequence of a related polarized partition relation, but is still enough to counterpoint the limitative results for non-weakly compact cardinals in the next section.

Theorem 2.1. *If κ is weakly compact, then $\kappa^+ \to (\kappa : \eta)_2^2$ for every $\eta < \kappa^+$.*

Proof. For the case when κ is weakly compact, Chudnovsky [4] stated without proof the following polarized partition relation, and proofs have since been provided by Wolfsdorf [31], Shelah, and Kanamori [15]: For every $\eta < \kappa$,

$$\binom{\kappa^+}{\kappa} \to \begin{bmatrix} \eta \\ \kappa \end{bmatrix}_2.$$

The desired result is now immediate, for given g:$[\kappa^+]^2 \to 2$, one can identify $\kappa^+ - \kappa$ with κ^+ in applying the polarized partition relation to $g \restriction \kappa \times (\kappa^+ - \kappa)$, say.

The following question seems to remain unanswered:

Question 2.2. If κ is weakly compact, does $\kappa^+ \to (\kappa + \kappa)_2^2$?

It may be expected that introducing a measure-theoretic overlay leads to stronger results. The following result is relevant for measurable cardinals, but it turns out that all that is needed is the existence of a Laver ideal. A good reference for the theory of ideals is Baumgartner, Taylor, and Wagon [2]. By a Laver ideal, we shall mean one satisfying the primary case of a class of strong saturation properties studied by Laver: a (non-trivial, κ-complete) ideal I over κ such that given κ^+ sets in $P(\kappa) - I$, there are κ^+ of them so that any $< \kappa$ of these has intersection $\notin I$. A Laver ideal I over κ is easily seen to be κ^+-saturated in the usual sense, and by standard arguments we can take I to be normal. A measurable cardinal trivially carries such an ideal, which indeed is dual to an ultrafilter. Laver [18] provided forcing constructions of such ideals over certain accessible cardinals, starting with a measurable cardinal in the ground model. He also provided [19] a construction of a Laver ideal over ω_1, starting with a huge cardinal, and derived some strong consequences in the partition calculus. The following result was first discovered by Laver and a proof is outlined in [19]; the result was rediscovered by the author, and with Laver's permission, a detailed proof is presented here to complete the section.

THEOREM 2.3. *If $\kappa^{<\kappa} = \kappa$ and there is a Laver ideal over κ, then $\kappa^+ \to (\kappa + \kappa + 1, \alpha)_2^2$ for every $\alpha < \kappa^+$.*

COROLLARY 2.4. *If κ is measurable, then $\kappa^+ \to (\kappa + \kappa + 1, \alpha)_2^2$ for every $\alpha < \kappa^+$.*

Proof of Theorem 2.3. Suppose that $[\kappa^+]^2 = J_0 \cup J_1$ is a partition into two cells. Whenever A is a set of ordinals with ordertype κ, let I_A be a normal Laver ideal over A in the appropriate sense (i.e., if A were identified with κ via the order-preserving bijection), and I_A^* its dual filter. Also, set $J_i(\alpha, A) = \{\xi \in A \mid \{\xi, \alpha\} \in J_i\}$ for $i < 2$.

Let us consider the following hypothesis:

$$\text{There is an } A \in [\kappa^+]^\kappa \text{ such that } [A]^2 \subseteq J_0 \text{ and } X = \{\alpha < \kappa^+ \mid J_0(\alpha, A) \notin I_A\} \text{ has cardinality } \kappa^+. \qquad (*)$$

The proof splits according to whether or not $(*)$ holds.

Case I. $(*)$ holds. Since I_A is a Laver ideal, let $Y \subseteq X - \sup(A)$ be of cardinality κ^+ such that whenever $s \in [Y]^{<\kappa}$, we have $\bigcap_{\alpha \in s} J_0(\alpha, A) \notin I_A$. For each $\alpha \in Y$, we try to define ordinals $x_\xi^\alpha \in A$ and $y_\xi^\alpha \in Y$ by induction on $\xi < \kappa$ for as long as possible, using the following joint schemes:

(i) $x_\xi^\alpha =$ least ordinal x such that: $x > x_\zeta^\alpha$ for $\zeta < \xi$, and $x \in J_0(\alpha, A) \cap \bigcap_{\zeta < \xi} J_0(y_\zeta^\alpha, A)$. (Such an x always exists since this last set is $\notin I_A$.)

(ii) $y_\xi^\alpha =$ least ordinal $y < \alpha$ (if it exists) such that: $y \in Y$ and $y > y_\zeta^\alpha$ for $\zeta < \xi$, $\{x_\zeta^\alpha \mid \zeta \leqslant \xi\} \subseteq J_0(y, A)$, and $\{\{y_\zeta^\alpha, y\} \mid \zeta < \xi\} \cup \{\{y, \alpha\}\} \subseteq J_0$.

If there were an α such that this induction proceeds through all ordinals $\xi < \kappa$, then $\{x_\xi^\alpha \mid \xi < \kappa\} \subseteq \bigcap_{\xi < \kappa} J_0(y_\xi^\alpha, A)$, and thus,

$$[\{x_\xi^\alpha \mid \xi < \kappa\} \cup \{y_\xi^\alpha \mid \xi < \kappa\} \cup \{\alpha\}]^2 \subseteq J_0.$$

So, we can assume that there is no such α. Let $\psi \colon \kappa^+ \leftrightarrow Y$ be the unique order-preserving bijection. Define $f(\gamma)=\sup\{\psi^{-1}(y_\xi^{\psi(\gamma)})\mid y_\xi^{\psi(\gamma)}$ is defined$\}$ and $g(\gamma)=\sup\{\xi\mid y_\xi^{\psi(\gamma)}$ is defined$\}$ for $\gamma<\kappa^+$. Then whenever $cf(\gamma)=\kappa$, we have $f(\gamma)<\gamma$ by our assumption. By the regressive function lemma, there is a stationary set $S_1\subseteq\{\gamma<\kappa^+\mid cf(\gamma)=\kappa\}$ and fixed $\delta<\kappa^+$ and $\rho<\kappa$ such that $f(\gamma)=\delta$ and $g(\gamma)=\rho$ for every $\gamma\in S_1$. Since $|\delta^\rho|\leqslant\kappa^{<\kappa}=\kappa$, there is a stationary $S_2\subseteq S_1$ and fixed s and t such that $\{y_\xi^{\psi(\gamma)}\mid y_\xi^{\psi(\gamma)}$ is defined$\}=s$ and $\{x_\xi^{\psi(\gamma)}\mid x_\xi^{\psi(\gamma)}$ is defined$\}=t$ for every $\gamma\in S_2$. Now if $\gamma_1<\gamma_2$ are both in S_2, then $\{\psi(\gamma_1),\psi(\gamma_2)\}\in J_0$ would inply that condition (ii) in the definition of the $y_\xi^{\psi(\gamma_2)}$ sequence could have been met, so that the sequence could have been extended beyond s, contrary to our assertions. Thus, $[\psi''S_2]^2\subseteq J_1$, which is more than enough to conclude the theorem for Case I.

Case II. $(*)$ fails. Here, we can apply some ideas and terminology of Prikry [21]. If B and C are sets of ordinals, write $B<C$ to indicate that every element of B is strictly less than any element of C. Suppose that: (a) $F\subseteq\{A\in[\kappa^+]^\kappa\mid[A]^2\subseteq J_0\}$ with $|F|\leqslant\kappa$; (b) $\eta<\kappa^+$; and (c) $X\in[\kappa^+]^{\kappa^+}$ such that $A<X$ for every $A\in F$. Then say that F is *(η, X)-extendible iff* there are $C_A\in I_A^*$ for $A\in F$ and $\{B_\delta\mid\delta<\eta\}\subseteq\{B\in[X]^\kappa\mid[B]^2\subseteq J_0\}$ such that: whenever $A\in F$ and $\delta<\rho<\eta$, then $C_A<B_\delta<B_\rho$, $C_A\times B_\delta\subseteq J_1$, and $B_\delta\times B_\rho\subseteq J_1$. In discussing this notion, the conditions (a), (b), and (c) on F, η, and X, respectively, will be implicitly assumed. To establish the theorem in Case II, it suffices to assume that there is no $H\in[\kappa^+]^{\kappa^+}$ such that $[H]^2\subseteq J_1$, and then to conclude that for every triple F, η, and X, F is (η, X)-extendible. This is a direct consequence of the following three lemmata:

LEMMA 1. *Assume there is no $H\in[\kappa^+]^{\kappa^+}$ such that $[H]^2\subseteq J_1$. Then for every pair F and X, F is $(1, X)$-extendible.*

LEMMA 2. *Suppose that $\eta<\kappa^+$ is a limit ordinal with $cf(\eta)<\kappa$. If for every triple F, X, and $\delta<\eta$, F is (δ, X)-extendible, then for every pair F and X, F is (η, X)-extendible.*

LEMMA 3. *Suppose that $\eta<\kappa^+$ with $cf(\eta)=\kappa$. If for every triple F, X, and $\delta<\eta$, F is (δ, X)-extendible, then for every pair F and X, F is (η, X)-extendible.*

A straightforward inductive construction using the κ-completeness of the I_A^*'s establishes Lemma 2, so it remains to verify Lemmas 1 and 3:

Proof of Lemma 1. Suppose that F and X are given. Since $|F|\leqslant\kappa$ and we are assuming that $(*)$ fails, it is easy to see that $Y=\{\alpha\in X\mid J_1(\alpha, A)\in I_A^*$ for every $A\in F\}$ still has cardinality κ^+. Let $F=\{A_\sigma\mid\sigma<\kappa\}$ be an enumeration in ordertype κ. We try to define ordinals $y_\xi^\alpha\in Y$ and $x_\xi^{\alpha,\sigma}\in A_\sigma$

for every $\sigma \in \xi$ by induction on $\xi < \kappa$ for as long as possible, using the following joint schemes:

(i) For each $\sigma < \xi$, $x_\xi^{\alpha,\sigma} =$ least ordinal x such that: $x > x_\zeta^{\alpha,\sigma}$ for $\sigma < \zeta < \xi$, and $x \in J_1(\alpha, A_\sigma) \cap \bigcap_{\zeta < \xi} J_1(y_\zeta^\alpha, A_\sigma)$. (Such an x always exists, since this last set is in I_A^*.)

(ii) $y_\xi^\alpha =$ least ordinal $y < \alpha$ (if it exists) such that: $y \in Y$ and $y > y_\zeta^\alpha$ for $\zeta < \xi$, $\{x_\zeta^{\alpha,\sigma} \mid \sigma < \zeta \leqslant \xi\} \subseteq J_1(y, A_\sigma)$ for $\sigma < \xi$, and $\{\{y_\xi^\alpha, y\} \mid \zeta < \xi\} \cup \{\{y, \alpha\}\} \subseteq J_0$.

Suppose first that:

There is an α such that this induction proceeds through all ordinals $\xi < \kappa$. $(**)$

Then $\{x_\xi^{\alpha,\delta} \mid \sigma < \xi < \kappa\} \subseteq \bigcap_{\xi < \kappa} J_1(y_\xi^\alpha, A_\sigma)$ for every $\sigma < \kappa$. This would establish that F is $(1, X)$-extendible to $\{y_\xi^\alpha \mid \xi < \kappa\}$, if only we can show that $\{x_\xi^{\alpha,\sigma} \mid \sigma < \xi < \kappa\} \in I_{A_\sigma}^*$ for every $\sigma < \kappa$. To this end, fix σ and $A = A_\sigma$, and let $g: A \leftrightarrow \kappa$ be the unique order-preserving bijection. It suffices to establish the following *Claim*: $\{\gamma \in A \mid \gamma = x_{g(\gamma)}^{\alpha,\sigma}\} \in I_A^*$.

The verification of this Claim uses the normality of I_A^*: First of all, since $\langle x_\xi^{\alpha,\sigma} \mid \sigma < \xi < \kappa \rangle$ is an ascending sequence from A, it is clear that $\gamma \leqslant x_{g(\gamma)}^{\alpha,\sigma}$ for every $\gamma \in A$ with $g(\gamma) > \sigma$. Secondly, a simple argument using normality shows that $B = \{\gamma \in A \mid \delta \in A \;\&\; \delta < \gamma$ implies $x_{g(\delta)}^{\alpha,\sigma} < \gamma\} \in I_A^*$. Thus, were the Claim false, $\{\gamma \in B \mid \gamma < x_{g(\gamma)}^{\alpha,\sigma}\} \notin I_A$, i.e., by the definition of the $x_\xi^{\alpha,\sigma}$'s, $\{\gamma \in B \cap J_1(\alpha, A) \mid \exists \delta < \gamma (\gamma \notin J_1(y_{g(\delta)}^\alpha, A))\} \notin I_A$. Then by normality, there would be a fixed $\bar{\delta} \in A$ such that $\{\gamma \in B \cap J_1(\alpha, A) \mid \gamma \notin J_1(y_{g(\bar{\delta})}^\alpha, A\} \notin I_A$, contradicting $J_1(y_{g(\bar{\delta})}^\alpha, A) \in I_A^*$. This verifies the Claim, and concludes the argument if $(**)$ were assumed.

Suppose now that $(**)$ fails. Then we can argue just as in the last part of the argument for Case I to find an $H \in [Y]^{\kappa^+}$ such that $[H]^2 \subseteq J_1$, contradicting the hypothesis of Lemma 1. Note that we took care to deal with the members of F in a gradual manner, so that by any stage $\xi < \kappa$, less than κ many ordinals appear in $\{x_\zeta^{\alpha,\sigma} \mid \sigma < \zeta \leqslant \xi\} \cup \{y_\zeta^\alpha \mid \zeta < \xi\}$. So again, by $\kappa^{<\kappa} = \kappa$ we can find many α's which yield the same construction. The proof of Lemma 1 is therefore complete.

Proof of Lemma 3. Since $cf(\eta) = \kappa$, write $\eta = \sum_{\xi < \kappa} \eta_\xi$ with each $\eta_\xi < \eta$. Suppose now that F and X are given. Again, by the failure of $(*)$, we can assume that $Y = \{\alpha \in X \mid J_1(\alpha, A) \in I_A^*$ for every $A \in F\}$ still has cardinality κ^+. Let $F = \{A_\sigma \mid \sigma < \kappa\}$ be an enumeration in ordertype κ. We try to define sets $T_\xi^\alpha \in [[Y]^\kappa]^{\eta_\xi}$, ordinals $x_\xi^{\alpha,\sigma} \in A_\sigma$ for every $\sigma < \xi$, and (inductively having fixed enumerations $T_\nu^\alpha = \{B_\sigma^\nu \mid \nu < \sigma < \kappa\}$ for $\nu < \xi$) ordinals $z_\xi^{\alpha,\nu,\sigma} \in B_\sigma^\nu$ for $\nu < \sigma < \xi$ for as long as possible, using the following joint schemes. This is an eleborate version of the previous proof which at each step provides a η_ξ-extension T_ξ^α, and this then necessitates carrying out the thinning procedure not only for members of F but also for members of T_σ^α with respect to all further apparitions T_ξ^α for $\sigma<\xi<\kappa$.

(i) For each $\sigma<\xi$, $x_\xi^{\alpha,\sigma}=$ least ordinal x such that: $x>x_\zeta^{\alpha,\sigma}$ for $\sigma<\zeta<\xi$, and $x\in J_1(\alpha, A_\sigma)\cap\bigcap_{\zeta<\xi}\bigcap_{B\in T_\zeta^\alpha}\bigcap_{\rho\in B}J_1(\rho, A_\sigma)$. (Such an x always exists, since it will be clear from the induction that $\bigcap_{B\in T_\zeta^\alpha}\bigcap_{\rho\in B}J_1(\rho, A_\sigma)\in I_A^*$ for every $\zeta<\xi$.)

(ii) For every pair $\nu<\sigma<\xi$, $z_\xi^{\alpha,\nu,\sigma}=$ least ordinal z such that: $z>z_\zeta^{\alpha,\nu,\sigma}$ for $\sigma<\zeta<\xi$, and $z\in J_1(\alpha, B_\sigma^\nu)\cap\bigcap_{\nu<\zeta<\xi}\bigcap_{\rho\in T_\zeta^\alpha}J_1(\rho, B_\sigma^\nu)$. (Such a z exists, as before.)

(iii) $T_\xi^\alpha=$ lexigraphically least member T (if it exists) of $[[Y]^\kappa]^{\eta_\xi}$ such that: $T>T_\zeta^\alpha$ for $\zeta<\xi$, $\bigcap_{\rho\in T}J_1(\rho, A)\in I_A^*$ for every $A\in F\cup\bigcup\{T_\zeta^\alpha\mid\zeta<\xi\}$, $\{x_\zeta^{\alpha,\sigma}\mid\sigma<\zeta\leqslant\xi\}\subseteq\bigcap_{\rho\in T}J_1(\rho, A_\sigma)$ for every $\sigma<\xi$, and $\{z_\zeta^{\alpha,\nu,\sigma}\mid\sigma<\zeta\leqslant\xi\}\subseteq\bigcap_{\rho\in T}J_1(\rho, B_\sigma^\nu)$ for every $\nu<\sigma<\xi$.

We now make the following Claim:

$\{\tau\mid\exists\alpha\in Y\,\exists\xi<\kappa(T_\xi^\alpha$ is defined & $\exists A\in T_\xi^\alpha(\tau\in A))\}$ has cardinality $\leqslant\kappa$.

That is, even if the schemes are carried out for every $\alpha\in Y$, at most κ many ordinals ever get involved.

The Claim is established by induction on $\xi<\kappa$: If as α ranges over Y at most κ ordinals ever get involved in all previous stages $\zeta<\xi$, note first that for $\alpha\in Y$, the $x_\xi^{\alpha,\sigma}$'s and $z_\xi^{\alpha,\nu,\sigma}$'s defined through clauses (i) and (ii) constitute less than κ ordinals defined from a fixed set of κ ordinals. Thus by $\kappa^{<\kappa}=\kappa$, as α ranges over Y, there are at most κ possible choices. Next, a look at clause (iii) indicates that T_ξ^α only depends on such a choice, and ordinals involved with T_ζ^α for $\zeta<\xi$. So again by induction there are at most κ possibilities for T_ξ^α, as α ranges over all of Y.

Now if there were an α such that the induction proceeds through all ordinals $\xi<\kappa$, then we can finish the argument analogously to the argument from (**) in Lemma 1. So suppose that this is not the case. Then surely there would be an $S_1\in[Y]^{\kappa^+}$ and a fixed $\rho<\kappa$ such that for every $\alpha\in S_1$, ρ is least such that T_ρ^α is left undefined. Now $\{x_\xi^{\alpha,\sigma}\mid\sigma<\xi\leqslant\rho\}\cup\{z_\xi^{\alpha,\nu,\sigma}\mid\sigma<\xi\leqslant\rho\}$ constitutes less than κ ordinals, which by the Claim is chosen from a fixed set of κ ordinals. Thus, there is an $S_2\in[S_1]^{\kappa^+}$ such that for every $\alpha\in S_2$, the choice was the same. Again as in the argument for the Claim, this implies that there is a fixed sequence $\langle T_\xi\mid\xi<\rho\rangle$ such that $\langle T_\xi^\alpha\mid\xi<\rho\rangle=\langle T_\xi\mid\xi<\rho\rangle$ for every $\alpha\in S_2$. Finally, by the hypothesis of Lemma 3, $F\cup\bigcup\{T_\xi\mid\xi<\rho\}$ is (η_ρ, S_2)-extendible. Thus, the defining clause (iii) for T would certainly be satisfied by some $T\in[[S_2]^\kappa]^{\eta_\rho}$ for any

$\alpha \in S_2$. This contradiction establishes Lemma 3, and thus the overall theorem.

The following question seems to remain unanswered:

Question 2.5. If κ is measurable, does $\kappa^+ \to (\kappa + \kappa + 2)_2^2$?

3. The Limitations

The limitative results concerning our partition relation are closely related to combinatorial principles derivable from the existence of morasses. Various such principles formulated by various people for various purposes are framed into a coherent scheme in Kanamori [15] (see also [13, 14]). Here, we discuss the immediately relevant principles and note some new connections.

Let us first consider the case of a successor cardinal κ with κ^- its predecessor. Rebholz [23] first established that if $V = L$, then for all successor κ, $\kappa^+ \not\to [\kappa : \kappa^-]_\kappa^2$. He established this result by first deriving the following principle, dubbed Rebholz's Principle in Kanamori [15], from the existence of morasses:

> There is a collection $\{f_\alpha \mid \alpha < \kappa^+\}$ of functions $f_\alpha : \alpha \to \alpha$, so that whenever $s \in [\kappa^+]^{\kappa^-}$ and ϕ is a regressive function with domain $= s$ (i.e., $\phi(\alpha) < \alpha$ for $\alpha \in s$), then $|\{\xi < \bigcap s \mid \forall \alpha \in s (f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$. $\quad (\mathrm{R}_\kappa)$

Clearly, (R_κ) is equivalent to the following principle if we compose each f_α with a bijection $\alpha \leftrightarrow \kappa$ for $\kappa \leqslant \alpha < \kappa^+$:

> There is a collection $\{g_\alpha \mid \alpha < \kappa^+\}$ of functions $g_\alpha : \alpha \to \kappa$, so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi : s \to \kappa$, then $|\{\xi < \bigcap s \mid \forall \alpha \in s (g_\alpha(\xi) \neq \phi(\alpha))| < \kappa$. $\quad (\mathrm{R}'_\kappa)$

This is perhaps a more convenient formulation; note that $\kappa^+ \not\to [\kappa : \kappa^-]_\kappa^2$ is immediately entailed by the function G: $[\kappa^+]^2 \to \kappa$ defined by $G(\beta, \alpha) = g_\alpha(\beta)$ for $\beta < \alpha$.

Rebholz actually dubbed his principle the Extended Prikry's Principle, after the following principle formulated by Prikry [22]:

> There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ so that whenever $s \in [\kappa^+]^{\kappa^-}$ and $\phi : s \to \kappa$, we have $|\{\xi < \kappa \mid \forall \alpha \in s (f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$. $\quad (\mathrm{P}_\kappa)$

By considering $\{g_\alpha \restriction \kappa \mid \alpha < \kappa^+\}$ with the g_α's as in (R'_κ), it is immediate that (R_κ) implies (P_κ). Prikry had devised his principle and established its consistency with the GCH by forcing, to delimit a polarized partition relation. With (P_κ), the function $F\colon \kappa^+ \times \kappa \to \kappa$ given by $F(\alpha, \beta) = f_\alpha(\beta)$ verifies

$$\binom{\kappa^+}{\kappa} \nrightarrow \begin{bmatrix} \kappa^- \\ \kappa \end{bmatrix}_\kappa .$$

Prikry's paper was significant for several for several reasons. Not only did it provide the first example of a consistency proof, rather than outright derivation, of a result in the Erdös–Rado partition calculus, but it was the first instance of a recurring phenomenon: a combinatorial principle is formulated to isolate salient features of a particular construction, and is first shown consistent by forcing—then specialists in L establish that it holds there, using the full structure of a morass.

All this was described in Kanamori [15], but Donder pointed out that (R_κ) and (P_κ) are actually equivalent if we use the following typically perspicuous lemma of Kunen [16]:

LEMMA 3.1. *For any regular λ, there are injections $i_\alpha\colon \alpha \to \lambda$ for every $\alpha < \lambda^+$ such that $\alpha < \beta < \lambda^+$ implies that $|\{\xi < \alpha \mid i_\alpha(\xi) \neq i_\beta(\xi)\}| < \lambda$.*

The lemma is proved by a straightforward inductive construction, and Kunen used it to provide a short proof of the Specker result that $\lambda^{<\lambda} = \lambda$ implies that there is a λ^+-Aronszajn tree (since $\langle \{i_\alpha \restriction \xi \mid \xi < \alpha < \lambda^+\}, \subseteq \rangle$ is such a tree).

THEOREM 3.2. (Donder). *The principles (R_κ) and (P_κ) are equivalent.*

Proof. If $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ is as provided by (P_κ) and $\{i_\alpha \mid \alpha < \kappa^+\}$ is as provided by Kunen's lemma with $\lambda = \kappa$, then $g_\alpha = f_\alpha \cdot i_\alpha$ satisfies (R'_κ): If $s \in [\kappa^+]^{\kappa-}$ and $\phi\colon s \to \kappa$, there is first of all a $t \in [\bigcap s]^{<\kappa}$ such that $\xi \notin t$ implies that $i_\alpha(\xi) = i_\beta(\xi)$ whenever $\alpha, \beta \in s$. Thus, $\{\xi < \bigcap s \mid \forall \alpha \in s(g_\alpha(\xi) \neq \phi(\alpha))\} \subseteq t \cup i_\eta^{-1}(\{\rho < \kappa \mid \forall \alpha \in s(f_\alpha(\rho) \neq \phi(\alpha))\})$, where η is any member of s, and this last set has cardinality less than κ by (P_κ).

This result simplifies the chart of implications at the end of Section 3 in Kanamori [15]. The partition relations corresponding to (R_κ) and (P_κ) are equivalent to the weaker versions of these principles where the $\phi\colon s \to \kappa$ only range over the constant functions. Thus, we can analogously prove:

THEOREM 3.3. $\binom{\kappa^+}{\kappa} \nrightarrow \left[\begin{smallmatrix}\kappa^- \\ \kappa\end{smallmatrix}\right]_\kappa$ is equivalent to $\kappa^+ \to [\kappa : \kappa^-]^2_\kappa$.

Incidentally, Kunen's lemma has another notable application. The following proposition may be called Weak Kurepa's Hypothesis for κ:

There is a collection $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq {}^\kappa\kappa^-$ such that $\alpha < \beta < \kappa^+$ implies $|\{\xi \mid f_\alpha(\xi) = f_\beta(\xi)\}| < \kappa$. (WKH_κ)

Hypothesis (WKH_κ) is a simple consequence of the well-known Kurepa hypothesis for κ, and so if $V=L$, then (WKH_κ) holds for every successor cardinal κ.

THEOREM 3.4. *The following are equivalent*:

(i) (WKH_κ).

(ii) $\binom{\kappa^+}{\kappa} \not\rightarrow \left[\begin{smallmatrix}2\\ \kappa\end{smallmatrix}\right]_{\kappa^-}$.

(iii) $\kappa^+ \not\rightarrow (\kappa : 2)^2_{\kappa^-}$.

(iv) *There is a collection* $\{f_\alpha \mid \alpha < \kappa^+\}$ *such that* $f_\alpha : \alpha \rightarrow \kappa^-$ *and* $\alpha < \beta < \kappa^+$ *implies* $|\{\xi < \alpha \mid f_\alpha(\xi) = f_\beta(\xi)\}| < \kappa$.

Proof. (i)$\leftrightarrow$(ii) and (iii)$\leftrightarrow$(iv) are the usual translations, and (iv)$\leftrightarrow$(i) is clear. (i)$\leftrightarrow$(iv) uses Kunen's lemma as before.

Thus, for example, if (iii) failed then κ^+ is inaccessible in L, since this is a well-known consequence of the failure of Kurepa's hypothesis for κ. As noted in Section 1, (iii) is a best possible negative result in an appropriate sense. Although strictly speaking, it is not comparable with the negative square-bracket relations which are our main preoccupation, the latter seem to have a more formidable content. Galvin and Gray mentioned (iv) to the author; Galvin also observed that if (iv) for $\kappa = \omega_1$ and (MA_{ω_2}) holds, then:

> There is a collection $\{f_\alpha \mid \alpha < \omega_2\}$ such that $f_\alpha : \alpha \rightarrow \omega$ and $\alpha < \beta < \omega_2$ implies that $\{\xi \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite.

The point is that this is equivalent to $\omega_2 \not\rightarrow (\omega : 2)^2_\omega$, which in turn violates the Continuum Hypothesis by the well-known Erdös–Rado partition theorem.

Let us now turn to the extent of the elaboration of the structure of the constructible universe needed to establish (R_κ) and (P_κ). Jensen established that if $V=L$, then (P_κ) holds for every successor κ, using the morass structure that he invented, and we now see from Theorem 3.2 that some of the technicalities of Rebholz [23] could have been avoided. Since then, there has been an ongoing investigation of morasses, which has sharpened the focus. In Devlin's notes [5], gap-1 morasses are defined as structures which satisfy the eight axioms (M0)–(M7), and a rather complicated example is constructed in L. Those structures satisfying only (M0)–(M5) have become known as coarse morasses. Donder [6] observed that a natural example of a coarse morass can be defined easily in L, just using Skolem hulls and least parameters, and without invoking the fine structure theory of Jensen. He goes on to derive several combinatorial consequences. In fact, all the principles for successor cardinals discussed in Kanamori [15] can already be derived from this natural coarse morass, since the strongest such principle, Burgess' principle, can be easily shown to hold for the natural Kurepa tree that Donder associates with the coarse morass. In particular, (R_κ) and (P_κ) are thus entailed.

We now turn to generalizations of (R_κ) and (P_κ) where we no longer assume that κ is a successor cardinal. The main interest in these generalizations lies in the consequent limitative results in the partition calculus which counterpoint the positive results available from large cardinals. With a κ^- no longer necessarily available, we can only consider the following weaker versions of (R_κ) and (P_κ):

> There is a collection $\{f_\alpha \mid \alpha < \kappa^+\}$ of functions $f_\alpha : \alpha \to \kappa$, so that whenever $s \in [\kappa^+]^\kappa$ and $\phi : s \to \kappa$, then $|\{\xi < \bigcap s \mid \forall \alpha \in s (f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$. (wR_κ)
>
> There is a collection $\{f_\alpha \mid \alpha < \kappa^+\}$ of functions $f_\alpha : \kappa \to \kappa$, so that whenever $s \in [\kappa^+]^\kappa$ and $\phi : s \to \kappa$, then $|\{\xi < \kappa \mid \forall \alpha \in s (f_\alpha(\xi) \neq \phi(\alpha))\}| < \kappa$. (wP_κ)

The principle (wR_κ) is the analogue of (R'_κ) rather than (R_κ), which is inconsistent notationally with Kanamori [15], but this will be more convenient for our purposes. In direct analogy with previous results,

$$(\mathrm{wR}_\kappa) \text{ implies } \kappa^+ \not\to [\kappa : \kappa]^2_\kappa,$$

and

$$(\mathrm{wP}_\kappa) \text{ implies } \binom{\kappa^+}{\kappa} \not\to \begin{bmatrix} \kappa \\ \kappa \end{bmatrix}_\kappa.$$

However, it is not clear that (wR_κ) is equivalent to (wP_κ), as the argument of Theorem 3.2 no longer works.

We shall first discuss the consistency of (wR_κ) via forcing, since it is typically easier to see as a generic overlay over a ground model rather than a direct construction assuming structural hypotheses. Then, recent results of Velleman and Donder are cited at the end of this section in connection with the axiom of constructibility. Indeed, the impetus for their work was to handle this kind of forcing in the author's papers, one featuring a new and rather elegant density argument first discovered by Shelah and later independently by the author. See [13] for the corresponding result on (wP_κ); the present argument incorporates an important use of Lemma 3.1.

THEOREM 3.5. *If the ground model V satisfies $\kappa^{<\kappa} = \kappa$, then there is a $\kappa^+ -$c.c. forcing extension in which (wR_κ) holds. Moreover, this forcing adds no new η sequences of ordinals for any $\eta < \kappa$; also, properties like the Mahloness of κ are preserved.*

Proof. For the duration of the proof, fix a collection of functions $\{i_\alpha \mid \alpha < \kappa^+\}$ satisfying Kunen's Lemma 3.1 with λ replaced by κ; i.e., for every $\alpha < \kappa^+$, $i_\alpha : \alpha \to \kappa$ is an injection, and $\alpha < \beta < \kappa^+$ implies $|\{\xi < \alpha \mid i_\alpha(\xi) \neq i_\beta(\xi)\}| < \kappa$.

Our forcing conditions will need to carry a strong side condition, embodied in the following concept: If $S \subseteq \{\langle s, \phi\rangle \mid s \in [\kappa^+]^{<\kappa} \;\&\; \phi : s \to \kappa\}$, then $h : \bigcup\{s \mid \langle s, \phi\rangle \in S\} \to \kappa$ is a *consistent map for S iff* for every $\langle s, \phi\rangle \in S$, there are infinitely many $\alpha \in s$ so that $h(\alpha) = \phi(\alpha)$. With this in mind, let us formulate the forcing notion Q_κ as consisting of pairs $\langle F, S\rangle$, where:

(a) F is a function: domain$(F) \to \kappa$ with $|F| < \kappa$, and there are $a_F \in [\kappa^+]^{<\kappa}$ and $\gamma_F < \kappa$ such that domain$(F) = \{\langle \alpha, i_\alpha^{-1}(\delta)\rangle \mid \alpha \in a_F \;\&\; \delta < \gamma_F \;\&\; \delta \in \text{Range}(i_\alpha)\}$.

(b) $S \subseteq \{\langle s, \phi\rangle \mid s \in [\kappa^+]^{<\kappa} \;\&\; \phi : s \to \kappa\}$ with $|S| < \kappa$ possessing a consistent map.

For $\langle F, S\rangle$, $\langle G, T\rangle \in Q_\kappa$, define $\langle G, T\rangle \leqslant \langle F, S\rangle$ *iff*:

(i) $G \supseteq F$ and $T \supseteq S$.

(ii) If $\delta \in \gamma_G - \gamma_F$ and $\langle s, \phi\rangle \in S$ is such that $\delta \in \text{range}(i_\alpha)$ for every $\alpha \in s$, then there is an $\alpha \in s$ such that: $\alpha \in a_G$ and $G(\alpha, i_\alpha^{-1}(\delta)) = \phi(\alpha)$.

(iii) Any consistent map for S can be extended to a consistent map for T.

Intuitively, F is a less than κ size approximation to a witness for (wR_κ), and S records the conditions that must henceforth be met by any extension of F. The particular way in which they must be met though the vehicle of the i_α's, as specified in (ii), will insure that amalgamations are possible in the coming argument for the κ^+ – c.c. Part (iii) is an important feature, which insures that the notion of forcing is *$<\kappa$-continuously closed*, i.e., if $\eta < \kappa$, $\alpha < \beta < \eta$ implies $\langle F_\beta, S_\beta\rangle \leqslant \langle F_\alpha, S_\alpha\rangle$, and $\langle F_\delta, S_\delta\rangle = \langle \bigcup_{\alpha<\delta} F_\alpha, \bigcup_{\alpha<\delta} S_\alpha\rangle$ for limit ordinals $\delta < \eta$, then there is a common extension, namely $\langle \bigcup_{\alpha<\eta} F_\alpha, \bigcup_{\alpha<\eta} S_\alpha\rangle \in Q_\kappa$. Thus, this notion of forcing does not add any new η sequences of ordinals for any $\eta < \kappa$, and, for example, preserves the Mahloness of κ by standard arguments. Also, for any $a \in [\kappa^+]^{<\kappa}$ and $\gamma < \kappa$, $\{\langle F, S\rangle \in Q_\kappa \mid a_F \supseteq a \;\&\; \gamma_F \geqslant \gamma\}$ is dense. (To see this, note first that given any $\langle G, T\rangle \in Q_\kappa$, a consistent map for T can be used to provide a $\langle F, T\rangle \leqslant \langle G, T\rangle$ such that $\gamma_F = \gamma_G + 1$. Thus, one-step extensions are always possible, and the rest follows from $<\kappa$-continuous closure.)

If $\mathscr{G}$ is any generic filter over V, define $\{f_\alpha^{\mathscr{G}} \mid \alpha < \kappa^+\}$ in $V[\mathscr{G}]$ by: $f_\alpha^{\mathscr{G}}(\xi) = \beta$ *iff* $\xi < \alpha \;\&\; \exists \langle F, S\rangle \in \mathscr{G}(F(\alpha, \xi) = \beta)$. The next task is to verify that this collection of functions satisfies (wR_κ). In fact, we can establish the following:

Whenever $t \in [\kappa^+]^\kappa \cap V[\mathscr{G}]$ and $\psi \in {}^t\kappa \cap V[\mathscr{G}]$, there is an initial segment s of t with $|s| < \kappa$ such that $|\{\xi < \kappa \mid \forall \alpha \in s(f_\alpha(\xi) \neq \psi(\alpha))\}| < \kappa$. (†)

This would more than suffice.

So, suppose that $\langle F, S\rangle \Vdash \dot{t} \in [\kappa^+]^\kappa \,\&\, \dot{\psi}\colon \dot{t} \to \kappa$. By induction, construct conditions $\langle F_n, S_n\rangle$, ordinals α_n, sets $t_n \in [\kappa^+]^{<\kappa}$, and functions $\psi_n\colon t_n \to \kappa$ as follows: Set $\langle F_0, S_0\rangle = \langle F, S\rangle$. Given $\langle F_n, S_n\rangle$, since $\dot{t}$ is forced to have cardinality κ and Q_κ is sufficiently closed, produce a condition $\langle F_{n+1}, S_{n+1}\rangle \leqslant \langle F_n, S_n\rangle$, an ordinal α_n, a set $t_n \in [\kappa^+]^{<\kappa}$, and a function $\psi_n\colon t_n \to \kappa$ such that:

$$\langle F_{n+1}, S_{n+1}\rangle \Vdash \dot{t} \cap \alpha_n = t_n \,\&\, \dot{\psi} \restriction t_n = \psi_n \,\&\, \alpha_n \in \dot{t} - \bigcup \{s \mid \langle s, \phi\rangle \in S_n\}.$$

By taking a trivial extension if necessary, we can assume that $\alpha_n \in \bigcup \{s \mid \langle s, \phi\rangle \in S_{n+1}\}$.

Finally, set $\langle G, T\rangle = \langle \bigcup F_n, \bigcup S_n\rangle$, $\beta = \sup \alpha_n$, $s = \bigcup t_n$, and $\phi = \bigcup \psi_n$. There is now a *Claim*: $\langle G, T \cup \{\langle s, \phi\rangle\}\rangle$ is a condition extending $\langle F_n, S_n\rangle$ for every n (but not necessarily extending $\langle G, T\rangle$). Since it would then be the case that $\langle G, T \cup \{\langle s, \phi\rangle\}\rangle \Vdash \dot{t} \cap \beta = s \,\&\, \dot{\psi} \restriction s = \phi$, this would suffice to establish (†): If by density $\langle G, T \cup \{\langle s, \phi\rangle\}\rangle \in \mathscr{G}$, then $\{\xi < \bigcap s \mid \forall \alpha \in s(f_\alpha(\xi) \neq \phi(\alpha))\}$ is contained in the less than κ size set $\{\xi < \bigcap s \mid \exists \alpha, \beta \in s(i_\alpha(\xi) < \gamma_G$ or $i_\alpha(\xi) \neq i_\beta(\xi))\}$, since if for every $\alpha \in s$, $i_\alpha(\xi) =$ a fixed $\delta \geqslant \gamma_G$, then by condition (ii) in the definition of the forcing partial order, there would be an $\alpha \in s$ such that $f_\alpha^{\mathscr{G}}(\xi) = \phi(\alpha)$.

To establish the Claim, it is necessary to show that for any n and any consistent map h for S_n, h can be extended to a consistent map for $T \cup \{\langle s, \phi\rangle\}$. So, fix such an n and h, and define consistent maps h_i for S_{n+i} by induction on $i \in \omega$ as follows: Set $h_0 = h$. Given h_i, since $\langle F_{n+i+1}, S_{n+i+1}\rangle \leqslant \langle F_{n+i}, S_{n+i}\rangle$, let $g_{i+1} \supseteq h_i$ be a consistent map for S_{n+i+1}. Remember that $\alpha_{n+i} \in \bigcup \{s \mid \langle s, \phi\rangle \in S_{n+i+1}\} - \bigcup \{s \mid \langle s, \phi\rangle \in S_{n+i}\}$, define h_{i+1} by

$$\begin{aligned} h_{i+1}(\xi) &= g_{i+1}(\xi) && \text{if} \quad \xi \neq \alpha_{n+i} \\ &= \phi(\alpha_{n+i}) && \text{if} \quad \xi = \alpha_{n+i}. \end{aligned}$$

Clearly $h_{i+1} \supseteq h_i$ is again a consistent map for S_{n+i+1} since only one value was changed. Finally, set $\bar{h} = \bigcup h_i$, so that $\bar{h}$ is a consistent map for $T = \bigcup_{j \in \omega} S_j$. Moreover, for each $i \in \omega$ we have $\bar{h}(\alpha_{n+i}) = \phi(\alpha_{n+i})$, so that $\bar{h}$ is actually a consistent map for $T \cup \{\langle s, \phi\rangle\}$. This establishes the Claim.

All that remains is to establish the κ^+ – c.c. for Q_κ. So, suppose that $\{\langle F_\alpha, S_\alpha\rangle \mid \alpha<\kappa^+\}\subseteq Q_\kappa$. Standard Δ-system arguments appealing to $\kappa^{<\kappa}=\kappa$ establish that there is a $W\in[\kappa^+]^{\kappa^+}$, a $\gamma<\kappa$, and a set $z\in[\kappa^+]^{<\kappa}$ such that:

(1) $\alpha\in W$ implies $\gamma_{F_\alpha}=\gamma$.

(2) $\alpha\neq\beta\in W$ implies $a_{F_\alpha}\cap a_{F_\beta}=z$.

(3) If $A=\{\langle\sigma, i_\sigma^{-1}(\delta)\mid\sigma\in z \;\&\; \delta<\gamma \;\&\; \delta\in\text{range}(i_\sigma)\}$, then $\alpha,\beta\in W$ implies $F_\alpha\restriction A=F_\beta\restriction A$.

Thus, for any $\alpha,\beta\in W$, $F_\alpha\cup F_\beta$ is still a function. It was the need to have something like (1) above to hold that the i_α's were introduced and condition (ii) in the definition of the forcing partial order was formulated.

To take care of the S_α's, first find $X\in[W]^{\kappa^+}$ and a $\mu<\kappa$ such that $\alpha\in X$ implies $|S_\alpha|=\mu$. For such α, write $S_\alpha=\{\langle s_\xi^\alpha,\phi_\xi^\alpha\rangle\mid\xi<\mu\}$. By a further Δ-system argument using $\kappa^{<\kappa}=\kappa$, one can find $Y\in[X]^{\kappa^+}$ and a T such that:

(4) $\alpha\neq\beta\in Y$ implies $\bigcup\{s\mid\langle s,\phi\rangle\in S_\alpha\}\cap\{s\mid\langle s,\phi\rangle\in S_\beta\}=T$.

(5) $\alpha,\beta\in Y$ implies $\langle\langle s_\xi^\alpha\cap T,\phi_\xi^\alpha\restriction T\rangle\mid\xi<\mu\rangle=\langle\langle s_\xi^\beta\cap T,\phi_\xi^\beta\restriction T\rangle\mid\xi<\mu\rangle$.

For $\alpha\in Y$, write $T_\alpha=\bigcup\{s\mid\langle s,\phi\rangle\in S_\alpha\}-T$. By $\kappa^{<\kappa}=\kappa$, there are at most κ structures $\langle\rho,<,A_\xi\rangle_{\xi<\mu}$, where $\rho<\kappa$ and the A_ξ's are unary predicates. Each $M_\alpha=\langle T_\alpha,<,s_\xi^\alpha\cap T_\alpha\rangle_{\xi<\mu}$ when transitized is isomorphic to one of these, so by cardinality considerations there is a $Z\in[Y]^{\kappa^+}$ such that:

(6) $\alpha,\beta\in Z$ implies there is an isomorphism $\pi_{\alpha\beta}:M_\alpha\to M_\beta$.

(7) $\phi_\xi^\alpha(\delta)=\phi_\xi^\beta(\pi_{\alpha\beta}(\delta))$ for $\delta\in s_\xi^\alpha\cap T_\alpha$.

It is now claimed that if $\alpha,\beta\in Z$, there $\langle F_\alpha\cup F_\beta, S_\alpha\cup S_\beta\rangle$ is a condition extending both $\langle F_\alpha,S_\alpha\rangle$ and $\langle F_\beta,S_\beta\rangle$, thereby completing the proof. Recall that $F_\alpha\cup F_\beta$ is a function, and moreover that $\gamma_{(F_\alpha\cup F_\beta)}=\gamma=\gamma_{F_\alpha}=\gamma_{F_\beta}$. Thus, it suffices mutatis mutandis to show that if h is a consistent map for S_α, then h can be extended to a consistent map for $S_\alpha\cup S_\beta$. Let $\pi_{\alpha\beta}:M_\alpha\to M_\beta$ be as in (6) and (7). Then it is straightforward using (4) and (5) that if $\bar h$ is defined by

$$\begin{aligned}\bar h(\xi)&=h(\xi) &&\text{if}\quad \xi\in\bigcup\{s\mid\langle s,\phi\rangle\in S_\alpha\}\\ &=h(\pi_{\alpha\beta}^{-1}(\xi)) &&\text{if}\quad \xi\in T_\beta,\end{aligned}$$

then $\bar h$ is a consistent map for $S_\alpha\cup S_\beta$. This completes the proof of the theorem.

Let us finally turn to the consequences of the axiom of constructibility. Velleman [28] and Shelah and Stanley [25] independently provided "black-box" approaches to Jensen's gap-1 morass by establishing an equivalence with a Martin axiom-type forcing principle. Thus, they provided a transfer principle of sorts for transforming forcing consistency results into constructions in L. Velleman's treatment was more succinct, and eventually led him [29] to a surprisingly simple combinatorial formulation of gap–1 morasses. The Shelah–Stanley version, on the other hand, also had an amplification that was intended to handle partial orders like the Q_κ used in the proof of Theorem 3.5. Velleman, however, pointed out a shortcoming in this intended application, and furthermore formulated the strong notion of a morass with "linear" limits and "built-in $\diamond$", and provided a Martin axiom-type equivalence whose specifications do take care of Q_κ. Velleman conjectured that such morasses exist in L at non-weakly compact cardinals, and this was confirmed by Donder [7] with some necessarily intricate analysis of constructibility. Even then, it was not clear that the necessary collection of dense sets would be met by the generic object for Q_κ, so Donder provided yet a further amplification to derive $\kappa^+ \not\rightarrow [\kappa : \kappa]^2_\kappa$, still based on the ideas of Theorem 3.5. We thus have the following characterization, with most of the work due to Donder, which serves as a fitting conclusion to this paper:

THEOREM 3.6. *If $V=L$ and $\kappa>\omega$ is regular, then the following are equivalent*:

(i) *κ is weakly compact.*

(ii) *$\kappa^+ \rightarrow (\kappa : \eta)^2_2$ for every $\eta<\kappa^+$.*

(iii) *$\kappa^+ \rightarrow [\kappa : \kappa]^2_\kappa$.*

(i) $\rightarrow$ (ii) *is Theorem* 2.1; (ii) $\rightarrow$ (iii) *is immediate.*

As the referee has emphasized, it is also possible for a non-weakly compact, strongly inaccessible cardinal κ to carry a Laver ideal. In fact, it can be verified that the κ-saturated ideal constructed by Kunen [16a, Sect. 3] is a Laver ideal. Thus, the equivalences of Theorem 3.6 do not hold just in ZFC, not even for inaccessible cardinals.

REFERENCES

1. J. BAUMGARTNER AND A. HAJNAL, A proof (involving Martin's Axiom) of a partition relation, *Fund. Math.* **78** (1973), 193–203.
2. J. BAUMGARTNER, A. TAYLOR, AND S. WAGON, Ideals over uncountable sets, *Dissertationes Math.* (Rozprawy Mat.) **197** (1982).
3. T. CARLSON, Doctoral thesis, University of Minnesota, 1978.
4. G. CHUDNOSKY, Combinatorial properties of compact cardinals, *in* "Infinite and Finite Sets" (A. Hajnal, Ed.), Colloguia Mathematica Societatis Janos Bolyai 10, Vol. 1, pp. 289–306, North-Holland, Amsterdam, 1975.

5. K. DEVLIN, "Aspects of Constructibility," Lecture Notes in Mathematics Vol. 354, Springer-Verlag, Berlin/New York/Heidelberg, 1973.
6. H.-D. DONDER, Coarse morasses in L, *in* "Set Theory and Model Theory" (R. Jensen and A. Prestel, Eds.), Lecture Notes in Mathematics Vol. 872, Springer-Verlag, Berlin/New York/Heidelberg, 1981.
7. H.-D. DONDER, Another look at gap-1 morasses, manuscript, 1982.
8. H.-D. DONDER, Prikry's principle for ω_2 and $0^{\#}$, manuscript, 1982.
9. P. ERDÖS AND R. RADO, A problem on ordered sets, *J. London Math. Soc.* **28** (1953), 426–438.
10. P. ERDÖS AND R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62** (1956), 427–489.
11. F. GALVIN, On a partition theorem of Baumgartner and Hajnal, *in* "Infinite and Finite Sets," (A. Hajnal, Ed.), Colloquia Mathematica Societatis Janos Bolyai 10, Vol. 1 pp. 711–729, North-Holland, Amsterdam, 1975.
12. A. HAJNAL, Some results and problems in set theory, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 277–298.
13. A. KANAMORI, Morass-level combinatorial principles, *in* "Patras Logic Symposium" (G. Metakides, Ed.), North-Holland, Amsterdam, 1982.
14. A. KANAMORI, On Silver's and related principles, *in* "Logic Colloquium '80" (D. van Dalen, D. Lascar, and J. Smiley, Eds.), North-Holland, Amsterdam, 1982.
15. A. KANAMORI, Morasses in combinatorial set theory, *in* "Surveys in Set Theory" (A. R. D. Mathias, Ed.), London Math. Soc. Lecture Note Series Vol. 87, Cambridge Univ. Press, London/New York, 1983.
16. K. KUNEN, Combinatorics, *in* "Handbook of Mathematical Logic" (J. Barwise, Ed.), pp. 371–401, North-Holland, Amsterdam, 1977.
16a. K.KUNEN, Saturated ideals, *J. Symbolic Logic* **43** (1978), 65–76.
17. R. LAVER, Partition relations for uncountable cardinals $\leqslant 2^{\aleph_0}$, *in* "Infinite and Finite Sets" (A. Hajnal, Ed.), Colloquia Mathematica Societas Janos Bolyai 10, Vol. 1, pp. 1029–1042, North-Holland, Amsterdam, 1975.
18. R. LAVER, A saturation property on ideals, *Compositio Math.* **36** (3) (1978), 233–242.
19. R. LAVER, An $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1, *in* "Logic Colloquium '80" (D. van Dalen, D. Lascar, and J. Smiley, Eds.) pp. 173–180, North-Holland, Amsterdam, 1982.
20. E. MILNER AND R. RADO, The pigeon-hole principle for ordinal numbers, *Proc. London Math. Soc.* (3) **15** (1965), 750–768.
21. K. PRIKRY, On a set-theoretic partition problem, *Duke Math. J.* **39** (1972), 77–83.
22. K. PRIKRY, On a problem of Erdös, Hajnal, and Rado, *Discrete Math.* **2** (1972), 51–59.
23. J. REBHOLZ, Consequences of morass and diamond, *Ann. of Math. Logic* **7** (1974), 361–385.
24. S. SHELAH, Notes on combinatorial set theory, *Israel J. Math.* **14** (1973), 262–277.
25. S. SHELAH AND L. STANLEY, S-Forcing, L: A "black-box" theorem for morasses, with applications to super-Souslin trees, *Israel J. Math.* **43** (1982), 185–224.
26. S. TODORCEVIC, Forcing positive partition relations, *Trans. Amer. Math. Soc.* **280** (1983), 703–720.
27. S. TODORCEVIC, Partition relations for partially ordered sets, preprint.
28. D. VELLEMAN, Morasses, diamond, and forcing, *Ann. of Math. Logic* **23** (1983), 199–281.
29. D. VELLEMAN, Simplified morasses, *J. Symbolic Logic* **49** (1984), 1001–1021.
30. N. WILLIAMS, "Combinatorial Set Theory," North-Holland, Amsterdam, 1977.
31. K. WOLFSDORF, Der beweis eines satzes von G. Chudnovsky, *Arch. Math. Logik Grundlag.* **20** (1980), 161–171.