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0. Introduction

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Set theory has entered its prime as an advanced and autonomous research field of mathematics with broad foundational significance, and this Handbook with its expanse and variety amply attests to the fecundity and sophistication of the subject. Indeed, in set theory's further reaches one sees tremendous progress both in its continuing development of its historical heritage, the investigation of the transfinite numbers and of definable sets of reals, as well as its analysis of strong propositions and consistency strength in terms of large cardinal hypotheses and inner models.

This introduction provides a historical and organizational frame for both modern set theory and this Handbook, the chapter summaries at the end being a final elaboration. To the purpose of drawing in the serious, mathematically experienced reader and providing context for the prospective researcher, we initially recapitulate the consequential historical developments leading to modern set theory as a field of mathematics. In the process we affirm basic concepts and terminology, chart out the motivating issues and driving initiatives, and describe the salient features of the field's internal practices. As the narrative proceeds, there will be a natural inversion: Less and less will be said about more and more as one progresses from basic concepts to elaborate structures, from seminal proofs to complex argumentation, from individual moves to collective enterprise. We try to put matters in a succinct yet illuminating manner, but be that as it may, according to one's experience or interest one can skim the all too familiar or too obscure. To the historian this account would not properly be history—it is, rather, a deliberate arrangement, in significant part to lay the ground for the coming chapters. To the seasoned set theorist there may be issues of under-emphasis or over-emphasis, of omissions or commissions. In any case, we take refuge in a wise aphorism: If it's worth doing, it's worth doing badly.

1. Beginnings

1.1. Cantor

Set theory was born on that day in December 1873 when Georg Cantor (1845-1918) established that *the continuum is not countable*—there is no one-to-one correspondence between the real numbers and the natural numbers $0, 1, 2, \ldots$. Given a (countable) sequence of reals, Cantor defined nested intervals so that any real in their intersection will not be in the sequence. In the course of his earlier investigations of trigonometric series Cantor had developed a definition of the reals and had begun to entertain infinite totalities of reals for their own sake. Now with his uncountability result Cantor embarked on a full-fledged investigation that would initiate an expansion of the very concept of number. Articulating cardinality as based on bijection (one-to-one correspondence) Cantor soon established positive results about the existence of bijections between sets of reals, subsets of the plane, and the like. By 1878 his investigations had led him to assert that there are only two infinite cardinalities embedded in the continuum: *Every infinite set of reals is either countable or in bijective correspondence with all the reals.* This was the Continuum Hypothesis (CH) in its nascent context, and the *continuum problem*, to resolve this hypothesis, would become a major motivation for Cantor's large-scale investigations of infinite numbers and sets.

In his magisterial *Grundlagen* of 1883 Cantor developed the *transfinite numbers* and the key concept of *well-ordering*, in large part to take a new, structured approach to infinite cardinality. The transfinite numbers follow the natural numbers $0, 1, 2, \ldots$ and have come to be depicted in his later notation in terms of natural extensions of arithmetical operations:

$$\omega, \omega+1, \omega+2, \ldots \omega+\omega (= \omega{\cdot}2), \\ \ldots \omega{\cdot}3, \ldots \omega{\cdot}\omega (= \omega^2), \ldots \omega^3, \ldots \omega^\omega, \ldots \omega^{\omega^\omega}, \ldots$$

A *well-ordering* on a set is a linear ordering of it according to which every non-empty subset has a least element. Well-orderings were to carry the sense of sequential counting, and the transfinite numbers served as standards for gauging well-orderings. Cantor developed cardinality by grouping his transfinite numbers into successive number classes, two numbers being in the same class if there is a bijection between them. Cantor then propounded a basic principle: "It is always possible to bring any *well-defined* set into the form of a *well-ordered* set." Sets are to be well-ordered, and they and their cardinalities are to be gauged via the transfinite numbers of his structured conception of the infinite.

The transfinite numbers provided the framework for Cantor's two approaches to the continuum problem, one through cardinality and the other through definable sets of reals, these each to initiate vast research programs. As for the first, Cantor in the *Grundlagen* established results that reduced the continuum problem to showing that the continuum and the countable transfinite numbers have a bijection between them. However, despite several announcements Cantor could never develop a workable correlation, an emerging problem being that he could not *define* a well-ordering of the reals.

As for the approach through definable sets of reals, Cantor formulated the key concept of a *perfect* set of reals (non-empty, closed, and containing no isolated points), observed that perfect sets of reals *are* in bijective correspondence with the continuum, and showed that every closed set of reals is either countable or else have a perfect subset. Thus, Cantor showed that "CH holds for closed sets". The *perfect set property*, being either countable or else having a perfect subset, would become a focal property as more and more definable sets of reals came under purview.

Almost two decades after his initial 1873 result, Cantor in 1891 subsumed it through his celebrated *diagonal* argument. In logical terms this argument turns on the use of the validity $\neg\exists y\forall x(Pxx \longleftrightarrow \neg Pyx)$ for binary predicates P parametrizing unary predicates and became, of course, fundamental to the development of mathematical logic. Cantor stated his new, general result in terms of functions, ushering in totalities of arbitrary functions into mathematics, but his result is cast today in terms of the power set $P(x) = \{y \mid y \subseteq x\}$ of a set x: *For any set x, $P(x)$ has a larger cardinality than x.* Cantor had been extending his notion of set to a level of abstraction beyond sets of reals and the like; this new result showed for the first time that there is a set of a larger cardinality than that of the continuum.

Cantor's *Beiträge* of 1895 and 1897 presented his mature theory of the transfinite, incorporating his concepts of *ordinal number* and *cardinal number* The former are the transfinite numbers now reconstrued as the "order-types" of well-orderings. As for the latter, Cantor defined the addition, multiplication, and exponentiation of cardinal numbers primordially in terms of set-theoretic operations and functions. Salient was the incorporation of "all" possibilities in the definition of exponentiation: If $\mathfrak{a}$ is the cardinal number of A and $\mathfrak{b}$ is the cardinal number of B then $\mathfrak{a}^{\mathfrak{b}}$ is the cardinal number of the totality, nowadays denoted ${}^{B}A$, of all functions from B into A. As befits the introduction of new numbers Cantor introduced a new notation, one using the Hebrew letter aleph, $\aleph$. $\aleph_0$ is to be the cardinal number of the natural numbers and the successive alephs

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$$

indexed by the ordinal numbers are now to be the cardinal numbers of the successive number classes from the *Grundlagen* and thus to exhaust all the infinite cardinal numbers. Cantor pointed out that the exponentiated $2^{\aleph_0}$ is the cardinal number of the continuum, so that CH could now have been stated as

$$2^{\aleph_0} = \aleph_1 .$$

However, with CH unresolved Cantor did not even mention the hypothesis in the *Grundlagen*, only in correspondence. Every well-ordered set has an aleph as its cardinal number, but where is $2^{\aleph_0}$ in the aleph sequence?

Cantor's great achievement, accomplished through almost three decades of prodigious effort, was to have brought into being the new subject of set theory as bolstered by the mathematical objectification of the actual infinite and moreover to have articulated a fundamental problem, the continuum problem. Hilbert made this the very first of his famous problems for the 20th Century, and he drew out Cantor's difficulty by suggesting the desirability of "actually giving" a well-ordering of the real numbers.

1.2. Zermelo

Ernst Zermelo (1871-1953), already estimable as an applied mathematician, turned to set theory at Göttingen under the influence of Hilbert. Zermelo analyzed Cantor's well-ordering principle by reducing it to the Axiom of Choice (AC), the abstract existence assertion that every set x has a *choice function*, i.e. a function f with domain x such that for every non-empty $y \in x$, $f(y) \in y$. Zermelo's 1904 proof of the Well-Ordering Theorem, that with AC *every set can be well-ordered*, would anticipate the argument two decades later for transfinite recursion:

With x a set to be well-ordered, let f be a choice function on the power set $P(x)$. Call $y \subseteq x$ an *f-set* if there is a well-ordering R of y such that for any $a \in y$, $a = f(\{b \in x \mid b \text{ does not } R\text{-precede } a\})$. The well-orderings of f-sets are thus determined by f, and f-sets cohere. It follows that the union of f-sets is again an f-set and must in fact be x itself.

Zermelo's argument provoked open controversy because of its appeal to AC, and the subsequent tilting toward the acceptance of AC amounted to a conceptual shift in mathematics toward arbitrary functions and abstract existence principles. Responding to his critics Zermelo in 1908 published a second proof of the Well-Ordering Theorem and then the first full-fledged axiomatization of set theory, one similar in approach to Hilbert's axiomatization of geometry and incorporating set-theoretic ideas of Richard Dedekind. This axiomatization duly avoided the emerging "paradoxes" like Russell's Paradox, which Zermelo had come to independently, and served to buttress the Well-Ordering Theorem by making explicit its underlying set-existence assumptions. Zermelo's axioms, now formalized, constitute the familiar theory Z, *Zermelo set theory*:

Extensionality (sets are equal if they contain the same members), Empty Set (there is a set having no members), Pairs (for any sets x and y there is a set $\{x, y\}$ consisting exactly of x and y), Union (for any set x there is a set $\bigcup x$ consisting exactly of those sets that are members of some member of x), Power Set (for any set x there is a set $P(x)$ consisting exactly of the subsets of x), Choice (for any set x consisting of non-empty, pairwise disjoint sets, there is a set c such that every member of x has exactly one member in c), Infinity (there is a certain, specified infinite set); and Separation (for any set x and "definite" property P, there is a set consisting exactly of those members of x having the property P).

Extensionality, Empty Set, and Pairs lay the basis for sets. Infinity and Power Set ensure sufficiently rich settings for set-theoretic constructions. Power Set legitimizes "all" for subsets of a given set, and Separation legitimizes "all" for elements of a given set satisfying a property. Finally, Union and Choice (formulated reductively in terms of the existence of a "transversal" set meeting each of a family of sets in one member) complete the encasing of the Well-Ordering Theorem.

Zermelo's axiomatization sought to clarify vague subject matter, and like strangers in a strange land, stalwarts developed a familiarity with sets guided hand-in-hand by the axiomatic framework. Zermelo's own papers, with work of Dedekind as an antecedent, pioneered the reduction of mathematical concepts and arguments to set-theoretic concepts and arguments from axioms. Zermelo's analysis moreover served to draw out what would come to be generally regarded as set-theoretic and combinatorial out of the presumptively logical, with Infinity and Power Set salient and the process being strategically advanced by the segregation of the notion of property to the Separation axioms.

Taken together, Zermelo's work in the first decade of the 20th Century initiated a major transmutation of the notion of set after Cantor. With AC Zermelo shifted the notion away from Cantor's inherently well-ordered sets, and with his axiomatization Zermelo ushered in a new abstract, prescriptive view of sets as structured solely by membership and governed and generated by axioms. Through his set-theoretic reductionism Zermelo made evident how his set theory is adequate as a basis for mathematics.

1.3. First Developments

During this period Cantor's two main legacies, the extension of number into the transfinite and the investigation of definable sets of reals, became fully incorporated into mathematics in direct initiatives. The axiomatic tradition would be complemented by another, one that would draw its life more directly from the mathematics.

The French analysts Emile Borel, René Baire, and Henri Lebesgue took on the investigation of definable sets of reals in what would be a typically "constructive" approach. Cantor had established the perfect set property for closed sets and formulated the concept of *content* for a set of reals, but he did not pursue these matters. With these as antecedents the French work would lay the basis for measure theory as well as *descriptive set theory*, the definability theory of the continuum.

Borel, already in 1898, developed a theory of *measure* for sets of reals; the formulation was axiomatic, and at this early stage bold and imaginative. The sets measurable according to his measure are the now well-known *Borel sets*. Starting with the open intervals (a, b) of reals assigned measure $b - a$, the Borel sets result when closing off under complements and countable unions, measures assigned in a corresponding manner.

Baire in his 1899 thesis classified those real functions obtainable by starting with the continuous functions and closing off under pointwise limits—the *Baire functions*—into classes indexed by the countable ordinal numbers, providing the first transfinite hierarchy after Cantor. Baire's thesis also introduced the now basic concept of *category.* A set of reals is *nowhere dense iff* its closure under limits includes no open set, and a set of reals is *meager* (or *of first category*) *iff* it is a countable union of nowhere dense sets—otherwise, it is *of second category.* Generalizing Cantor's 1873 argument, Baire established the Baire Category Theorem: *Every non-empty open set of reals is of second category.* His work also suggested a basic property: A set of reals A has the *Baire property iff* there is an open set O such that the symmetric difference $(A-O)\cup(O-A)$ is meager. Straightforward arguments show that every Borel set has the Baire property.

Lebesgue's 1902 thesis is fundamental for modern integration theory as the source of his concept of measurability. Lebesgue's concept of measurable set subsumed the Borel sets, and his analytic definition of measurable function subsumed the Baire functions. In simple terms, any *arbitrary* subset of a Borel measure zero set is a Lebesgue measure zero, or *null*, set, and a set is *Lebesgue measurable* if it is the union of a Borel set and a null set, in which case the measure assigned is that of the Borel set. It is this "completion" of Borel measure through the introduction of arbitrary subsets which gives Lebesgue measure its complexity and applicability and draws in wider issues of constructivity and set theory. Lebesgue's subsequent 1905 paper was the seminal paper of descriptive set theory: He correlated the Borel sets with the Baire functions, thereby providing a transfinite hierarchy for the Borel sets, and then applied Cantor's diagonalization argument to show both that this hierarchy is proper (new sets appear at each level) and that there is a Lebesgue measurable set which is not Borel.

As descriptive set theory was to develop, a major concern became the extent of the *regularity properties*, those indicative of well-behaved sets of reals, of which prominent examples were Lebesgue measurability, having the Baire property, and having the perfect set property. Significantly, the context was delimited by early explicit uses of AC in the role of providing a well-ordering of the reals: In 1905 Giuseppe Vitali established that there is a non-Lebesgue measurable set, and in 1908 Felix Bernstein established that there is a set without the perfect set property. Thus, Cantor's early contention that the reals are well-orderable precluded the universality of his own perfect set property, and it would be that his new, enumerative approach to the continuum would steadily provide focal examples and counterexamples.

The other, more primal Cantorian legacy, the extension of number into the transfinite, was considerably advanced by Felix Hausdorff, whose work was first to suggest the rich possibilities for a mathematical investigation of the uncountable. A mathematician *par excellence*, he took that sort of mathematical approach to set theory and extensional, set-theoretic approach to mathematics that would come to dominate in the years to come. In a

1908 paper, Hausdorff provided an elegant analysis of scattered linear orders (those having no dense sub-ordering) in a transfinite hierarchy. He first stated the Generalized Continuum Hypothesis (GCH)

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for every } \alpha\,.$$

He emphasized cofinality (the *cofinality* $\mathrm{cf}(\kappa)$ of a cardinal number κ is the least cardinal number λ such that a set of cardinality κ is a union of λ sets each of cardinality less than κ) and the distinction between *singular* ($\mathrm{cf}(\kappa) < \kappa$) and *regular* ($\mathrm{cf}(\kappa) = \kappa$) cardinals. And for the first time broached a "large cardinal" concept, a regular limit cardinal $> \aleph_0$. Hausdorff's work around this time on sets of real functions ordered under eventual domination and having no uncountable "gaps" led to the first plausible mathematical proposition that entailed the denial of CH.

Hausdorff's 1914 text, *Grundzüge der Mengenlehre*, broke the ground for a generation of mathematicians in both set theory and topology. Early on, he defined an ordered pair of sets in terms of (unordered) pairs, formulated functions in terms of ordered pairs, and ordering relations as collections of ordered pairs. He in effect capped efforts of logicians by making these moves in mathematics, completing the set-theoretic reduction of relations and functions. He then presented Cantor's and Zermelo's work systematically, and of particular interest, he used a well-ordering of the reals to provide what is now known as Hausdorff's Paradox. The source of the later and better known Banach-Tarski Paradox, Hausdorff's Paradox provided an implausible decomposition of the sphere and was the first, and a dramatic, synthesis of classical mathematics and the new Zermelian abstract view.

A decade after Lebesgue's seminal 1905 paper, descriptive set theory came into being as a distinct discipline through the efforts of the Russian mathematician Nikolai Luzin. He had become acquainted with the work of the French analysts while in Paris as a student, and in Moscow he began a formative seminar, a major topic of which was the "descriptive theory of functions". The young Pole Wacław Sierpiński was an early participant while he was interned in Moscow in 1915, and undoubtedly this not only kindled the decade-long collaboration between Luzin and Sierpiński but also encouraged the latter's involvement in the development of a Polish school of mathematics and its interest in descriptive set theory. In an early success, Luzin's student Pavel Aleksandrov (and independently, Hausdorff) established the groundbreaking result that *the Borel sets have the perfect set property*, so that "CH holds for the Borel sets".

In the work that really began descriptive set theory, another student of Luzin's, Mikhail Suslin, investigated the *analytic sets* after finding a mistake in Lebesgue's paper. In a brief 1917 note Suslin formulated these sets in terms of an explicit operation $\mathcal{A}$ drawn from Aleksandrov's work and announced two fundamental results: *a set B of reals is Borel iff both B and its complement $\mathbb{R}-B$ are analytic;* and *there is an analytic set which is not Borel.* This was to be his sole publication, for he succumbed to typhus in a Moscow epidemic in

1919 at the age of 25. In an accompanying note Luzin announced that *every analytic set is Lebesgue measurable and has the perfect set property*, the latter result attributed to Suslin. Luzin and Sierpiński in joint papers soon provided proofs, in work that shifted the emphasis to the *co-analytic sets*, complements of analytic sets, and provided for them a basic *tree representation* based on well-foundedness (having no infinite branches) from which the main results of the period flowed.

After this first wave in descriptive set theory had crested, Luzin and Sierpiński in 1925 extended the domain of study to the *projective sets*. For $Y \subseteq \mathbb{R}^{k+1}$, the *projection of* Y is $pY = \{\langle x_1, ..., x_k\rangle \mid \exists y(\langle x_1, ..., x_k, y\rangle \in Y)\}$. Suslin had essentially noted that *a set of reals is analytic iff it is the projection of a Borel subset of* $\mathbb{R}^2$. Luzin and Sierpiński took the geometric operation of projection to be basic and defined the projective sets as those sets obtainable from the Borel sets by the iterated applications of projection and complementation. The corresponding hierarchy of projective subsets of $\mathbb{R}^k$ is defined, in modern notation, as follows: For $A \subseteq \mathbb{R}^k$,

$$A \text{ is } \mathbf{\Sigma}^1_1 \quad \textit{iff} \quad A = pY \text{ for some Borel set } Y \subseteq \mathbb{R}^{k+1} \ ,$$

A is *analytic* as for $k = 1$, and for $n > 0$,

$$\begin{aligned} A \text{ is } \mathbf{\Pi}^1_n \quad &\textit{iff} \quad \mathbb{R}^k - A \text{ is } \mathbf{\Sigma}^1_n \ , \\ A \text{ is } \mathbf{\Sigma}^1_{n+1} \quad &\textit{iff} \quad A = pY \text{ for some } \mathbf{\Pi}^1_n \text{ set } Y \subseteq \mathbb{R}^{k+1}, \text{ and} \\ A \text{ is } \mathbf{\Delta}^1_n \quad &\textit{iff} \quad A \text{ is both } \mathbf{\Sigma}^1_n \text{ and } \mathbf{\Pi}^1_n \ . \end{aligned}$$

($\mathbf{\Sigma}^1_n$ is also written $\utilde{\Sigma}^1_n$; $\mathbf{\Pi}^1_n$ is also written $\utilde{\Pi}^1_n$; and $\mathbf{\Delta}^1_n$ is also written $\utilde{\Delta}^1_n$. One can formulate these concepts with continuous images instead of projections, e.g. A is $\mathbf{\Sigma}^1_{n+1}$ iff A is the continuous image of some $\mathbf{\Pi}^1_n$ set $Y \subseteq \mathbb{R}$. If the basics of continuous functions are in hand, this obviates the need to have different spaces.)

Luzin and Sierpiński recast Lebesgue's use of the Cantor diagonal argument to show that the projective hierarchy is proper, and soon its basic properties were established. However, this investigation encountered obstacles from the beginning. Whether the $\mathbf{\Pi}^1_1$ subsets of $\mathbb{R}$, the co-analytic sets at the bottom of the hierarchy, have the perfect set property and whether the $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable remained unknown. Besides the regularity properties, the properties of *separation, reduction,* and especially *uniformization* relating sets to others were studied, but there were accomplishments only at the first projective level. The one eventual success and a culminating result of the early period was the Japanese mathematician Motokiti Kondô's 1937 result, the $\mathbf{\Pi}^1_1$ Uniformization Theorem: *Every* $\mathbf{\Pi}^1_1$ *relation can be uniformized by a* $\mathbf{\Pi}^1_1$ *function.* This impasse with respect to the regularity properties would be clarified, surprisingly, by penetrating work of Gödel involving metamathematical methods.

In modern set theory, what has come to be taken for the "reals" is actually *Baire space*, the set of functions from the natural numbers into the natural numbers (with the product topology). Baire space, the "fundamental domain" of a 1930 Luzin monograph, is homeomorphic to the irrational reals and so equivalent for all purposes having to do measure, category, and perfect sets. Already by then it had become evident that a set-theoretic study of the continuum is best cast in terms of Baire space, with geometric intuitions being augmented by combinatorial ones.

During this period AC and CH were explored by the new Polish school, most notably by Sierpiński, Alfred Tarski, and Kazimierz Kuratowski, no longer as underlying axiom and primordial hypothesis but as part of ongoing mathematics. Sierpiński's own earliest publications, culminating in a 1918 survey, not only dealt with specific constructions but also showed how deeply embedded AC was in the informal development of cardinality, measure, and the Borel hierarchy. Even more than AC, Sierpiński investigated CH, and summed up his researches in a 1934 monograph. It became evident how having not only a well-ordering of the reals but one as given by CH whose initial segments are countable led to striking, often initially counter-intuitive, examples in analysis and topology.

1.4. Replacement and Foundation

In the 1920s, fresh initiatives in axiomatics structured the loose Zermelian framework with new features and corresponding axioms, the most consequential moves made by John von Neumann (1903-1957) in his doctoral work, with anticipations by Dmitry Mirimanoff in an informal setting. Von Neumann effected a Counter-Reformation of sorts that led to the incorporation of a new axiom, the Axiom of Replacement: *For any set x and property $P(v, w)$ functional on x (i.e. for any $a \in x$ there is exactly one b such that $P(a, b)$), $\{b \mid P(a, b)$ for some $a \in x\}$ is a set.* The transfinite numbers had been central for Cantor but peripheral to Zermelo; von Neumann reconstrued them as *bona fide* sets, the ordinals, and established their efficacy by formalizing transfinite recursion, the method for defining sets in terms of previously defined sets applied with transfinite indexing.

Ordinals manifest the basic idea of taking precedence in a well-ordering simply to be membership. A set x is *transitive iff* $\bigcup x \subseteq x$, so that x is "closed" under membership, and x is an *ordinal iff* x is transitive and well-ordered by $\in$. Von Neumann, as had Mirimanoff before him, established the key instrumental property of Cantor's ordinal numbers for ordinals: *Every well-ordered set is order-isomorphic to exactly one ordinal with membership.* Von Neumann took the further step of ascribing to the ordinals the role of Cantor's ordinal numbers. To establish the basic ordinal arithmetic results that affirm this role, von Neumann saw the need to establish the Transfinite Recursion Theorem, the theorem that validates definitions by transfinite recursion. The proof was anticipated by the Zermelo 1904 proof, but Replacement was necessary even for the very formulation, let alone the proof, of the theorem. Abraham Fraenkel and Thoralf Skolem had independently proposed Replacement to ensure that a specific collection resulting from a simple recursion be a set, but it was von Neumann's formal incorporation of transfinite recursion as method which brought Replacement into set theory. With the ordinals in place von Neumann completed the restoration of the Cantorian transfinite by defining the *cardinals* as the *initial ordinals*, i.e. those ordinals not in bijective correspondence with any of its predecessors. The infinite initial ordinals are now denoted

$$\omega = \omega_0, \omega_1, \omega_2, \ldots, \omega_\alpha, \ldots,$$

so that ω is to be the set of natural numbers in the ordinal construal. It would henceforth be that we take

$$\omega_\alpha = \aleph_\alpha$$

conflating extension with intension, with the left being a von Neumann ordinal and the right being the Cantorian cardinal concept. Every infinite set x, with AC, is well-orderable and hence in bijective correspondence with a unique initial ordinal ω_α, and the cardinality of x is $|x| = \aleph_\alpha$. It has become customary to use the lower case Greek letters to denote ordinals; $\alpha < \beta$ to denote $\alpha \in \beta$ construed as ordering; On to denote the ordinals; and the middle letters $\kappa, \lambda, \mu, \ldots$ to denote the initial ordinals in their role as the infinite cardinals, with κ^+ denoting the cardinal successor of κ.

Von Neumann provided a new axiomatization of set theory, one that first incorporated what we now call proper classes. A *class* is the totality of all sets that satisfy a specified property, so that membership in the class amounts to satisfying the property, and von Neumann axiomatized the ways to have these properties. Only sets can be members, and so the recourse to possibly proper classes, classes not represented by sets, avoids the contradictions arising from formalizing the known paradoxes. Actually, von Neumann took functions to be primitive in an involved framework, and Paul Bernays in 1930 re-constituted the von Neumann axiomatization with sets and classes as primitive. Classes would not remain a formalized component of modern set theory, but the informal use of classes as objectifications of properties would become increasingly liberal, particularly to convey large-scale issues in set theory.

Von Neumann (and before him Mirimanoff, Fraenkel, and Skolem) also considered the salutary effects of restricting the universe of sets to the *well-founded sets*. The well-founded sets are the sets in the class $\bigcup_\alpha V_\alpha$, where the "ranks" V_α are defined by transfinite recursion:

$$V_0 = \emptyset;\ V_{\alpha+1} = P(V_\alpha);\ \text{and } V_\delta = \textstyle\bigcup_{\alpha<\delta} V_\alpha \text{ for limit ordinals } \delta\,.$$

Von Neumann entertained the Axiom of Foundation: *Every nonempty set x has an $\in$-minimal element, i.e. a $y \in x$ such that $x \cap y$ is empty.* (With AC this is equivalent to having no infinite $\in$-descending sequences.) This axiom amounts to the assertion that the *cumulative hierarchy* exhausts the universe V of sets:

$$V = \bigcup_\alpha V_\alpha \, .$$

In modern terms, the ascribed well-foundedness of $\in$ leads to a ranking function $\rho \colon V \to \mathrm{On}$ defined recursively by $\rho(x) = \bigcup\{\rho(y) + 1 \mid y \in x\}$, so that $V_\alpha = \{x \mid \rho(x) < \alpha\}$, and one can establish results for all sets by induction on rank.

Zermelo in a 1930 paper offered his final axiomatization of set theory as well as a striking, synthetic view of a procession of models that would have a modern resonance. Proceeding in what we would now call a second-order context, Zermelo amended his 1908 axiomatization Z by adjoining both Replacement and Foundation while leaving out Infinity and AC, the latter being regarded as part of the underlying logic. The now standard axiomatization of set theory

ZFC, *Zermelo-Fraenkel with Choice*,

is recognizable if we inject Infinity and AC, the main difference being that ZFC is a first-order theory (as discussed below). "Fraenkel" acknowledges the early suggestion by Fraenkel to adjoin Replacement; and the Axiom of Choice is explicitly mentioned.

ZF, *Zermelo-Fraenkel*,

is ZFC without AC and is a base theory for the investigation of weak Choice-type propositions as well as propositions that contradict AC.

Zermelo herewith completed his transmutation of the notion of set, his abstract view stabilized by further axioms that structured the universe of sets. Replacement and Foundation focused the notion of set, with the first providing the means for transfinite recursion and induction and the second making possible the application of those means to get results about *all* sets, they appearing in the cumulative hierarchy. Foundation is the one axiom unnecessary for the recasting of mathematics in set-theoretic terms, but the axiom is also the salient feature that distinguishes investigations specific to set theory as a field of mathematics. With Replacement and Foundation in place Zermelo was able to provide natural models of his axioms, each a V_κ where κ is an *inaccessible* cardinal (regular and *strong limit*: if $\lambda < \kappa$, then $2^\lambda < \kappa$), and to establish algebraic isomorphism, initial segment, and embedding results for his models. Finally, Zermelo posited an endless procession of such models, each a set in the next, as natural extensions of their cumulative hierarchies.

Inaccessible cardinals are at the modest beginnings of the theory of *large cardinals*, now a mainstream of modern set theory devoted to the investigation of strong hypotheses and consistency strength. The journal volume containing Zermelo's paper also contained Stanisław Ulam's seminal paper on *measurable cardinals*, which would become focal among large cardinals. In modern terminology, a *filter over* a set Z is a family of subsets of Z closed under the taking of supersets and of intersections. (Usually excluded from consideration as trivial are $\{X \subseteq Z \mid A \subseteq X\}$ for some set $A \subseteq Z$, the *principal* filters.) An *ultrafilter U over Z* is a maximal filter over Z, i.e. for any $X \subseteq Z$, either $X \in U$ or else $Z - X \in U$. For a cardinal λ, a filter is λ-*complete* if it is closed under the taking of intersections of fewer than λ members. Finally, an uncountable cardinal κ is *measurable iff* there is a κ-complete ultrafilter over κ. In a previous, 1929 note Ulam had been the first to construct, using a well-ordering of the reals, an ultrafilter over ω. Measurability thus generalizes a property of ω, and Ulam showed moreover that *measurable cardinals are inaccessible*. In this work, Ulam was motivated by measure-theoretic considerations, and he viewed his work as about $\{0, 1\}$-valued measures, the measure 1 sets being the sets in the ultrafilter. To this day, ultrafilters of all sorts in large cardinal theory are also called measures.

A decade later Tarski provided a systematic development of these concepts in terms of ideals. An *ideal over* a set Z is a family of subsets of Z closed under the taking of subsets and of unions. This is the "dual" notion to filters; if I is an ideal (resp. filter) over Z, then $\breve{I} = \{Z - X \mid X \in I\}$ is its dual filter (resp. ideal). An ideal is λ-*complete* if its dual filter is. A more familiar conceptualization in mathematics, Tarski investigated a general notion of ideal on a Boolean algebra in place of the power set algebra $P(Z)$. Although filters and ideals in large cardinal theory are most often said to be *on* a cardinal κ, they are more properly *on* the Boolean algebra $P(\kappa)$. Moreover, the measure-theoretic terminology has persisted: For an ideal $I \subseteq P(Z)$, the I-*measure zero* (negligible) sets are the members of I, the I-*positive measure* (non-negligible) sets are the members of $P(Z) - I$, and the I-*measure one* (all but negligible) sets are the members of the dual filter $\{Z - X \mid X \in I\}$.

Returning to the axiomatic tradition, Zermelo's 1930 paper was in part a response to Skolem's advocacy of the idea of framing Zermelo's 1908 axioms in *first-order logic*, the logic of formal languages based on the quantifiers $\forall$ and $\exists$ interpreted as ranging over the *elements* of a domain of discourse. First-order logic had emerged in 1917 lectures of Hilbert as a delimited system of logic amenable to mathematical investigation. Entering from a different, algebraic tradition, Skolem in 1920 had established a seminal result for semantic methods with the Löwenheim-Skolem Theorem, that a countable collection of first-order sentences, if satisfiable, is satisfiable in a countable domain. For this he introduced what we now call Skolem functions, functions added formally for witnessing $\exists x$ assertions. For set theory Skolem in 1923 proposed formalizing Zermelo's axioms in the first-order language with $\in$ and $=$ as binary predicate symbols. Zermelo's "definite" properties were to be those expressible in this first-order language in terms of given sets, and the Axiom of Separation was to become a *schema* of axioms, one for each first-order formula. As an argument against taking set theory as a foundation for mathematics, Skolem pointed out what has come to be called *Skolem's Paradox*: Zermelo's 1908 axioms cast in first-order logic is a countable collection of sentences, and so if they are satisfiable at all, they are satisfiable in a countable domain. Thus, we have the paradoxical existence of countable models for Zermelo's axioms although they entail the existence of uncountable sets. Zermelo found this antithetical and repugnant. However, strong currents were at work leading to a further, subtler transmutation of the notion of set as based on first-order logic and incorporating its relativism of set-theoretic concepts.

2. New Groundwork

2.1. Gödel

Kurt Gödel (1906-1978) substantially advanced the mathematization of logic by submerging metamathematical methods into mathematics. The main vehicle was the direct coding, "the arithmetization of syntax", in his celebrated 1931 Incompleteness Theorem, which worked dialectically against a program of Hilbert's for establishing the consistency of classical mathematics. But starting an undercurrent, the earlier 1930 Completeness Theorem for first-order logic clarified the distinction between the formal syntax and semantics of first-order logic and secured its key instrumental property with the Compactness Theorem.

Tarski in the early 1930s provided his systematic "definition of truth", exercising philosophers to a surprising extent ever since. Tarski simply schematized truth as a correspondence between formulas of a formal language and set-theoretic assertions about an intended structure interpreting the language and provided a recursive definition of the *satisfaction* relation, when a formula holds in the structure, in set-theoretic terms. The eventual effect of Tarski's mathematical formulation of semantics would be not only to make mathematics out of the informal notion of satisfiability, but also to enrich ongoing mathematics with a systematic method for forming mathematical analogues of several intuitive semantic notions. Tarski would only be explicit much later about satisfaction-in-a-structure for arbitrary structures, this leading to his notion of logical consequence. For coming purposes, the following affirms notation and concepts in connection with Tarski's definition.

For a first-order language, a structure N interpreting that language (i.e. a specification of a domain of discourse as well as interpretations of the function and predicate symbols), a formula $\varphi(v_1, v_2, \ldots, v_n)$ of the language with the (free) variables as displayed, and $a_1, a_2, \ldots, a_n$ in the domain of N,

$$N \models \varphi[a_1, a_2, \ldots, a_n]$$

asserts that the formula φ is satisfied in N according to Tarski's recursive definition when v_i is interpreted as a_i. A subset y of the domain of N is *first-order definable over N iff* there is a $\psi(v_1, v_2, \ldots, v_{n+1})$ and $a_1, a_2, \ldots, a_n$ in the domain of N such that

$$y = \{z \in N \mid N \models \psi[a_1, a_2, \ldots, a_n, z]\}.$$

(The first-order definability of k-ary relations is analogously formulated with v_{n+1} replaced by k variables.)

Through Tarski's recursive definition and an "arithmetization of syntax" whereby formulas are systematically coded by natural numbers, the satisfaction relation $N \models \varphi[a_1, a_2, \ldots, a_n]$ for *sets* N is definable in set theory. On the other hand, by Tarski's result on the "undefinability of truth", the satisfaction relation for V itself is not first-order definable over V.

Set theory was launched as a distinctive field of mathematics by Gödel's construction of the class L leading to the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. In a brief 1939 account Gödel informally presented L essentially as is done today: For any set x let $\mathrm{def}(x)$ denote the collection of subsets of x first-order definable over the structure $\langle x, \in \rangle$ with domain x and the membership relation restricted to it. Then define:

$$L_0 = \emptyset;\ L_{\alpha+1} = \mathrm{def}(L_\alpha),\ L_\delta = \bigcup\{L_\alpha \mid \alpha < \delta\} \text{ for limit ordinals } \delta;$$

and the *constructible universe*

$$L = \bigcup\nolimits_\alpha L_\alpha \, .$$

Gödel pointed out that L "can be defined and its theory developed in the formal systems of set theory themselves." This is actually the central feature of the construction of L. L is definable in ZF via transfinite recursion based on the formalizability of $\mathrm{def}(x)$, which was reaffirmed by Tarski's definition of satisfaction. With this, one can formalize the Axiom of Constructibility $V = L$, i.e. $\forall x(x \in L)$. To set a larger context, we affirm the following for a class X: for a set-theoretic formula φ, φ^X denotes φ with its quantifiers restricted to X and this extends to set-theoretic terms t (like $\bigcup x$, $P(x)$, and so forth) through their definitions to yield t^X. X is an *inner model iff* X is a transitive class containing all the ordinals such that φ^X is a theorem of ZF for every axiom φ of ZF. What Gödel did was to show in ZF that L is an inner model which satisfies AC and GCH. He thus established a relative consistency which can be formalized as an assertion: Con(ZF) implies Con(ZFC + GCH).

In the approach via $\mathrm{def}(x)$ it is necessary to show that $\mathrm{def}(x)$ remains unaltered when applied in L with quantifiers restricted to L. Gödel himself would never establish this *absoluteness of first-order definability* explicitly. In a 1940 monograph, Gödel worked in Bernays' class-set theory and used eight binary operations producing new classes from old to generate L set by set via transfinite recursion. This veritable "Gödel numbering" with ordinals eschewed $\mathrm{def}(x)$ and made evident certain aspects of L. Since there is a direct, definable well-ordering of L, choice functions abound in L, and AC holds there. Of the other axioms the crux is where first-order logic impinges, in Separation and Replacement. For this, "algebraic" closure under Gödel's eight operations ensured "logical" Separation for *bounded* formulas, formulas having only quantifiers expressible in terms of $\forall v \in w$, and then the full exercise of Replacement (in V) secured all of the ZF axioms in L.

Gödel's proof that L satisfies GCH consisted of two separate parts. He established the implication $V = L \to \mathrm{GCH}$, and, in order to apply this implication within L, that $(V = L)^L$. This latter follows from the aforementioned absoluteness of def(x), and in his monograph Gödel gave an alternate proof based on the absoluteness of his eight binary operations.

Gödel's argument for $V = L \rightarrow$ GCH rests, as he himself wrote in his 1939 note, on "a generalization of Skolem's method for constructing enumerable models." This was the first significant use of Skolem functions since Skolem's own to establish the Löwenheim-Skolem theorem, and with it, Skolem's Paradox. Ironically, though Skolem sought through his paradox to discredit set theory based on first-order logic as a foundation for mathematics, Gödel turned paradox into method, one promoting first-order logic. Gödel specifically established his "Fundamental Theorem":

> For infinite γ, every constructible subset of L_γ
> belongs to some L_β for a β of the same cardinality as γ.

For infinite α, L_α has the same cardinality as that of α. It follows from the Fundamental Theorem that in the sense of L, the power set of L_{ω_α} is included in $L_{\omega_{\alpha+1}}$, and so GCH follows in L.

The work with L led, further, to the resolution of difficulties in descriptive set theory. Gödel announced, in modern terms: *If $V = L$, then (a) there is a $\mathbf{\Delta}^1_2$ set of reals that is not Lebesgue measurable, and (b) there is a $\mathbf{\Pi}^1_1$ set of reals without the perfect set property.* Thus, the early descriptive set theorists were confronting an obstacle insurmountable in ZFC! When eventually confirmed and refined, the results were seen to turn on a "good" $\mathbf{\Sigma}^1_2$ well-ordering of the reals in L defined via reals coding well-founded structures and thus connected to the well-founded tree representation of $\mathbf{\Pi}^1_1$ sets. Gödel's results (a) and (b) constitute the first real synthesis of abstract and descriptive set theory, in that the axiomatic framework is brought to bear on the investigation of definable sets of reals.

Gödel brought into set theory a method of construction and of argument which affirmed several features of its axiomatic presentation. Most prominently, he showed how first-order definability can be formalized and used to achieve strikingly new mathematical results. This significantly contributed to a lasting ascendancy for first-order logic which, in addition to its sufficiency as a logical framework for mathematics, was seen to have considerable operational efficacy. Moreover, Gödel's work buttressed the incorporation of Replacement and Foundation into set theory, the first immanent in the transfinite recursion and arbitrary extent of the ordinals, and the second as underlying the basic cumulative hierarchy picture that anchors L.

In later years Gödel speculated about the possibility of deciding propositions like CH with large cardinal hypotheses based on the heuristics of *reflection*, and later, *generalization.* In a 1946 address he suggested the consideration of "stronger and stronger axioms of infinity" and reflection down from V: "Any proof of a set-theoretic theorem in the next higher system above set theory (i.e. any proof involving the concept of truth ...) is replaceable by a proof from such an axiom of infinity." In a 1947 expository article on the continuum problem Gödel presumed that CH would be shown independent from ZF and speculated more concretely about possibilities with large cardinals. He argued that the axioms of set theory do not "form a system closed in itself" and so the "very concept of set on which they are based suggests their extension by new axioms that assert the existence of still further iterations of the operation of 'set of'." In an unpublished footnote toward a 1966 revision of the article, Gödel acknowledged "extremely strong axioms of infinity of an entirely new kind", generalizations of properties of ω "supported by strong arguments from analogy." These heuristics would surface anew in the 1960s, when the theory of large cardinals developed a self-fueling momentum of its own, stimulated by the emergence of forcing and inner models.

2.2. Infinite Combinatorics

For decades Gödel's construction of L stood as an isolated monument in the axiomatic tradition, and his methodological advances would only become fully assimilated after the infusion of model-theoretic techniques in the 1950s. In the mean time, the direct investigation of the transfinite as extension of number was advanced, gingerly at first, by the emergence of *infinite combinatorics.*

The 1934 Sierpiński monograph on CH discussed earlier having considerably elaborated its consequences, a new angle in the combinatorial investigation of the continuum was soon broached. Hausdorff in 1936 reactivated his early work on gaps in the orderings of functions to show that the reals can be partitioned into $\aleph_1$ Borel sets, answering an early question of Sierpiński. Hausdorff had newly cast his work in terms of functions from ω to ω, the members of Baire space or the "reals", under the ordering of *eventual dominance*: $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Work on this structure and definable sets of reals in the 1930s, and particularly of Fritz Rothberger through the 1940s, isolated what is now called the *dominating number* $\mathfrak{d}$, the least cardinality of a subset of Baire space cofinal in $\leq^*$. $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$, but absent CH $\mathfrak{d}$ assumed an independent significance as a pivotal cardinal. Rothberger established incisive results which we now cast as about the relationships to other pivotal cardinals, results which provided new understandings about the structure of the continuum but would become vacuous with the blanket assumption of CH. The investigation of $\mathfrak{d}$ and other "cardinal characteristics (or invariants) of the continuum" would blossom with the advent of forcing.

Taking up another thread, Frank Ramsey in 1930, addressing a problem of formal logic, established a generalization of the pigeonhole principle for finite sets, and in a move transcending purpose and context he also established an infinite version implicitly applying the now familiar Kőnig's Lemma for trees. In modern terms, for ordinals α, β, and δ and $n \in \omega$ the *partition relation*

$$\beta \longrightarrow (\alpha)^n_\delta$$

asserts that for any partition $f\colon [\beta]^n \to \delta$ of the n-element subsets of β into δ cells, there is an $H \subseteq \beta$ of order type α *homogeneous* for the partition, i.e. all the n-element subsets of H lie in the same cell. Ramsey's theorem for finite sets is: *For any $n, k, i \in \omega$ there is an $r \in \omega$ such that $r \longrightarrow (k)^n_i$.* The "Ramsey numbers", the least possible r's for various n, k, i, are unknown except in a few basic cases. The (infinite) Ramsey's Theorem is: *$\omega \longrightarrow (\omega)^n_i$ for every $n, i \in \omega$.*

A *tree* is a partially ordered set T such that the predecessors of any element are well-ordered. The *αth level* of T consists of those elements whose predecessors have order-type α, and the *height* of T is the least α such that the αth level of T is empty. A *chain* of T is a linearly ordered subset, and an *antichain* is a subset consisting of pairwise incompatible elements. A *cofinal branch* of T is a chain with elements at every non-empty level of T. Finally, for a cardinal κ, a *κ-tree* is a tree of height κ each of whose levels has cardinality less than κ, and κ has the *tree property iff* every κ-tree has a cofinal branch. Kőnig's Lemma, of 1927, is the assertion that *ω has the tree property.*

The first systematic study of transfinite trees was carried out in Djuro Kurepa's 1935 thesis, and several properties emerging from his investigations, particularly for ω_1-trees as the first broaching context, would later become focal in the combinatorial study of the transfinite. An *Aronszajn tree* is an ω_1-tree without a cofinal branch, i.e. a counterexample to the tree property for ω_1. Kurepa acknowledged and gave Nachman Aronszajn's proof that *there is an Aronszajn tree.* A *Suslin tree* is an ω_1-tree with no uncountable chains or antichains. Kurepa reduced a hypothesis growing out of a 1920 question of Suslin about the characterizability of the ordering of the reals to a combinatorial property of ω_1, Suslin's Hypothesis (SH): *There are no Suslin trees.* Finally, a *Kurepa tree* is an ω_1-tree with at least ω_2 cofinal branches, and *Kurepa's Hypothesis* deriving from a later 1942 paper of Kurepa's is the assertion that such trees exist. Much of this would be rediscovered, and both Suslin's Hypothesis and Kurepa's Hypothesis would be resolved decades later with the advent of forcing, several of the resolutions in terms of large cardinal hypotheses. Kurepa's work also anticipated another development from a different quarter.

Paul Erdős, although an itinerant mathematician for most of his life, was the prominent figure of a strong Hungarian tradition in combinatorics, and through some seminal results he introduced major initiatives into the detailed combinatorial study of the transfinite. Erdős and his collaborators simply viewed the transfinite numbers as a combinatorially rich source of intrinsically interesting problems, the concrete questions about graphs and mappings having a natural appeal through their immediacy. One of the earliest advances was an 1943 paper of Erdős and Tarski which concluded enticingly with an intriguing list of six combinatorial problems, the positive solution to any, as it was to turn out, amounting to the existence of a large cardinal. In a footnote various implications were noted, one of them being essentially that *for inaccessible κ, the tree property for κ implies $\kappa \longrightarrow (\kappa)^2_2$*, a generalization of Ramsey's $\omega \longrightarrow (\omega)^2_2$ drawing out the Kőnig Lemma property needed.

The detailed investigation of partition relations began in earnest in the 1950s, with a 1956 paper of Erdős and Richard Rado's being representative. For a cardinal κ, set $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. What became known as *the* Erdős-Rado Theorem asserts: *For any infinite cardinal κ and $n \in \omega$,*

$$\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa \,.$$

This was established using the basic tree argument underlying Ramsey's results, whereby a homogeneous set is not constructed recursively, but a tree is constructed such that its branches provide homogeneous sets, and a counting argument ensures that there must be a homogeneous set of sufficient cardinality. The Erdős-Rado Theorem is the transfinite analogue of Ramsey's theorem for finite sets, with both having the form, given α, δ and n there is a β such that $\beta \longrightarrow (\alpha)^n_\delta$. However, while what the Ramsey numbers are is largely unknown, the $\beth_n(\kappa)^+$ are known to be optimal. Kurepa in effect had actually established the case $n = 1$ and shown that $\beth_1(\kappa)^+$ is the least possible, and the $\beth_n(\kappa)^+$ was also shown to be the least possible in the general case by a "negative stepping up" lemma.

Still among the Hungarians, Géza Fodor in 1956 established a now basic fact about the uncountable that has become woven into its sense, so operationally useful and ubiquitous it has become in infinite combinatorics. For a cardinal λ and a set $C \subseteq \lambda$, C is *closed unbounded* (or "club") in λ *iff* C contains its limit (or "accumulation") points, i.e. those $0 < \alpha < \lambda$ such that $\sup(C \cap \alpha) = \alpha$, and is cofinal, i.e. $\bigcup C = \lambda$. The use of "closed" and "unbounded" are as for $\langle \lambda, < \rangle$ with the order topology. A set $S \subseteq \lambda$ is *stationary in λ iff* for any C closed unbounded in λ, $S \cap C$ is not empty. For regular $\lambda > \omega$, the intersection of fewer than λ sets closed unbounded in λ is again closed unbounded in λ, and so the closed unbounded subsets of λ generate a λ-complete filter, the *closed unbounded filter*, denoted $\mathcal{C}_\lambda$. The nonstationary subsets of λ constitute the dual *nonstationary ideal*, denoted NS_λ. Now Fodor's (or Regressive Function or "Pressing Down") Lemma: *For regular $\lambda > \omega$, if a function f is regressive on a set $S \subseteq \lambda$ stationary in λ, i.e. $f(\alpha) < \alpha$ for every $\alpha \in S$, then there is a $T \subseteq S$ stationary in λ on which f is constant.*

Fodor's Lemma is a basic fact and its proof a simple exercise now, but then it was the culmination of a progression of results beginning with a seminal 1929 observation of Aleksandrov that a regressive function on ω_1 must be constant on an uncountable set. The subsets of a regular $\lambda > \omega$ naturally separate out into the nonstationary sets, the stationary sets, and among them the closed unbounded sets as the negligible, non-negligible, and all but negligible sets according to NS_λ. Fodor's Lemma is intrinsic to stationarity, and can be cast as a substantive characterization of the concept. It would be that far-reaching generalizations of stationarity, e.g. stationary towers, would become important in modern set theory.

2.3. Definability

Descriptive set theory was to become transmuted by the turn to definability following Gödel's work. After his fundamental work on recursive function theory in the 1930s, Stephen Kleene expanded his investigations of effectiveness and developed a general theory of definability for relations on ω. In the early 1940s Kleene investigated the *arithmetical relations* on reals, those relations obtainable from the recursive relations by applications of number quantifiers. Developing canonical representations he classified these relations into a hierarchy according to quantifier complexity and showed that the hierarchy is proper. In the mid-1950s Kleene investigated the *analytical relations*, those relations obtainable from the arithmetical relations by applications of function ("real") quantifiers. Again he worked out representation and hierarchy results, and moreover he established an elegant theorem that turned out to be an effective version of Suslin's characterization of the Borel sets.

Kleene was developing what amounted to the effective content of classical descriptive set theory, unaware that his work had direct antecedents in the papers of Lebesgue, Luzin, Sierpiński, and Tarski. Kleene's student John Addison then established that there is an exact correlation between the hierarchies of classical and effective descriptive set theory (as described below). The development of effective descriptive set theory considerably clarified the classical context, injected recursion-theoretic techniques into the subject, and placed definability considerations squarely at its forefront. Not only were new approaches to classical problems provided, but results and questions could now be formulated in a refined setting.

Second-order arithmetic is the two-sorted structure

$$\mathcal{A}^2 = \langle \omega, {}^{\omega}\omega, ap, +, \times, <, 0, 1 \rangle,$$

where ω and ${}^{\omega}\omega$ (Baire space or the "reals") are two separate domains connected by the binary operation $ap\colon {}^{\omega}\omega \times \omega \to \omega$ of *application* given by $ap(x, m) = x(m)$, and $+, \times, <, 0, 1$ impose the usual arithmetical structure on ω. The underlying language has two sorts of variables, those ranging over ω and those ranging over ${}^{\omega}\omega$, and corresponding *number quantifiers* $\forall^0$, $\exists^0$ and *function quantifiers* $\forall^1$, $\exists^1$.

For relations $A \subseteq ({}^{\omega}\omega)^k$,

A is *arithmetical* *iff* A is definable over $\mathcal{A}^2$ by a formula without function quantifiers .
A is *analytical* *iff* A is definable over $\mathcal{A}^2$.

Through the manipulation of quantifiers the analytical sets can be classified in the *analytical hierarchy*, the levels of which are the (lightface) Σ^1_n, Π^1_n, and

Δ^1_n classes defined as follows: For relations $A \subseteq ({}^\omega\omega)^k$ and $n > 0$,

$$A \in \Sigma^1_n \quad \textit{iff} \quad \forall \mathbf{w}(A(\mathbf{w}) \leftrightarrow \exists^1 x_1 \forall^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, \ldots, x_n)) \text{ , and}$$
$$A \in \Pi^1_n \quad \textit{iff} \quad \forall \mathbf{w}(A(\mathbf{w}) \leftrightarrow \forall^1 x_1 \exists^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, \ldots, x_n))$$

for some arithmetical $R \subseteq ({}^\omega\omega)^{k+n}$, where Q is $\exists^1$ if n is odd and $\forall^1$ if n is even in the first and *vice versa* in the second. Finally,

$$A \in \Delta^1_n \quad \textit{iff} \quad A \in \Sigma^1_n \cap \Pi^1_n \ .$$

The correlation of the effective ("lightface") and classical ("boldface") hierarchies was established by Addison in 1958 through the simple expedient of relativization to real parameters. For $a \in {}^\omega\omega$, *second-order arithmetic in* a is the expanded structure

$$\mathcal{A}^2(a) = \langle \omega, {}^\omega\omega, ap, +, \times, <, 0, 1, a \rangle$$

where a is regarded as a binary relation on ω. Replacing $\mathcal{A}^2$ by $\mathcal{A}^2(a)$ in the preceding, we get the corresponding relativized notions: *arithmetical in* a, *analytical in* a, $\Sigma^1_n(a)$, $\Pi^1_n(a)$, and $\Delta^1_n(a)$. The correlation of the hierarchies is then as follows: *Suppose that* $A \subseteq ({}^\omega\omega)^k$ *and* $n > 0$. *Then* $A \in \boldsymbol{\Sigma}^1_n$ *iff* $A \in \Sigma^1_n(a)$ *for some* $a \in {}^\omega\omega$, *and similarly for* $\boldsymbol{\Pi}^1_n$. Loosely speaking, a projective set can be analyzed with a real parameter coding the construction of the underlying Borel set, $\exists^1$ corresponding to projection, and $\forall^1$ through $\neg\exists^1\neg$ corresponding to complementation.

Joseph Shoenfield in 1961 advanced the study of projective sets into the new definability context by providing a tree representation for $\boldsymbol{\Sigma}^1_2$ sets based on well-foundedness as charted out to ω_1. The classical Luzin-Sierpiński tree representation of $\boldsymbol{\Pi}^1_1$ sets turned, in the new terms, on the f of the function quantifier $\forall f$ imputing infinite branches through a tree arithmetical in a for some $a \in {}^\omega\omega$ that must be cut off. This well-foundedness can be cast as having an order-preserving ranking function into ω_1, which Shoenfield pointed out can be recast as having an infinite branch through a tree built on the countable ordinals.

T is a *tree on* $\omega \times \kappa$ *iff* (a) T consists of pairs $\langle s, t \rangle$ where s is a finite sequence drawn from ω and t is a finite sequence drawn from κ of the same length, and (b) if $\langle s, t \rangle \in T$, s' is an initial segment of s and t' is a initial segment of t of the same length, then also $\langle s', t' \rangle \in T$. For such T, $[T]$ consists of pairs $\langle f, g \rangle$ corresponding to infinite branches, i.e. f and g are ω-sequences such that for any finite initial segment s of f and finite initial segment t of g of the same length, $\langle s, t \rangle \in T$. In modern terms, $A \subseteq {}^\omega\omega$ is κ*-Suslin iff* there is a tree on $\omega \times \kappa$ such that $A = p[T] = \{f \mid \exists g(\langle f, g \rangle \in [T])\}$. $[T]$ is a closed set in the space of $\langle f, g \rangle$'s where $f \colon \omega \to \omega$ and $g \colon \omega \to \kappa$, and so otherwise complicated sets of reals, if shown to be κ-Suslin, are newly comprehended as projections of closed sets. The analytic ($\boldsymbol{\Sigma}^1_1$) sets are exactly the ω-Suslin sets.

Shoenfield established that *every* $\mathbf{\Sigma}^1_2$ *set is* ω_1*-Suslin*, and his proof, emphasizing constructibility, showed that if $A \subseteq {}^\omega\omega$ is Σ^1_2, then $A = p[T]$ for a tree T on $\omega \times \omega_1$ such that $T \in L$. Shoenfield applied well-foundedness in the $\forall$ sense (no infinite descending sequences) and the $\exists$ sense (there is a ranking function) to establish that Σ^1_2 *relations are absolute (or "correct") for* L: For any $\mathbf{w} \in L$, $\mathcal{A}^2 \models \exists^1 x \forall^1 y \varphi[x, y, \mathbf{w}]$ *iff* $(\mathcal{A}^2 \models \exists^1 x \forall^1 y \varphi[x, y, \mathbf{w}])^L$ when φ has no function quantifiers.

Many substantive propositions of classical analysis as well as of metamathematical investigation are Σ^1_2 or Π^1_2, and if they can be established from $V = L$ (or just CH), then they can be established in ZF alone. It would be that in the years to come more and more projective sets of reals would be comprehended through κ-Suslin representations for larger and larger cardinals κ.

András Hajnal and Azriel Levy, in their theses of the mid-1950s, developed generalizations of L that were to become basic in a richer setting. For a set A, Hajnal formulated the *constructible closure* $L(A)$ *of* A, i.e. the smallest inner model M such that $A \in M$, and Levy formulated the *class* $L[A]$ *of sets constructible relative to* A, i.e. the smallest inner model M such that for every $x \in M$, $A \cap x \in M$. To formulate $L(A)$, define: $L_0(A) =$ the smallest transitive set $\supseteq \{A\}$ (to ensure that the resulting class is transitive); $L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(A))$; $L_\delta(A) = \bigcup_{\alpha<\delta} L_\alpha(A)$ for limit $\delta > 0$; and finally $L(A) = \bigcup_\alpha L_\alpha(A)$. To formulate $L[A]$, first let $\mathrm{def}^A(x)$ denote the collection of subsets of x first-order definable over $\langle x, \in, A \cap x\rangle$, i.e. $A \cap x$ is now allowed as a predicate in the definitions. Then define: $L_0[A] = \emptyset$; $L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A])$; $L_\delta[A] = \bigcup_{\alpha<\delta} L_\alpha[A]$ for limit $\delta > 0$; and finally $L[A] = \bigcup_\alpha L_\alpha[A]$. With the "trace" $\overline{A} = A \cap L[A]$ one has $L_\alpha[\overline{A}] = L_\alpha[A]$ for every α and so $L[\overline{A}] = L[A]$.

$L(A)$ realizes the algebraic idea of building up a model starting from a set of generators, and $L[A]$ the idea of building up a model using A construed as a predicate. $L(A)$ may not satisfy AC since it may not have a well-ordering of A, yet $L[A]$ always satisfies that axiom. This distinction was only to surface later, as both Hajnal and Levy took A to be a set of ordinals, when $L(A) = L[A]$, and used these models to establish conditional independence results of the sort: if the failure of CH is consistent, then so is that failure together with $2^\lambda = \lambda^+$ for sufficiently large cardinals λ. In the coming expansion of the 1960s, both Hajnal and Levy would be otherwise engaged, with Hajnal becoming a major combinatorial set theorist and collaborator with Erdős, and Levy, a pioneer in the investigation of independence results.

2.4. Model-Theoretic Techniques

Model theory began in earnest with the appearance in 1949 of the method of diagrams in Abraham Robinson's thesis and the related method of constants in Leon Henkin's thesis, which gave a new proof of the Gödel Completeness Theorem. Tarski had set the stage with the formulation of formal languages and semantics in set-theoretic terms, and with him established at the University of California at Berkeley, a large part of the development in the 1950s and 1960s would take place there. Tarski and his students carefully laid out satisfaction-in-a-structure; *theories* (deductively closed collections of sentences) and their *models*; algebratization with *Skolem functions* and *hulls*; and *elementary substructures and embeddings.* $j\colon \mathcal{A} \to \mathcal{B}$ is an *elementary embedding* if for any $a_1, \ldots, a_n$ from the domain of $\mathcal{A}$, $\langle a_1, \ldots, a_n \rangle$ satisfies in $\mathcal{A}$ the same formulas that $\langle j(a_1), \ldots, j(a_n) \rangle$ does in $\mathcal{B}$; and when j is the identity $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$, denoted $\mathcal{A} \prec \mathcal{B}$. The construction of models freely used transfinite methods and soon led to new questions in set theory, but also set theory was to be decisively advanced by the infusion of model-theoretic methods.

A precursory result was a 1949 generalization by Andrzej Mostowski of the Mirimanoff-von Neumann result that every well-ordered set is order-isomorphic to exactly one ordinal with membership. A binary relation R on a set X is *extensional* if distinct members of X have distinct R-predecessors, and *well-founded* if every non-empty $Y \subseteq X$ has an R-minimal element (or, assuming AC, there is no infinite R-descending sequence). *If R is an extensional, well-founded relation on a set X, then there is a unique transitive set T and an isomorphism of $\langle X, R \rangle$ onto $\langle T, \in \rangle$, i.e. a bijection $\pi\colon X \to T$ such that for any $x, y \in X$, $x\, R\, y$ iff $\pi(x) \in \pi(y)$.* $\langle T, \in \rangle$ is the *transitive collapse* of X, and π the *collapsing isomorphism.* Thus, the linearity of well-orderings has been relaxed to analogues of Extensionality and Foundation, and transitive sets become canonical representatives as ordinals are for well-orderings. Well-founded relations other than membership had surfaced much earlier, most notably in the Luzin-Sierpiński tree representation of $\mathbf{\Pi}^1_1$ sets. The general transitive collapse result would come to epitomize how well-foundedness made possible a coherent theory of models of set theory.

After Richard Montague applied reflection phenomena to establish that ZF is not finitely axiomatizable, Levy also formulated reflection principles and established their broader significance. The 1960 Montague-Levy Reflection Principle for ZF asserts: *For any (first-order) formula $\varphi(v_1, \ldots, v_n)$ and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,*

$$\varphi[x_1, \ldots, x_n] \textit{ iff } V_\alpha \models \varphi[x_1, \ldots, x_n]\,.$$

Levy showed that this schema is equivalent to the conjunction of the Replacement schema together with Infinity in the presence of the other axioms of ZF. Moreover, he formulated reflection principles in local form that characterized the *Mahlo* cardinals, conceptually the least large cardinals after the inaccessible cardinals. Also William Hanf and Dana Scott posited analogous reflection principles for higher-order formulas, leading to what are now called the *indescribable cardinals*. The model-theoretic reflection idea thus provided a coherent scheme for viewing the bottom of an emerging hierarchy of large cardinals as a generalization of Replacement and Infinity.

In those 1946 remarks by Gödel where he broached the heuristic of reflection, Gödel also entertained the concept of ordinal definable set. A set x is *ordinal definable iff* there are ordinals $\alpha_1, \ldots, \alpha_n$ and a formula $\varphi(v_0, \ldots, v_n)$ such that $\forall y(y \in x \leftrightarrow \varphi[y, \alpha_1, \ldots, \alpha_n])$. This ostensible dependence on the satisfaction relation for V can be formally recast through a version of the Reflection Principle for ZF, so that one can define the class OD of ordinal definable sets. With $\mathrm{tc}(y)$ denoting the smallest transitive superset of y, let $\mathrm{HOD} = \{x \mid \mathrm{tc}(\{x\}) \subseteq \mathrm{OD}\}$, the class of *hereditarily ordinal definable sets*.

As noted by Gödel, HOD is an inner model in which AC, though not necessarily CH, holds. The basic results about this inner model were to be rediscovered several times. In these several ways reflection phenomena both as heuristic and as principle became incorporated into set theory, bringing to the forefront what was to become a basic feature of the study of well-foundedness.

The set-theoretic generalization of first-order logic allowing transfinitely indexed logical operations was to clarify the size of measurable cardinals. Extending familiarity by abstracting to a new domain, Tarski in 1962 formulated the *strongly compact* and *weakly compact* cardinals by ascribing natural generalizations of the key compactness property of first-order logic to the corresponding infinitary languages. These cardinals had figured in that 1943 Erdős-Tarski paper in equivalent combinatorial formulations that were later seen to imply that *a strongly compact cardinal is measurable,* and *a measurable cardinal is weakly compact.* Tarski's student Hanf then established, using the satisfaction relation for infinitary languages, that *there are many inaccessible cardinals (and Mahlo cardinals) below a weakly compact cardinal. A fortiori, the least inaccessible cardinal is not measurable.* This breakthrough was the first result about the size of measurable cardinals since Ulam's original 1930 paper and was greeted as a spectacular success for metamathematical methods. Hanf's work radically altered size intuitions about problems coming to be understood in terms of large cardinals and ushered in model-theoretic methods into the study of large cardinals beyond the Mahlo cardinals.

Weak compactness was soon seen to have a variety of characterizations, most notably *κ is weakly compact iff $\kappa \to (\kappa)^2_2$ iff $\kappa \to (\kappa)^n_\lambda$ for every $n \in \omega$ and $\lambda < \kappa$ iff κ is inaccessible and has the tree property,* and this was an early, significant articulation of the large cardinal extension of context for effecting known proof ideas and methods.

The concurrent emergence of the *ultraproduct construction* in model theory set the stage for the development of the modern theory of large cardinals. The ultraproduct construction was brought to the forefront by Tarski and his students after Jerzy Łoś's 1955 adumbration of its fundamental theorem. The new method of constructing concrete models brought set theory and model theory even closer together in a surge of results and a lasting interest in ultrafilters.

The ultraproduct construction was driven by the algebraic idea of making a structure out of a direct product of structures as modulated (or "reduced") by a filter. The particular case when all the structures are the same, the *ultrapower*, was itself seen to be substantive. To briefly describe a focal case for set theory, let N be a set, construed as a structure with $\in$, and U an ultrafilter over a set Z. On ${}^Z N$, the set of functions from Z to N, define

$$f =_U g \quad \textit{iff} \quad \{i \in Z \mid f(i) = g(i)\} \in U \ ,$$

The filter properties of U imply that $=_U$ is an equivalence relation on ${}^Z N$, so with $(f)_U$ denoting the corresponding equivalence class of f, set ${}^Z N/U = \{(f)_U \mid f \in {}^Z N\}$. Next, the filter properties of U show that a binary relation E_U on ${}^Z N/U$ can be unambiguously defined by

$$(f)_U \ E_U \ (g)_U \quad \textit{iff} \quad \{i \in Z \mid f(i) \in g(i)\} \in U \ .$$

$=_U$ is thus a *congruence* relation, one that preserves the underlying structure; this sort of preservation is crucial in ultraproduct and classical, antecedent constructions with filters. (For example, in the space L^∞ in which two bounded measurable functions are equated when they agree on a set in the filter of full measure sets, the algebraic structure of $+$ and $\times$ have many of the properties that $+$ and $\times$ for the real numbers have. If the filter is extended to an ultrafilter, we get an ultrapower.) The *ultrapower* of N by U is then defined to be the structure $\langle {}^Z N/U, E_U\rangle$. The crux of the construction is the fundamental *Łoś's Theorem*: *For a formula* $\varphi(v_1, \ldots, v_n)$ *and* $f_1, \ldots, f_n \in {}^Z N$,

$$\langle {}^Z N/U, E_U\rangle \models \varphi[(f_1)_U, \ldots, (f_n)_U] \quad \textit{iff} \quad \{i \in Z \mid \mathcal{N} \models \varphi[f_1(i), \ldots, f_n(i)]\} \in U \ .$$

Satisfaction in the ultrapower is thus reduced to satisfaction on a large set of coordinates, large in the sense of U. The proof is by induction on the complexity of φ using the filter properties of U, the ultrafilter property for the negation step, and AC for the existential quantifier step.

E_U is an extensional relation, and crucially, well-founded when U is $\aleph_1$-complete. In that case by Mostowski's theorem there is a collapsing isomorphism π of the ultrapower onto its transitive collapse $\langle M, \in\rangle$. Moreover, if for $x \in N$, c_x is the constant function: $N \to \{x\}$ and $j_U \colon N \to M$ is defined by $j_U(x) = \pi((c_x)_U)$, then j_U is an elementary embedding, i.e. for any formula $\varphi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in N$,

$$\langle N, \in\rangle \models \varphi[a_1, \ldots, a_n] \quad \textit{iff} \quad \langle M, \in\rangle \models \varphi[j_U(a_1), \ldots, j_U(a_n)]$$

by Łoś's Theorem. When we have well-foundedness, the ultrapower is identified with its transitive collapse and denoted $\mathrm{Ult}(N, U)$.

All of the foregoing is applicable, and will be applied, with proper classes N, as long as we replace the equivalence class $(f)_U$ by sets

$$(f)^0_U = \{g \in (f)_U \mid g \text{ has minimal rank }\}$$

("Scott's trick"), and take Łoś's Theorem as a schema for formulas.

The model theorist H. Jerome Keisler established penetrating connections between combinatorial properties of ultrafilters and of their ultraproducts, and in particular took the ultrapower of a measurable cardinal κ by a κ-complete ultrafilter over κ to provide a new proof of Hanf's result that there are many large cardinals below a measurable cardinal. With Ulam's concept shown in a new light as providing well-founded ultrapowers, Dana Scott then struck on the idea of taking the ultrapower of the entire universe V by a κ-complete ultrafilter over a measurable κ, exploiting the resulting well-foundedness to get an elementary embedding $j\colon V \to \mathrm{Ult}(V,U)$. Importantly, κ is the *critical point*, i.e. $j(\alpha) = \alpha$ for every $\alpha < \kappa$ yet $\kappa < j(\kappa)$: Taking e.g. the identity function id : $\kappa \to \kappa$, $\{\xi < \kappa \mid \alpha < \xi < \kappa\} \in U$ for every $\alpha < \kappa$, so that $\kappa \leq \pi((\mathrm{id})_U) < j(\kappa)$ by Łoś's Theorem. If $V = L$, then $\mathrm{Ult}(V,U) = L$ by the definability properties of L, but this confronts $\kappa < j(\kappa)$, e.g. if κ were the least measurable cardinal. (One could also appeal to the general fact that $U \notin \mathrm{Ult}(V,U)$; that one "loses" the ultrafilter when taking the ultrapower would become an important theme in later work.) With this Scott established that *if there is a measurable cardinal, then* $V \neq L$. Large cardinal assumptions thus assumed a new significance as a means for "maximizing" possibilities away from Gödel's delimitative construction.

The ultrapower construction provided one direction of a new characterization that established a central structural role for measurable cardinals: *There is an elementary embedding* $j\colon V \to M$ *for some* M *with critical point* δ *iff* δ *is a measurable cardinal.* Keisler provided the converse direction: With j as hypothesized, $U_j \subseteq P(\delta)$ defined "canonically" by $X \in U_j$ *iff* $\delta \in j(X)$ is a δ-complete ultrafilter over δ. Generating ultrafilters thus via "ideal" elements would become integral to the theory of ultrafilters and large cardinals.

This characterization, when viewed with the focus on elementary embeddings, raises a point that will be even more germane, and thus will be emphasized later, in connection with strong hypotheses. That a $j\colon V \to M$ is elementary is not formalizable in set theory because of the appeal to the satisfaction relation for V, let alone the assertion that there is such a class j. Thus the "characterization" is really one of giving a formalization, one that provides operative sense through the ultrapower construction. In any event Ulam's original concept was thus made intrinsic to set theory with the categorical imperative of elementary embeddings. In any event ZFC is never actually transcended in consistency results; one can always work in a sufficiently large V_α through the Reflection Principle for ZF.

In Scott's $j\colon V \to M = \mathrm{Ult}(V,U)$ the concreteness of the ultrapower construction delivered ${}^\kappa M \subseteq M$, i.e. M is closed under the taking of arbitrary (in V) κ-sequences, so that in particular $V_{\kappa+1} \cap M = V_{\kappa+1}$. Through this agreement strong reflection conclusions can be drawn. U is *normal iff* $\pi((\mathrm{id})_U) = \kappa$, the identity function is a "least non-constant" function, a property that can be easily arranged. For such U, since κ is inaccessible, it is so in M and hence by Łoś's Theorem $\{\xi < \kappa \mid \xi \text{ is inaccessible}\} \in U$—the inaccessible cardinals below κ have measure one. An analogous argument applies to any $V_{\kappa+1}$ property of κ like weak compactness, and so, as would typify large cardinal hypotheses, measurability articulates its own sense of transcendence over "smaller" large cardinals.

Normality went on to become staple to the investigation of ideals and large cardinals. Formulated for an ideal I over a cardinal λ, I is *normal iff* whenever a function f is regressive on an $S \in P(\lambda) - I$, there is a $T \in P(S) - I$ on which f is constant. Fodor's Lemma is then just the assertion that the nonstationary ideal NS_λ is normal for regular $\lambda > \omega$, and a multitude of "smallness" properties other than nonstationarity has been seen to lead to normal ideals.

Through model-theoretic methods set theory was brought to the point of entertaining elementary embeddings into well-founded models. It was soon to be transfigured by a new means for getting well-founded *extensions* of well-founded models.

3. The Advent of Forcing

3.1. Cohen

Paul Cohen (1934-2007) in April 1963 established the independence of AC from ZF and the independence of CH from ZFC. That is, Cohen established that Con(ZF) implies Con(ZF + ¬AC) and Con(ZFC) implies Con(ZFC + ¬CH). Already prominent as an analyst, Cohen had ventured into set theory with fresh eyes and a open-mindedness about possibilities. These results solved two central problems of set theory. But beyond that, Cohen's proofs were the inaugural examples of a new technique, *forcing*, which was to become a remarkably general and flexible method for extending models of set theory. Forcing has strong intuitive underpinnings and reinforces the notion of set as given by the first-order ZF axioms with prominent uses of Replacement and Foundation. If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's method of forcing began its transformation into a modern, sophisticated one.

Cohen's approach was to start with a model M of ZF and adjoin a set G that witnesses some desired new property. This would have to be done in a minimal fashion in order that the resulting extension also model ZF, and so Cohen devised special conditions on both M and G. To be concrete, Cohen started with a countable transitive model $\langle M, \in \rangle$ of ZF. The ordinals of M would then coincide with the predecessors of some ordinal ρ, and M would be the cumulative hierarchy $M = \bigcup_{\alpha<\rho} V_\alpha \cap M$. Cohen recursively defined in M a system of terms (or "names") to denote members of the new model, working with a ramified language. In a streamlined rendition, for each $x \in M$ let $\check{x}$ be a corresponding constant; let $\dot{G}$ be a new constant; and for each $\alpha < \rho$ introduce quantifiers $\forall_\alpha$ and $\exists_\alpha$. Then define: $\dot{M}_0 = \{\dot{G}\}$, and for limit ordinals $\delta < \rho$, $\dot{M}_\delta = \bigcup_{\alpha<\delta} \dot{M}_\alpha$. At the successor stage, let $\dot{M}_{\alpha+1}$ be the collection of constants $\check{x}$ for $x \in V_\alpha \cap M$ and class terms corresponding to formulas allowing parameters from $\dot{M}_\alpha$ and quantifiers $\forall_\alpha$ and $\exists_\alpha$—a syntactical analogue of the operator def(x) for Gödel's L. Once a set G is provided from the outside, a model $M[G] = \bigcup_{\alpha<\rho} M_\alpha[G]$ would be determined by the terms.

But what properties can be imposed on G to ensure that $M[G]$ be a model of ZF? Cohen's key idea was to tie G closely to M through a partially ordered system of sets in M called *conditions* that would approximate G. While G may not be a member of M, G is to be a subset of some $Y \in M$ (with $Y = \omega$ a basic case), and these conditions would "force" some assertions about the eventual $M[G]$ e.g. by deciding some of the membership questions, whether $x \in G$ or not, for $x \in Y$. The assertions are to be just those expressible in the ramified language, and Cohen developed a corresponding *forcing relation* $p \Vdash \varphi$, "p forces φ", between conditions p and formulas φ, a relation with properties reflecting his approximation idea. For example, if $p \Vdash \varphi$ and $p \Vdash \psi$, then $p \Vdash \varphi \wedge \psi$. The conditions are ordered according to the constraints they impose on the eventual G, so that if $p \Vdash \varphi$, and q is a stronger condition, then $q \Vdash \varphi$. It was crucial to Cohen's approach that the forcing relation, like the ramified language, be definable in M.

The final ingredient which gives this whole scaffolding life is the incorporation of a certain kind of set G. Stepping out of M and making the only use of its countability, Cohen enumerated the formulas of the ramified language in a countable sequence and required that G be completely determined by a sequence of stronger and stronger conditions $p_0, p_1, p_2, \ldots$ such that for every formula φ of the ramified language exactly one of φ or $\neg\varphi$ is forced by some p_n. Such a G is called a *generic* set. The language is congenial; with the forcing conditions naturally topologized, a generic set meets every open dense set in M and is thus generic in a classical topological sense.

Cohen was able to show that the resulting $M[G]$ does indeed satisfy the axioms of ZF: Every assertion about $M[G]$ is already forced by some condition; the forcing relation is definable in M; and so the ZF axioms holding in M, most crucially Replacement and Foundation, can be applied to the ramified terms and language to derive corresponding forcing assertions about the ZF axioms holding in $M[G]$.

Cohen first described the case when $G \subseteq \omega$ and the conditions p are functions from some finite subset of ω into $\{0, 1\}$ and $p \Vdash \dot{n} \in \dot{G}$ if $p(n) = 1$ and $p \Vdash \dot{n} \notin \dot{G}$ if $p(n) = 0$. Today, a G so adjoined to M is called a *Cohen real over M*. If subsets of ω are identified with reals as traditionally construed, that G is generic can be extrinsically characterized by saying that G meets every open dense set of reals lying in M.

Generally, a $G \subseteq \kappa$ analogously adjoined with conditions of cardinality less than κ is called a *Cohen subset of κ*. Cohen established the independence of CH by adjoining a set which in effect is a sequence of many Cohen reals. It was crucial that the cardinals in the ground model and generic extension coincide, and with two forcing conditions said to be *incompatible* if they have no common, stronger condition, Cohen to this end drew out and relied on the important *countable chain condition* (c.c.c.): Any antichain, i.e. collection of mutually incompatible conditions, is countable.

Cohen established the independence of AC by a version of the above scheme, where in addition to $\dot{G}$ there are also new constants $\dot{G}_i$ for $i \in \omega$, and $\dot{G}$ is interpreted by a set X of Cohen reals, each an interpretation of some $\dot{G}_i$. The point is that X is not well-orderable in the extension, since there are permutations of the forcing conditions that induce a permutation of the G_i's yet leave X fixed.

Several features of Cohen's arguments would quickly be reformulated, reorganized, and generalized, but the thrust of his approach through definability and genericity would remain. Cohen's great achievement lies in devising a concrete procedure for extending well-founded models of set theory in a minimal fashion to well-founded models of set theory with new properties but without altering the ordinals.

The extent and breadth of the expansion of set theory described henceforth dwarfs all that has been described before, both in terms of the numbers of people involved and the results established, and we are left to paint with even broader strokes. With clear intimations of a new and concrete way of building models, set theorists rushed in and, with forcing becoming method, were soon establishing a cornucopia of relative consistency results, truths in a wider sense, with some illuminating classical problems of mathematics. Just in the first weeks after Cohen's discovery, Solomon Feferman, who had been extensively consulted by Cohen as he was coming up with forcing, established further independences elaborating ¬AC and about definability; Levy soon joined in this work and pursued both directions, formulating the "Levy collapse" of an inaccessible cardinal; and Stanley Tennenbaum established the failure of Suslin's Hypothesis by generically adjoining a Suslin tree. Soon, ZFC became quite unlike Euclidean geometry and much like group theory, with a wide range of models being investigated for their own sake.

3.2. Method of Forcing

Robert Solovay above all epitomized this period of sudden expansion in set theory with his mathematical sophistication and central results about and with forcing, and in the areas of large cardinals and descriptive set theory. Following initial graduate study in differential topology, Solovay turned to set theory after hearing a May 1963 lecture by Cohen. Just weeks after, Solovay elaborated the independence of CH by characterizing the possibilities for the size of 2^κ for regular κ and made the first exploration of a range of cardinals. Building on this William Easton in late 1963 established the definitive result for powers of regular cardinals: *Suppose that* GCH *holds and F is a class function from the class of regular cardinals to cardinals such that for regular $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$ and the cofinality* $\mathrm{cf}(F(\kappa)) > \kappa$. *Then there is a (class) forcing extension preserving cofinalities in which $2^\kappa = F(\kappa)$ for every* *regular* κ. Thus, as Solovay had seen locally, the *only* restriction beyond monotonicity on the power function for regular cardinals is that given by a well-known constraint, the classical Zermelo-Kőnig inequality that $\mathrm{cf}(2^\kappa) > \kappa$ for any cardinal κ. Easton's result enriched the theory of forcing with the introduction of proper classes of forcing conditions, the basic idea of a product analysis, and the now familiar concept of *Easton support*. The result focused interest on the possibilities for powers of *singular* cardinals and the Singular Cardinals Hypothesis (SCH), which asserts that 2^κ for singular κ is the least possible with respect to the powers 2^μ for $\mu < \kappa$ as given by monotonicity and the Zermelo-Kőnig inequality. This requires in particular that for singular strong limit cardinals κ, $2^\kappa = \kappa^+$. With Easton's models satisfying SCH, the *Singular Cardinals Problem*, to determine the range of possibilities for powers of singular cardinals, would become a major stimulus for the further development of set theory much as the continuum problem had been for its early development.

In the Spring of 1964 Solovay established a result remarkable for its mathematical depth and revelatory of what standard of argument was possible with forcing: *If there is an inaccessible cardinal, then in a* ZF *inner model of a forcing extension the Principle of Dependent Choices* (DC) *holds and every set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.* Solovay's inner model is precluded from having a well-ordering of the reals, but DC is a choice principle implying the regularity of ω_1 and sufficient for the formalization of the traditional theory of measure and category on the real numbers. Thus, Solovay's work vindicated the early descriptive set theorists in the sense that the regularity properties can consistently hold for all sets of reals in a *bona fide* model for the classical mathematical analysis of the reals. To prove his result Solovay applied the Levy collapse of an inaccessible cardinal to make it ω_1. For the Lebesgue measurability he introduced a new kind of forcing beyond Cohen's direct ways of adjoining new sets of ordinals or collapsing cardinals, that of adding a *random real* given by forcing with the Borel sets of positive measure as conditions and p stronger than q when $p - q$ is null. In contrast to Cohen reals, a random real meets every measure one subset of the unit interval lying in the ground model. Solovay's work not only opened the door to a wealth of different forcing arguments, but to this day his original definability arguments remain vital to descriptive set theory.

The perfect set property, central to Cantor's direct approach to the continuum problem through definability, led to the first acknowledged instance of a new phenomenon in set theory: the derivation of *equi-consistency* results between large cardinal hypotheses and combinatorial propositions about low levels of the cumulative hierarchy. Forcing showed just how relative the Cantorian concept of cardinality is, since bijective functions could be adjoined to models of set theory and powers like $2^{\aleph_0}$ can be made arbitrarily large with relatively little disturbance. For instance, large cardinals were found to satisfy substantial propositions even after they were "collapsed" to ω_1

or ω_2, i.e. a bijective function was adjoined to render the cardinal the first or second uncountable cardinals respectively. Conversely, such propositions were found to entail large cardinal propositions in an L-like inner model, mostly pointedly the very same initial large cardinal hypothesis. Thus, for some large cardinal property $\varphi(\kappa)$ and proposition ψ, there is a direction $\mathrm{Con}(\exists\kappa\varphi(\kappa)) \rightarrow \mathrm{Con}(\psi)$ established by a collapsing forcing argument, and a converse direction $\mathrm{Con}(\psi) \rightarrow \mathrm{Con}(\exists\kappa\varphi(\kappa))$ established by witnessing $\varphi(\kappa)$ in an inner model.

Solovay's result provided the forcing direction from an inaccessible cardinal to the proposition that every set of reals has the perfect set property and ω_1 is regular. But Ernst Specker in 1957 had in effect established that if this obtains, then ω_1 (of V) is inaccessible in L. Thus, Solovay's use of an inaccessible cardinal was actually necessary, and its collapse to ω_1 complemented Specker's observation. The emergence of such equi-consistency results is a subtle realization of earlier hopes of Gödel for deciding propositions via large cardinals. Forcing, however, quickly led to the conclusion that there could be no direct implication for CH itself: Levy and Solovay, also in 1964, established that measurable cardinals neither imply nor refute CH, with an argument generalizable to other inaccessible large cardinals. Rather, CH and many other propositions would be reckoned with in terms of consistency, the methods of forcing and inner models being the operative modes of argument.

Building on his Lebesgue measurability result Solovay in 1965 reactivated the classical descriptive set theory program of investigating the extent of the regularity properties (in the presence of AC) by providing characterizations in terms of forcing and definability concepts for the $\mathbf{\Sigma}^1_2$ sets, the level at which Gödel established from $V = L$ the failure of the properties. This led to the consistency relative to ZFC of the Lebesgue measurability of all $\mathbf{\Sigma}^1_2$ sets. Also, the characterizations showed that the regularity properties for $\mathbf{\Sigma}^1_2$ sets follow from existence of a measurable cardinal. Thus, although measurable cardinals do not decide CH, they do establish the perfect set property for $\mathbf{\Sigma}^1_2$ sets so that "CH holds for the $\mathbf{\Sigma}^1_2$ sets". A coda after many years: Although Solovay's use of an inaccessible cardinal for universal Lebesgue measurability seemed *ad hoc* at the time, in 1979 Saharon Shelah established in a *tour de force* that if ZF + DC and all Σ^1_3 sets of reals are Lebesgue measurable, then ω_1 is inaccessible in L.

In a separate initiative, Solovay in 1966 established the equi-consistency of the existence of a measurable cardinal and the "real-valued" measurability of $2^{\aleph_0}$, i.e. that there is a (countably additive) measure extending Lebesgue measure to all sets of reals. For the forcing direction, Solovay starting with a measurable cardinal adjoined random reals and applied the Radon-Nikodym Theorem of analysis, and for the converse direction, he starting with a real-valued measure enlisted the inner model constructed relative to the ideal of measure zero sets. This consistency result provided context for an extended investigation of the possibilities for the continuum as structured by such a measure. Through this work the concept of *saturated ideal*, first studied by

Tarski, was brought to prominence as a generalization of having a measurable cardinal applicable to the low levels of the cumulative hierarchy. For an ideal over a cardinal κ, I is λ-*saturated iff* for any $\{X_\alpha \mid \alpha < \lambda\} \subseteq P(\kappa) - I$ there are $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in P(\kappa) - I$ (i.e. the corresponding Boolean algebra has no antichains of cardinality λ). The ideal of measure zero sets is $\aleph_1$-saturated, and Solovay showed that if I is any κ-complete λ-saturated ideal over κ for some $\lambda < \kappa$, then $L[I] \models$ "κ is measurable".

Solovay's work also brought to the foreground the concept of *generic ultrapower* and *generic elementary embedding*. For an ideal I over κ, forcing with the members of $P(\kappa) - I$ as conditions and p stronger than q when $p - q \in I$ engenders an ultrafilter on the ground model $P(\kappa)$. With this one can construct an ultrapower of the ground model in the generic extension and a corresponding elementary embedding. It turns out that the κ^+-saturation of the ideal ensures that this generic ultrapower is well-founded. Thus, a synthesis of forcing and ultrapowers is effected, and this raised enticing possibilities for having such large cardinal-type structure low in the cumulative hierarchy.

The development of the theory of forcing went hand in hand with this procession of central results. Solovay had first generalized forcing to arbitrary partial orders of conditions, proceeding in terms of incompatible members and dense sets and Levy's concept of *generic filter*. In his work on the Baire property for his 1964 model, Solovay came to the idea of assigning values to formulas from a complete Boolean algebra. Loosely speaking, the value would be the supremum of all the conditions forcing it. Working independently, Solovay and Scott developed the idea of recasting forcing entirely in terms of *Boolean-valued models*. This approach showed how to replace Cohen's ramified languages by a more direct induction on rank and how to avoid his dependence on a countable model. Boolean-valued functions play the role of sets, and formulas involving these functions are assigned Boolean-values by recursion respecting logical connectives and quantifiers. By establishing in ZFC that e.g. there is a complete Boolean algebra assigning the formula expressing $\neg$CH Boolean value one, a semantic construction was replaced by a syntactic one that directly secured relative consistency.

Still, the view of forcing as a way of actually extending models held the reservoir of sense and the promise of discovery, and after Shoenfield popularized an approach to the forcing relation that captured the gist of the Boolean-valued approach, forcing has been generally cast as a matter of partial orders and generic filters. Boolean algebras would nonetheless underscore and enhance the setting: partial orders are to have a maximum element 1; one is attuned to the *separativity* of partial orders, the property that ensures that they are densely embedded in their canonical Boolean completions; Boolean-values are used when illuminating; and embedding results for forcing partial orders are cast, as most algebraically informative, in terms of Boolean algebras.

By the 1970s there would be a further assimilation of both the syntactic and semantic approaches in that generic extensions would be "taken" of V. In this the current approach then, a partial order $\langle P, < \rangle$ of conditions is specified to a purpose, with $p < q$ for p being stronger than q. A class V^P of P-names defined recursively is used in forcing assertions, with a canonical name $\check{x}$ corresponding to $x \in V$. A $D \subseteq P$ is *dense* if for any $p \in P$ there is a $d \in D$ with $d \leq p$. An $F \subseteq P$ is a *filter* if (i) if $p \in F$ and $p \leq q$, then $q \in F$, and (ii) if $p_1, p_2 \in F$ then there is an $r \in F$ with $r \leq p_1$ and $r \leq p_2$. Finally, $G \subseteq P$ is a *V-generic filter* if G is a filter such that for every dense $D \subseteq P$, $G \cap D \neq \emptyset$. One posits such a G and takes a generic extension $V[G]$, its properties argued for on the basis of combinatorial properties of P. For inner or transitive set models M, one proceeds analogously to define *M-generic filters* meeting every dense set belonging to M and takes generic extensions $M[G]$.

In this one goes against the sense of V as *the* universe of all sets and Tarski's "undefinability of truth", but actually V has become *schematic* for a ground model. Generic extensions of inner models M are taken with M-generic G, and moreover, successive iterated extensions are taken, exacerbating any preoccupation with a single universe of sets. As the techniques of forcing were advanced, the methodology was itself soon to be woven into set theory as part of its postulations.

Solovay and Tennenbaum earlier in 1965 had established the consistency of Suslin's Hypothesis, that there are no Suslin trees, illuminating a classical question from 1920 with a ground-breaking use of iterated forcing to keep "killing Suslin trees" in intermediate extensions. D. Anthony Martin pointed out that the Solovay-Tennenbaum argument actually established the consistency of a closure of forcing extensions of a certain kind, an instrumental "axiom" now known as Martin's Axiom (MA): *For any* c.c.c. *partial order P and collection $\mathcal{D}$ of fewer than $2^{\aleph_0}$ dense subsets of P, there is a filter $G \subseteq P$ meeting every member of $\mathcal{D}$.* Thus method became axiom, and many consistency results could now be simply stated as direct consequences of a single umbrella proposition. CH technically implies MA, but the Solovay-Tennenbaum argument established the consistency of MA with the continuum being arbitrarily large.

While classical results with CH had worked on an $\aleph_0 \,/\, \aleph_1$ dichotomy, MA established a $<2^{\aleph_0} / 2^{\aleph_0}$ dichotomy. For example, Martin and Solovay established that MA implies that the union of fewer than $2^{\aleph_0}$ Lebesgue measure zero sets is again Lebesgue measure zero. Sierpiński in 1925 had established that every $\mathbf{\Sigma}^1_2$ set of reals is the union of $\aleph_1$ Borel sets. Hence, MA and $2^{\aleph_0} > \aleph_1$ implies that every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable. Many further results plied the $<2^{\aleph_0} / 2^{\aleph_0}$ dichotomy to show that under MA inductive arguments can be carried out in $2^{\aleph_0}$ steps that previously succeeded under CH in $\aleph_1$ steps. The continuum problem was newly illuminated as a matter of method, by showing that CH as a construction principle could be generalized to $2^{\aleph_0}$ being arbitrarily large.

Glancing across the wider landscape, forcing provided new and diverse ways of adjoin generic reals and other sets, and these led to new elucidations, for example about cardinal characteristics, or invariants, of the continuum and combinatorial structures and objects, like ultrafilters over ω. The work on Suslin's Hypothesis in hand and the possibilities afforded by Martin's Axiom, the investigation of general topological notions gathered steam. With Mary Ellen Rudin and her students at Wisconsin breaking the ground, new questions were raised for general topological spaces about separation properties, compactness-type covering properties, separability and metrizability, and corresponding cardinal characteristics.

3.3. $0^\#$, $L[U]$, and $L[\mathcal{U}]$

The infusion of forcing into set theory induced a broad context extending beyond its applications and sustained by model-theoretic methods, a context which included central developments about large cardinals having their source in Scott's 1961 result that measurable cardinals contradict $V = L$. Haim Gaifman invented *iterated ultrapowers* and established seminal results about and with the technique, results which most immediately stimulated definitive work in the formative theses of Silver and Kunen.

Jack Silver in his 1966 Berkeley thesis provided a structured sense of transcendence over L in terms of the existence of a special set of natural numbers $0^\#$ ("zero sharp") which refined an earlier formulation of Gaifman and was quickly investigated by Solovay in terms of definability. Mostowski and Andrzej Ehrenfeucht in 1956 had developed theories whose models have *indiscernibles*, implicitly ordered members of the domain all of whose n-tuples satisfy the same formulas. They had applied Ramsey's Theorem in compactness arguments to get models generated by indiscernibles, models consequently having many automorphisms. Silver applied partition properties satisfied by measurable cardinals to produce indiscernibles *within* given structures, particularly in the initial segment $\langle L_{\omega_1}, \in \rangle$ of the constructible universe. With definability and Skolem hull arguments, Silver was able to isolate a canonical collection of sentences to be satisfied by indiscernibles, a theory whose models cohere to get L itself as generated by canonical ordinal indiscernibles—a dramatic accentuation of the original Gödel generation of L. $0^\#$ is that theory coded as a real, and as Solovay emphasized, $0^\#$ is the only possible real to satisfy a certain Π^1_2 relation, one whose complexity arises from its asserting that to every countable well-ordering there corresponds a well-founded model of the coded theory. The canonical class, closed and unbounded, of ordinal indiscernibles is often called the *Silver indiscernibles*. Having these indiscernibles substantiates $V \neq L$ in drastic ways: Each indiscernible ι has various large cardinal properties and satisfies $L_\iota \prec L$, so that by a straightforward argument the satisfaction relation for L is definable from $0^\#$. The theory of $0^\#$ was seen to relativize, and for reals $a \in {}^\omega\omega$ the analogous $a^\#$ for the inner model $L[a]$ would play focal roles in descriptive set theory as based on definability.

Kunen's main large cardinal results emanating from his 1968 Stanford thesis would be the definitive structure results for inner models of measurability. For U a normal κ-complete ultrafilter over a measurable cardinal κ, the inner model $L[U]$ of sets constructible relative to U is easily seen with $\overline{U} = U \cap L[U]$ to satisfy $L[U] \models$ "$\overline{U}$ is a normal κ-complete ultrafilter". With no presumption that κ is measurable (in V) and taking $U \in L[U]$ from the beginning, call $\langle L[U], \in, U \rangle$ a *κ-model iff* $\langle L[U], \in, U \rangle \models$ "U is a normal κ-complete ultrafilter over κ". Solovay observed that in a κ-model, the GCH holds above κ by a version of Gödel's argument for L and that κ is the only measurable cardinal by a version of Scott's argument. Silver then established that the full GCH holds, thereby establishing the relative consistency of GCH and measurability; Silver's proof turned on a local structure $L_\alpha[U]$ being *acceptable* in the later parlance of inner model theory.

Kunen made Gaifman's technique of iterated ultrapowers integral to the subject of inner models of measurability. For a κ-model $\langle L[U], \in, U \rangle$, the ultrapower of $L[U]$ by U with corresponding elementary embedding j provides a $j(\kappa)$-model $\langle L[j(U)], \in, j(U) \rangle$, and this process can be repeated. At limit stages, one can take the direct limit of models, which when well-founded can be identified with the transitive collapse. Indeed, by Gaifman's work these iterated ultrapowers are always well-founded, i.e. κ-models are *iterable*. Kunen showed that the λth iterate of a κ-model for any regular $\lambda > \kappa^+$ is of form $\langle L[\mathcal{C}_\lambda], \in, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda] \rangle$, where $\mathcal{C}_\lambda$ again is the closed unbounded filter over λ, so that remarkably, constructing relative to a filter definable in set theory leads to an inner model of measurability. With this, there can be *comparison* of κ-models and κ'-models by iterating them up to a sufficiently large λ. This comparison possibility let to the structure results: (1) *for any κ-model and κ'-model with $\kappa < \kappa'$, the latter is an iterated ultrapower of the former,* and (2) *for any κ, there is at most one κ-model.* It then followed that if κ is measurable and U_1 and U_2 are any κ-complete ultrafilters over κ, then $L[U_1] = L[U_2]$. These various results argued forcefully for the coherence and consistency of the concept of measurability. And it would be that iterability and comparison would remain as basic features in inner model theory in its subsequent development.

Kunen's contribution to the theory of iterated ultrapowers was that iterated ultrapowers can be taken of an inner model M with respect to an ultrafilter U even if $U \notin M$, as long U is an *M-ultrafilter*, i.e. U in addition to having M related ultrafilter properties also satisfies an "amenability" condition for M. A crucial dividend was a characterization of the existence of $0^\#$ that secured its central importance in inner model theory. With $0^\#$, any increasing shift of the Silver indiscernibles provides an elementary embedding $j \colon L \to L$. Kunen established conversely that such an embedding generates indiscernibles, so that $0^\#$ *exists iff there is a (non-identity) elementary embedding* $j \colon L \to L$. Starting with such an embedding Kunen defined the corresponding ultrafilter U over the critical point and showed that U is an L-ultrafilter with which the iterated ultrapowers of L are well-founded.

The successive images of the critical point were seen to be indiscernibles for L, giving $0^{\#}$. As inner model theory was to develop, this sharp analysis would become schematic: the "sharp" of an inner model M would encapsulate transcendence over M, and the *non-rigidity* of M, that there is a (non-identity) elementary embedding $j\colon M \to M$, would provide equivalent structural sense.

William Mitchell in 1972, just after completing a pioneering Berkeley thesis on Aronszajn trees, provided the first substantive extension of Kunen's inner model results and brought to prominence a new large cardinal hypothesis. For normal κ-complete ultrafilters U and U' over κ, define the *Mitchell order* $U' \lhd U$ *iff* $U' \in \mathrm{Ult}(V, U)$, i.e. there is an $f\colon \kappa \to V$ representing U' in the ultrapower, so that $\{\alpha < \kappa \mid f(\alpha)$ is a normal α-complete ultrafilter over $\alpha\} \in U$ and κ is already a limit or measurable cardinals. $U \lhd U$ always fails, and generally, $\lhd$ is a well-founded relation by a version of Scott's argument that measurable cardinals contradict $V = L$. Consequently, to each U can be recursively assigned a rank $o(U) = \sup\{o(U') + 1 \mid U' \lhd U\}$, and to a cardinal κ, the supremum $o(\kappa) = \sup\{o(U) + 1 \mid U$ is a normal κ-complete ultrafilter over $\kappa\}$. By a cardinality argument, if $2^\kappa = \kappa^+$ then $o(\kappa) \leq \kappa^{++}$.

The hypothesis $o(\kappa) = \delta$ provided an "order" of measurability calibrated by δ, with larger δ corresponding to stronger assumptions on κ. For the investigation of these orders, Mitchell devised the concept of a *coherent* sequence of ultrafilters ("measures") and was able to establish canonicity results for inner models $L[\mathcal{U}] \models$ "$\mathcal{U}$ is a coherent sequence of ultrafilters". A coherent sequence $\mathcal{U}$ is a doubly indexed system of normal α-complete ultrafilters $\mathcal{U}(\alpha, \beta)$ over α such that $\mathcal{U}(\kappa, \beta) \lhd \mathcal{U}(\kappa, \beta')$ for $\beta < \beta'$ at the κth level, and the earlier levels contain just enough ultrafilters necessary to represent these $\lhd$ relationships in the respective ultrapowers. (Technically, if $j\colon V \to \mathrm{Ult}(V, \mathcal{U}(\kappa, \beta'))$, then $j(\mathcal{U}){\restriction}\{(\alpha, \beta) \mid \alpha \leq \kappa\} = \mathcal{U}{\restriction}\{(\alpha, \beta) \mid \alpha < \kappa \vee (\alpha = \kappa \wedge \beta < \beta')\}$, i.e. $j(\mathcal{U})$ through κ is exactly $\mathcal{U}$ "below" (κ, β').)

Mitchell first affirmed that these $L[\mathcal{U}]$'s are *iterable* in that arbitrary iterated ultrapowers via ultrafilters in $\mathcal{U}$ and its successive images are always well-founded. He then effected a comparison: *Any $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ have respective iterated ultrapowers $L[\mathcal{W}_1]$ and $L[\mathcal{W}_2]$ such that $\mathcal{W}_1$ is an initial segment of $\mathcal{W}_2$ or vice versa.* This he achieved through a process of *coiteration* of least differences: At each stage, one finds the lexicographically least coordinate at which the current iterated ultrapowers of $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ differ and takes the respective ultrapowers by the differing ultrafilters; the difference is eliminated as ultrafilters never occur in their ultrapowers. Note that this iteration process is external to $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$, further drawing out the advantages of working externally to models as Kunen had first done with his M-ultrafilters. With this coiteration, Mitchell established that in $L[\mathcal{U}]$ the only normal α-complete ultrafilters over α for any α are those that occur in $\mathcal{U}$ and other propositions like GCH that showed these models to be L-like. Coiteration would henceforth be embedded in inner model theory, and with his models $L[\mathcal{U}]$ modeling $o(\kappa) = \delta$ for $\delta < \kappa^{++L[\mathcal{U}]}$, $\exists\kappa(o(\kappa) = \kappa^{++})$ would become the delimitative proposition of his analysis.

3.4. Constructibility

These various results were set against a backdrop of an increasing articulation of Gödel's original notion of constructibility. Levy in 1965 had put forward the appropriate hierarchy for the first-order formulas of set theory: A formula is Σ_0 and Π_0 if it is bounded, i.e. having only quantifiers expressible in terms of $\forall v \in w$ and $\exists v \in w$, and recursively, a formula is Σ_{n+1} if it is of the form $\exists v_1 \dots \exists v_k \varphi$ where φ is Π_n and is Π_{n+1} if it is of the form $\forall v_1 \dots \forall v_k \varphi$ where φ is Σ_n. Two basic points about discounting bounded quantifiers are that Σ_0 formulas are absolute for transitive structures, i.e. they hold in such structures just in case they hold in V, and that if φ is Σ_n (resp. Π_n) then $\exists v \in w\varphi$ and $\forall v \in x\varphi$ are equivalent in ZFC to Σ_n (resp. Π_n) formulas by uses of Replacement. Levy wove in Shoenfield's Σ^1_2 absoluteness result to establish the Shoenfield-Levy Absoluteness Lemma: *For any* Σ_1 *sentence* σ, ZF + DC $\vdash \sigma \longleftrightarrow \sigma^L$. Levy actually showed that L here can be replaced by a countable L_γ fixed for all σ, and as such the lemma can be seen as a refinement of the Reflection Principle for ZF, one that was to find wide use in the burgeoning field of admissible set theory.

Gödel's original GCH result with L was newly seen in light of the structured context for definability. For N and M construed as structures with $\in$, $j\colon N \to M$ is a Σ_n*-elementary embedding iff* for any Σ_n $\varphi(v_1, \dots, v_k)$ and $x_1, \dots, x_k \in N$, $N \models \varphi[x_1, \dots, x_k]$ *iff* $M \models \varphi[j(x_1), \dots, j(x_k)]$. N is a Σ_n*-elementary substructure* of M, denoted $N \prec_n M$, *iff* the identity map is Σ_n-elementary. Analysis of the satisfaction relation established that being an L_α is a Σ_1 property, and this led to the Condensation Lemma:

> If α is a limit ordinal and $N \prec_1 L_\alpha$,
> then the transitive collapse of N is L_β for some $\beta \leq \alpha$.

Operatively, one applies this lemma with Skolem's algebraic approach to logic by taking N to be a Σ_1 *Skolem hull* in L_α: For any Σ_0 formula $\varphi(v_1, \dots, v_n, v_{n+1})$ and $x_1, \dots, x_n \in L_\alpha$, if $\langle L_\alpha, \in \rangle \models \varphi[x_1, \dots x_n, y]$ for $y \in L_\alpha$, let $f_\varphi(x_1, \dots, x_n)$ be such a y. Then let N be the algebraic closure of some subset of L_α under these Skolem functions. The road from the Condensation Lemma to Gödel's Fundamental Theorem for the consistency of GCH is short. Generally, the lemma articulates a crucial hierarchical cohesion, and its various emanations would become fundamental to all inner model theory.

The consummate master of constructibility was to be Ronald Jensen, whose first systematic analysis transformed the subject with the introduction of the *fine structure theory* for L. Jensen's work is distinguished by the persistent pursuit of internal logical structure, the sophistication of the local apparatus developed, and a series of remarkable successes with reverberations throughout the whole expanse of set theory. After his 1964 Bonn dissertation on models of arithmetic, Jensen moved with strength into investigations with forcing and of definability, two directions that would steadily complement each other in his work. He, like Solovay, saw the great potential of forcing, and he soon derived the Easton results independently. In the direction of definability he in 1965 worked out with Carol Karp a theory of primitive recursive set functions, and with these he began his investigation of L. By 1966 he had realized the importance of Σ_n Uniformization for $n > 1$, central to fine structure, although notably he had no particular application for it in mind at that time.

In 1968 Jensen made a major breakthrough by showing that $V = L$ *implies the failure of Suslin's Hypothesis*, i.e. (there is a Suslin tree)^L, applying L for the first time after Gödel to establish a relative consistency result about a classical proposition. The initial breakthrough had been when Tennenbaum had adjoined a Suslin tree with forcing and Thomas Jech had provided another forcing argument; Jensen at first pitched his construction in the guise of a forcing argument, one in fact like Jech's. This is the paradigmatic case of what would become a recurring phenomenon: A combinatorial existence assertion is first shown to be relatively consistent with ZFC using forcing, and then the assertion is shown to hold in L, the minimal inner model.

The lack of cofinal branches in Suslin trees is complemented by their abundance in Kurepa trees. Inspired by Jensen's construction the ubiquitous Solovay established: (there is a Kurepa tree)^L. Here too the relative consistency of the proposition had been established first through forcing.

Jensen isolated the combinatorial features of L that enabled these constructions and together with Kunen in 1969 worked out a larger theory. The focus was mainly on two combinatorial principles of Jensen's for a regular cardinal κ, $\diamondsuit_\kappa$ ("diamond") and a strengthening, $\diamondsuit^+_\kappa$ ("diamond plus"). Stating the first,

$\diamondsuit_\kappa$ There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ such that for any $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\}$ is stationary in κ .

Just $\diamondsuit$ is implicitly $\diamondsuit_{\omega_1}$. $\diamondsuit_\kappa$ implies $\bigcup_{\alpha<\kappa} \kappa^{|\alpha|} = \kappa$ (so that $\diamondsuit$ implies CH) as every bounded subset of κ occurs in a $\diamondsuit_\kappa$ sequence. Indeed, a $\diamondsuit_\kappa$ sequence is an enumeration of the bounded subsets of κ that can accommodate every $X \subseteq \kappa$ in anticipatory constructions where $X \cap \alpha$ appearing in the enumeration for many α's suffices. Within a few years $\diamondsuit$ would be on par with CH as a construction principle with wide applications in topology, algebra, and analysis. (Another coda of Shelah's after many years: In 2007 he established that for successors $\lambda^+ > \omega_1$, $2^\lambda = \lambda^+$ actually implies $\diamondsuit_{\lambda^+}$, so that the two are equivalent.)

Jensen abstracted his Suslin tree result to: (1) *if* $V = L$, *then* $\diamondsuit_\kappa$ *holds from every regular* $\kappa > \omega$, and (2) *if* $\diamondsuit_{\omega_1}$ *holds, then there is a Suslin tree.* Solovay's result was abstracted to higher, κ-*Kurepa trees*, κ-trees with at least κ^+ cofinal branches, in terms of a new cardinal concept, *ineffability*, arrived at independently by Jensen and Kunen: *If* $V = L$ *and* $\kappa > \omega$ *is regular,* *then $\diamondsuit^+_\kappa$ holds iff κ is not ineffable.* Ineffable cardinals, stronger than weakly compact cardinals, would soon be seen to have a range of involvements and an elegant theory. As for "higher" Suslin trees, they would involve the use of a new combinatorial principle, one that first figured in a sophisticated forcing argument.

The crowning achievement of the investigation of Suslin's Hypothesis was its joint consistency with CH, Con(ZFC) $\rightarrow$ Con(ZFC + CH + SH), established by Jensen. In the Solovay-Tennenbaum consistency proof for SH, cofinal branches had been adjoined iteratively to Suslin trees as they arose and direct limits were taken at limit stages, a limiting process that conspired to adjoin new reals so that CH fails. Jensen, with considerable virtuosity for the time, devised a way to kill Suslin trees less directly and effected the iteration according to a curtailed tree-like indexing—so that no new reals are ever adjoined. That indexing is captured by the $\kappa = \omega_1$ case of the combinatorial principle $\square_\kappa$ ("square"):

$\square_\kappa$ There is a sequence $\langle C_\alpha \mid \alpha \text{ a limit ordinal } < \kappa^+\rangle$ such that for $\alpha < \kappa^+$:

(a) $C_\alpha \subseteq \alpha$ is closed unbounded in α,
(b) for β a limit point of C_α, $C_\alpha \cap \beta = C_\beta$, and
(c) for $\omega \leq \mathrm{cf}(\alpha) < \kappa$, the order-type of C_α is less than κ.

$\square_\omega$ is immediate, as witnessed by any ladder system, i.e. a sequence $\langle C_\alpha \mid \alpha \text{ a limit ordinal } < \omega_1\rangle$ such that C_α is of order-type ω and cofinal in α. $\square_\kappa$ for $\kappa > \omega$ brings out the tension between the desired (b) and the needed (c). As such, $\square_\kappa$ came to guide many a construction of length κ^+ based on components of cardinality $< \kappa$.

$\square_\kappa$ can be adjoined by straightforward forcing with initial approximations; Jensen established: *If $V = L$, then $\square_\kappa$ holds for every κ.* As for higher Suslin trees, a *κ-Suslin tree* is expectedly a κ-tree with no chains or antichains of cardinality κ. Jensen established, generalizing his result for $\kappa = \omega_1$: (1) *for any κ, $\diamondsuit_{\kappa^+}$ and $\square_\kappa$ imply that there is a κ^+-Suslin tree,* and, for limit cardinals κ, the characterization (2) *there is a κ-Suslin tree iff κ is not weakly compact.* It is a notable happenstance that Suslin's early, 1920 speculation would have such extended ramifications in modern set theory.

Jensen's results that $\square_\kappa$ holds in L and (2) above were the initial applications of his fine structure theory. Unlike Gödel who had focused with L on relative consistency, Jensen regarded the investigation of how the constructible hierarchy grows by examining its behavior at arbitrary levels as of basic and intrinsic interest. And with his fine structure theory Jensen developed a considerable and intricate machinery for this investigation. A pivotal question became: when does an ordinal α first get "singularized", i.e. what is the least β such that there is in $L_{\beta+1}$ an unbounded subset of α of smaller order-type, and what definitional complexity does this set have? One is struck by the contrast between Jensen's attention to such local questions as this one, at the heart of his proof of $\square_\kappa$, and how his analysis could lead to major large-scale results of manifest significance.

For a uniform development of his fine structure theory, Jensen switched from the hierarchy of L_α's to a hierarchy of J_α's, the *Jensen hierarchy*, where $J_{\alpha+1}$ is the closure of $J_\alpha \cup \{J_\alpha\}$ under the "rudimentary" functions (the primitive recursive set functions without the recursion scheme). For $L[A]$, there is an analogous hierarchy of J^A_α where one also closes off under the function $x \longmapsto A \cap x$. For a set N, construed as a structure with $\in$ and possibly with some $A \cap N$ as a predicate, a relation is $\Sigma_n(N)$ *iff* it is first-order definable over N by a Σ_n formula. For every α, both $\langle J_\xi \mid \xi < \alpha\rangle$ and a well-ordering $<_L$ of L restricted to J_α are Σ_1 definable over J_α *uniformly*, in that the same formula works for all the J_α's.

In these terms, fine structure addresses the classical issue of Skolem functions through definability. For $(k+1)$-ary relations R and S,

$$R \text{ is } \textit{uniformized} \text{ by } S \textit{ iff}$$
$$S \subseteq R \text{ and } \forall \mathbf{w}(\exists y R(\mathbf{w}, y) \longleftrightarrow \exists! y S(\mathbf{w}, y)),$$

where $\exists!$ is "there exists exactly one". This amounts to the assertion that S refines R to a function on the same $\mathbf{w}$'s and is thus a form of AC. In systematic applications of the Condensation Lemma one deduces, toward the construction of Σ_1 Skolem hulls, that $\Sigma_0(J_\alpha)$, and hence $\Sigma_1(J_\alpha)$, relations are uniformizable by $\Sigma_1(J_\alpha)$ relations that choose $<_L$-least witnesses. Weaving together all such relations into one universal one, one gets a Skolem function h Σ_1 definable over J_α uniformly, with the property that for any $X \subseteq J_\alpha$ an application of h to X yields an elementary substructure of J_α.

What about $\Sigma_2(J_\alpha)$ relations? Choosing $<_L$-least witnesses as before leads only to a $\Sigma_3(J_\alpha)$ uniformizing relation, since asserting that no predecessor in the Σ_1 definable well-ordering satisfies the Σ_2 formula adds to the quantifier complexity. Jensen saw that a palatable analysis of definability stable through various transformations would require a $\Sigma_2(J_\alpha)$ uniformizing relation. He achieved this by applying the basic elements of fine structure: As a measure of the lack of closure under definability, let the *(first) projectum* $\rho_\alpha \leq \alpha$ be the least γ for which there is a $\Sigma_1(J_\alpha)$ subset of γ which is not a member of J_α. There is then a $\Sigma_1(J_\alpha)$ map of a subset of J_{ρ_α} onto J_α, essentially a Skolem function as in the previous paragraph. The $\Sigma_1(J_\alpha)$ definitions involved here can be construed as depending on one parameter in J_α, and one can fix the $<_L$-least possibility—the *standard parameter*. With this one can consider the *projectum structure* $\langle J_{\rho_\alpha}, A_\alpha\rangle$ where A_α the *standard code*—the $<_L$-least among certain *master codes*—a predicate that codes Σ_1 satisfaction for J_α so that the part of any $\Sigma_2(J_\alpha)$ relation in J_{ρ_α} can be taken to be a $\Sigma_1(\langle J_{\rho_\alpha}, A_\alpha\rangle)$ relation. The relation can then be uniformized by a $\Sigma_1(\langle J_{\rho_\alpha}, A_\alpha\rangle)$ function, one that can subsequently be projected up to be a $\Sigma_2(J_\alpha)$ uniformizing function with the available $\Sigma_1(J_\alpha)$ mapping of a subset of J_{ρ_α} onto J_α.

The foregoing sets out the salient features of fine structure theory, and Jensen carried out this analysis in general to establish for every $n \geq 1$ the

Σ_n Uniformization Theorem: *For every* α*, every* $\Sigma_n(J_\alpha)$ *relation can be uniformized by a* $\Sigma_n(J_\alpha)$ *relation.* In truth, as often with the thrust of method, fine structure would become autonomous in that it would be the actual fine structure workings of this lemma, rather than just its statement, which would be used. Jensen also gave expression to canonicity with what is now known as the *Downward Extension of Embeddings Lemma*, which for the foregoing situation asserts that if $e\colon N \to \langle J_{\rho_\alpha}, A_\alpha\rangle$ is Σ_0-elementary, then N itself is the projectum structure of a unique J_β and e can be extended uniquely to a Σ_1-elementary $\overline{e}\colon J_\beta \to J_\alpha$. Jensen moved forward with this fine structure theory to uncover and articulate the combinatorial structure of the constructible universe.

4. Strong Hypotheses

4.1. Large Large Cardinals

With elementary embedding having emerged as a systemic concept in set theory, Solovay and William Reinhardt at Berkeley in the late 1960s formulated inter-related large cardinal hypotheses stronger than measurability. Reinhardt conceived *extendibility*, and he and Solovay independently, *supercompactness*. A cardinal κ is γ*-supercompact iff* there is an elementary embedding $j\colon V \to M$ for some inner model M, with critical point κ and $\gamma < j(\kappa)$ such that ${}^\gamma M \subseteq M$, i.e. M is closed under the taking of arbitrary γ-sequences. κ is *supercompact iff* κ is γ-supercompact for every γ. Evidently, the heuristics of generalization and reflection were at work here, as κ is measurable *iff* κ is κ-supercompact, and stronger closure properties imposed on the target model M ensures stronger reflection properties. For example, if κ is 2^κ-supercompact with witnessing $j\colon V \to M$, then $M \models$ "κ is measurable", since ${}^{2^\kappa}M \subseteq M$ implies that every ultrafilter over κ is in M, and so if $U_j \subseteq P(\kappa)$ is defined canonically from j by $X \in U_j$ *iff* $\kappa \in j(X)$, then $\{\xi < \kappa \mid \xi \text{ is measurable}\} \in U_j$ by Łoś's Theorem. Supercompactness was initially viewed as an ostensible strengthening of Tarski's strong compactness in that, with the focus on elementary embedding, reflection properties were directly incorporated. Whether strong compactness is actually equivalent to supercompactness became a new "identity crisis" issue.

Reinhardt entertained a *prima facie* extension of these ideas, that there is a (non-identity) elementary embedding $j\colon V \to V$. With suspicions soon raised, Kunen dramatically established in 1970 that this is inconsistent with ZFC by applying an Erdős-Hajnal partition relation result, a combinatorial contingency making prominent use of the Axiom of Choice. This contingency pointed out a specific lack of closure of the target model: *For any elementary embedding* $j\colon V \to M$ *with critical point* κ*, let* λ *be the supremum of* $\kappa < j(\kappa) < j^2(\kappa) < \ldots$*. Then,* $V_{\lambda+1} \not\subseteq M$. This lack of closure has essentially stood as the weakest known to this day.

A net of hypotheses consistency-wise stronger than supercompactness was soon cast across the conceptual space delimited by Kunen's inconsistency. For $n \in \omega$, κ is *n-huge iff* there is an elementary embedding $j\colon V \to M$, for some inner model M, with critical point κ such that ${}^{j^n(\kappa)}M \subseteq M$. κ is *huge iff* κ is 1-huge. If κ is huge, then $V_\kappa \models$ "there are many supercompact cardinals". Thematically close to Kunen's inconsistency were several hypotheses articulated for further investigation, e.g. there is a (non-identity) elementary embedding $j\colon V_\lambda \to V_\lambda$ for some λ.

The appearance of proper classes in these various formulations raises issues about legitimacy. By Tarski's "undefinability of truth", the satisfaction relation for V is not definable in ZFC, and the elementary embedding characterization of measurability already suffers from this shortcoming. However, the γ-supercompactness of κ can be analogously formulated in terms of the existence of a "normal" ultrafilter over the set $P_\kappa\gamma = [\gamma]^{<\kappa} = \{x \subseteq \gamma \mid |x| < \kappa\}$. Similarly, n-hugeness can also be recast. As for Kunen's inconsistency, his argument can be regarded as establishing: *There is no (non-identity) elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$ for any λ.*

The details on γ-supercompactness drew out new, generalizing concepts for filters (and so, for ideals). Suppose that Z is a set and F a filter over $P(Z)$ (so $F \subseteq P(P(Z))$). Then F is *fine iff* for any $a \in Z$, $\{x \in P(Z) \mid a \in x\} \in F$. F is *normal iff* whenever f is a function satisfying $\{x \in P(Z) \mid f(x) \in x\} \in F$, i.e. f is a choice function on a set in F, there is an $a \in Z$ such that $\{x \in P(Z) \mid f(x) = a\} \in F$, i.e. f is constant on a set in F. When Z is a cardinal κ and $\kappa = \{x \in P(\kappa) \mid x \in \kappa\} \in F$, then this new normality reduces to the previous concept. With an analogous reduction to filters over $P_\kappa\gamma = [\gamma]^{<\kappa} = \{x \in P(\gamma) \mid |x| < \kappa\}$, we have the formulation: κ is γ-supercompact *iff* there is a κ-complete, fine, normal ultrafilter over $P_\kappa\gamma$. This inspired a substantial combinatorial investigation of filters over sets $P_\kappa\gamma$, and a general, structural approach to filters over sets $P(Z)$.

Whether it is in these large cardinal hypotheses or the transition from V to $V[G]$ in forcing, the appeal to the satisfaction relation for V is liberal and unabashed in modern set-theoretic practice. Yet ZFC remains parsimoniously the official theory and this carries with it the necessary burden of formalization. On the other hand, it is the formalization that henceforth carries the operative sense; for example, the ultrafilter characterization of γ-supercompactness delivers through the concreteness of the ultrapower construction critical properties that become part of the concept in its use. It has become commonplace in modern set theory that informal assertions and schematic procedures often convey an incipient intentional sense, but formalization refines that sense with workable structural articulations.

Although large large cardinals were developed particularly to investigate the possibilities for elementary embeddings and were quickly seen to have a simple but elegant basic theory, what really intimated their potentialities were new forcing results in the 1970s and 1980s, especially from supercompactness, that established new relative consistencies, even of assertions low in the cumulative hierarchy. The earliest, orienting result along these lines addressed the singular cardinals problem. The "Prikry-Silver" result provided the first instance of a failure of the Singular Cardinal Hypothesis by drawing together two results of independent significance, themselves crucial as methodological advances.

Karel Prikry in his 1968 Berkeley thesis had set out a simple but elegant notion of forcing that changed the cofinality of a measurable cardinal while not collapsing any cardinals. With U a normal κ-complete ultrafilter over κ, *(basic) Prikry forcing* for U has as conditions $\langle p, A\rangle$ where p is a finite subset of κ and $A \in U$. For conditions $\langle p, A\rangle$ and $\langle q, B\rangle$, the first is stronger than the second if $p \supseteq q$ and $\alpha \in p - q$ implies $\alpha > \max(q)$, and $A \cup (p - q) \subseteq B$. A condition thus specifies a finite initial part of a new ω-cofinalizing subset of κ, and further members are to be added on top from a set large in the sense of being in U. Applying a partition property available for normal ultrafilters, Prikry established that for any condition $\langle p, A\rangle$ and forcing statement, there is a $B \subseteq A$ such that $B \in U$ and $\langle p, B\rangle$ decides the statement, i.e. extending p is unnecessary. Hence, e.g. the κ-completeness of U implies that V_κ remains unchanged in the forcing extension yet the cofinality of κ now becomes ω.

Prikry forcing may at first have seemed a curious possibility for singularization. However, that a Prikry generic sequence also generates the corresponding U in simple fashion and also results from indiscernibles made them a central feature of measurability. Prikry forcing would be generalized in various directions and for a variety of purposes. With the capabilities made available for changing cofinalities, equi-consistency connections would eventually be established between large cardinals on the one hand and formulations in connection with the Singular Cardinals Problem on the other.

Silver in 1971 first established the relative consistency of having a measurable cardinal κ satisfying $2^\kappa > \kappa^+$, a proposition that Kunen had shown to be substantially stronger than measurability. Forcing over the model constructed by Silver with Prikry forcing yielded the first counterexample to the Singular Cardinals Hypothesis by providing a singular strong limit cardinal κ satisfying $2^\kappa > \kappa^+$.

To establish his result, Silver provided a technique for extending elementary embeddings into generic extensions and thereby preserving large cardinal properties. To get at what is at issue, suppose that $j\colon V \to M$ is an elementary embedding, P is a notion of forcing, and G is V-generic for P. To extend (or "lift") j to an elementary embedding for $V[G]$, the natural scheme would be to get a M-generic G' for $j(P)$ and extend j to an elementary embedding from $V[G]$ into $M[G']$. But for this to work with the forcing terms, it would be necessary to enforce

$$(*) \qquad \forall p \in G\,(j(p) \in G')\,.$$

For getting a measurable cardinal κ satisfying $2^\kappa = \kappa^{++}$, Silver started with an elementary embedding as above with critical point κ and devised a P for adjoining κ^{++} Cohen subsets of κ. In order to establish a close connection between P and $j(P)$ toward securing $(*)$, he took P to be a uniform iteration of forcings to adjoin λ^{++} Cohen subsets of λ for every inaccessible cardinal λ up to and including κ itself. Then with the shift from κ to $j(\kappa)$, $j(P)$ can be considered a two-stage iteration of P followed by a further iteration Q. Now with G V-generic for P, G is also M-generic for P, and in $M[G]$ one should devise an H $M[G]$-generic for Q such that the combined generic $G' = G * H$ satisfies $(*)$.

But how is this to be arranged? Silver was able to control the $j(p)$'s for $p \in G$ by a single, *(strong) master condition* $q \in Q$, and build in $V[G]$ an H $M[G]$-generic over Q with $q \in H$ to satisfy $(*)$. For getting both q and H, he needed that M be closed under arbitrary κ^{++}-sequences. Thus he established: *If κ is κ^{++}-supercompact, then there is a forcing extension in which κ is measurable and* $2^\kappa = \kappa^{++}$. (To mention an elegant coda, work of Woodin and Gitik in the 1980s showed that having a measurable cardinal satisfying $2^\kappa > \kappa^+$ is equi-consistent with having a κ with $o(\kappa) = \kappa^{++}$ in the Mitchell order.) Silver's preparatory "reversed Easton" forcing with Easton support and master condition constructions of generic filters would become staple ingredients for the generic extension of elementary embeddings.

What about the use of very strong hypotheses in consistency results? A signal, 1972 result of Kunen brought into play the strongest hypothesis to that date for establish a consistency result about the low levels of the cumulative hierarchy. Earlier, Kunen had established that having a κ-complete κ^+-saturated ideal over a successor cardinal κ had consistency strength stronger than having a measurable cardinal. Kunen now showed: *If κ is huge, then there is forcing extension in which $\kappa = \omega_1$ and there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1.* With a $j\colon V \to M$ with critical point κ, $\lambda = j(\kappa)$, and ${}^\lambda M \subseteq M$ as given by the hugeness of κ, Kunen collapsed κ to ω_1 and followed it was a collapse of λ to ω_2 in such a way so as to be able to define a saturated ideal. Crucially, the first collapse was a "universal" collapse P iteratively constructed so that the second collapse can be absorbed into $j(P)$ in a way consistent with j applied to P, and this required ${}^\lambda M \subseteq M$. Hence, a sufficient algebraic argument was contingent on a closure property for an elementary embedding, one plucked from the emerging large cardinal hierarchy. In the years to come, Kunen's argument would be elaborated and emended to become the main technique for getting various sorts of saturated ideals over accessible cardinals. As for the proposition that there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 itself, Kunen's result set an initial high bar for the stalking of its consistency strength, but definitive work of the 1980s would show that far less than hugeness suffices.

4.2. Determinacy

The investigation of the determinacy of infinite games is the most distinctive and intriguing development of modern set theory, and the correlations eventually achieved with large cardinals the most remarkable and synthetic. Notably, the mathematics of games first came to the attention of pioneers of set theory as an application of the emerging subject. Zermelo in a 1913 note discussed chess and worked with the concepts of *winning strategy* and *determined game*, and Kőnig in the paper that introduced his well-known tree lemma extended Zermelo's work to games with infinitely many positions. Von Neumann, lauding set-theoretic formulation, established the crucial minimax theorem, the result that really began the mathematical theory of games, and by the mid-1940s he and Oskar Morgenstern had codified the theory and its analysis of economic behavior, stimulating research for decades to come.

The investigation of infinitely long games that can be cast in a simple, abstract way would draw game-theoretic initiatives back into set theory. For a set X and $A \subseteq {}^{\omega}X$, let $G_X(A)$ denote the following "infinite two-person game with perfect information": There are two players, I and II. I initially chooses an $x(0) \in X$; then II chooses an $x(1) \in X$; then I chooses an $x(2) \in X$; then II chooses an $x(3) \in X$; and so forth:

$$\begin{array}{lccccc} I\ : & x(0) & & x(2) & & \ldots \\ II: & & x(1) & & x(3) & \ldots \end{array}$$

Each player before making each of his moves is privy to the sequence of previous moves ("perfect information"); and the players together specify an $x \in {}^{\omega}X$. I wins $G_X(A)$ if $x \in A$, and otherwise II wins. A *strategy* is a function that tells a player what move to make given the sequence of previous moves. A *winning strategy* is a strategy such that if a player plays according to it he always wins no matter what his opponent plays. A is *determined* if either I or II has a winning strategy in $G_X(A)$.

David Gale and James Stewart in 1953 initiated the study of these games and observed that *if $A \subseteq {}^{\omega}X$ is open (in the product topology) then A is determined.* The simple argument turned on how membership is secured at a finite stage, and a basic stratagem in the further investigations of determinacy would be the reduction to such "open games". Focusing on the basic case $X = \omega$ and noting that a strategy then can itself be construed as a real, Gale and Stewart showed by diagonalizing through all strategies that *assuming* AC *there is an undetermined $A \subseteq {}^{\omega}\omega$.* Determinacy itself would come to be regarded as a regularity property, but there were basic difficulties from the beginning. Gale and Stewart asked whether all Borel sets of reals are determined, and in the decade that followed only sets very low in the Borel hierarchy were shown to be determined.

Infinitely long games involving reals had been considered as early as in the 1920s by mathematicians of the Polish school. With renewed interest in the subject in the 1950s, and with determinacy increasingly seen to be potent in its consequences, Jan Mycielski and Hugo Steinhaus in 1962 proposed the following axiom, now known as the *Axiom of Determinacy* (AD):

$$\text{Every } A \subseteq {}^{\omega}\omega \text{ is determined.}$$

With AD contradicting AC they proposed from the beginning that in the ZFC context the axiom should hold in some inner model. Solovay pointed out that the natural candidate $L(\mathbb{R})$, the constructible closure of the reals $\mathbb{R} = {}^{\omega}\omega$, observing that if AD holds then $\mathrm{AD}^{L(\mathbb{R})}$, i.e. AD holds in $L(\mathbb{R})$. Further restricted hypotheses would soon be applied to the tasks at hand: Projective Determinacy (PD) asserts that every projective $A \subseteq {}^{\omega}\omega$ is determined; $\mathbf{\Sigma}^1_n$-determinacy, that every $\mathbf{\Sigma}^1_n$ set A is determined; and so forth.

By 1964, games to specific purposes had been devised to show that for $A \subseteq {}^{\omega}\omega$ there is a closely related $B \subseteq {}^{\omega}\omega$ (a continuous preimage) so that if B is determined, then A is Lebesgue measurable, and similarly for the Baire property and the perfect set property. Moreover, AD does imply a limited choice principle, that *every countable set consisting of sets of reals has a choice function.* Thus, the groundwork was laid for the reign of AD in $L(\mathbb{R})$ to enforce the regularity properties for all sets of reals there as well as a local choice principle, and unfettered uses of AC relegated to the universe at large.

In 1967 two results drew determinacy to the foreground of set theory, one about the transfinite and the other about definable sets of reals. Solovay established that AD *implies that* ω_1 *is measurable*, injecting emerging large cardinal techniques into a novel setting without AC. David Blackwell provided a new proof via the determinacy of open games of a classical result of Kuratowski that *the* $\mathbf{\Pi}^1_1$ *sets have the reduction property.* These results stimulated interest because of their immediacy and new approach to proof, that of devising a game and appealing to the existence of winning strategies to deduce a dichotomy. Martin in particular saw the potentialities at hand and soon made incisive contributions to investigations with and of determinacy. He initially made a simple but crucial observation based on the construal of strategies as reals that would have myriad applications; he showed that *under* AD *the filter over the Turing degrees generated by the cones is an ultrafilter.*

After seeing Blackwell's argument, Martin and Addison quickly and independently came to the idea of *assuming* determinacy hypotheses and pointed out that $\mathbf{\Delta}^1_2$-determinacy implies that $\mathbf{\Sigma}^1_3$ sets have the reduction property. Then Martin and Yiannis Moschovakis independently in 1968 extended the reduction property through the projective hierarchy by playing games and assuming PD, realizing a methodological goal of the classical descriptive set theorists by carrying out an inductive propagation. This was Martin's initial application of his ultrafilter on Turing cones, and the idea of ranking ordinal-valued functions via ultrafilters, so crucial in later arguments, first occurred here.

Already in 1964 Moschovakis had abstracted a property stronger and more intrinsic than reduction, the prewellordering property, from the classical analysis of $\mathbf{\Pi}^1_1$ sets. A relation $\preceq$ is a *prewellordering* if it is a well-ordering except possibly that there could be distinct x and y such that $x \preceq y$ and $y \preceq x$. While a well-ordering of a set A corresponds to a bijection of A into an ordinal, a prewellordering corresponds to a surjection onto an ordinal—a stratification of A into well-ordered layers. A class $\mathbf{\Gamma}$ of sets of reals has the *prewellordering property* if for any $A \in \mathbf{\Gamma}$ there is a prewellordering of A such that both it and its complement are in $\mathbf{\Gamma}$ in a strong sense. and this property supplanted the reduction property in the Martin-Moschovakis First Periodicity Theorem, which implied that under PD the prewellordering property holds periodically for the projective classes: $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_3$, $\mathbf{\Sigma}^1_4, \ldots$.

As for Solovay's result, he in fact established that *under* AD *the closed unbounded filter* $\mathcal{C}_{\omega_1}$ *is an ultrafilter* by using a game played with countable ordinals and simulating it with reals. Martin provided an alternate proof using his ultrafilter on Turing cones, and then Solovay in 1968 used Martin's result to establish that *under* AD ω_2 *is measurable.* With an apparent trend set, quite unexpected was the next advance. Martin in 1970 established that under AD the ω_n's for $3 \leq n < \omega$ are all singular with cofinality ω_2! This was a by-product of Martin's incisive analysis of $\mathbf{\Sigma}^1_3$ sets under AD.

Martin and Solovay had by 1969 established results about the $\mathbf{\Sigma}^1_3$ sets assuming $a^{\#}$ exists for every $a \in {}^{\omega}\omega$, and Martin went on to make explicit a "Martin-Solovay" tree representation for $\mathbf{\Sigma}^1_3$ sets. Just as Shoenfield had dualized the classical tree representation of $\mathbf{\Pi}^1_1$ sets by reconstruing well-foundedness as having order-preserving ranking functions, so too Martin was able to dualize the Shoenfield tree. For this he used the existence of sharps to order ordinal-valued functions and secure important *homogeneity* properties to establish that *if* $a^{\#}$ *exists for every* $a \in {}^{\omega}\omega$*, then every* $\mathbf{\Sigma}^1_3$ *set is* ω_2*-Suslin.* This careful analysis with indiscernibles led to the aforementioned singularity of the ω_n's for $3 \leq n < \omega$ under AD.

Martin also reactivated the earlier project of securing more and more determinacy by establishing that if there is measurable cardinal, then $\mathbf{\Pi}^1_1$-determinacy holds, or in refined terms, *if* $a^{\#}$ *exists, then* $\Pi^1_1(a)$*-determinacy holds.* The proof featured a remarkably simple reduction to an open game, based on indiscernibles and homogeneity properties, of form $G_X(A)$ for a set X of ordinals. This ground-breaking proof served both to make plausible the possibility of getting PD from large cardinals as well as getting $\mathbf{\Delta}^1_1$-determinacy, Borel Determinacy, in ZFC—both directions to be met with complete success in later years.

The next advance would be by way of what would become the central structural concept in the investigation of the projective sets under determinacy. The classical issue of uniformization had been left unaddressed by the prewellordering property, and so Moschovakis in 1971 isolated a strengthening abstracted from the proof of the classical, Kondô $\mathbf{\Pi}^1_1$ Uniformization Theorem. A *scale* on a set $A \subseteq {}^{\omega}\omega$ is an ω-sequence of ordinal-valued functions on A satisfying convergence and continuity properties, and a class $\mathbf{\Gamma}$ of sets of reals has the *scale property* if for any $A \in \mathbf{\Gamma}$ there is a scale on A whose corresponding graph relations are in $\mathbf{\Gamma}$ in a strong sense. Having a scale on A corresponds to having $A = p[T]$ for a tree T in such a way that, importantly, from A is definable a member of A through a minimization process ("choosing the honest leftmost branch").

Instead of carrying out a tree dualizing procedure directly à la Shoenfield and Martin-Solovay, Moschovakis used a game argument to establish the

Second Periodicity Theorem, which implied that under PD the scale property, and therefore uniformization, holds for the same projective classes as for prewellordering: $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_3$, $\mathbf{\Sigma}^1_4, \ldots$.

In the early 1970s Moschovakis, Martin, and Alexander Kechris proceeded with scales to provide a detailed analysis of the projective sets under PD in terms of Borel sets and as projections of trees, based on the *projective ordinals* $\boldsymbol{\delta}^1_n$ $(= \underset{\sim}{\delta}^1_n)$ = the supremum of the lengths of the $\mathbf{\Delta}^1_n$ prewellorderings. For example, the $\mathbf{\Sigma}^1_{2n+2}$ sets are exactly the $\boldsymbol{\delta}^1_{2n+1}$-Suslin sets. The further analysis would be based on Moschovakis's Coding Lemma, which with determinacy provides for an arbitrary set meeting the layers of a prewellordering a appropriately definable subset meeting those same layers, and his Third Periodicity Theorem, which with determinacy asserts that when winning strategies exist there are appropriately definable such strategies. The projective ordinals themselves were subjected to considerable scrutiny, with penetrating work of Kunen particularly advancing the theory, and were found to be measurable and to satisfy strong partition properties. However, where exactly the $\boldsymbol{\delta}^1_n$ for $n \geq 5$ are in the aleph hierarchy would remain a mystery until the latter 1980s, when Steve Jackson in a *tour de force* settled the question with a deep analysis of the ultrafilters and partition properties involved. As an otherwise complete structure theory for projective sets was being worked out into the 1970s, Martin in 1974 returned to a bedrock issue for the regularity properties and established in ZFC that $\mathbf{\Delta}^1_1$-determinacy, Borel Determinacy, holds.

4.3. Silver's Theorem and Covering

In mid-1974 Silver established that *if* κ *is a singular cardinal with* $\mathrm{cf}(\kappa) > \omega$ *and* $2^\lambda = \lambda^+$ *for* $\lambda < \kappa$*, then* $2^\kappa = \kappa^+$. This was a dramatic event and would stimulate dramatic developments. There had been precious little in the way of results provable in ZFC about cardinal arithmetic, and in the early ruminations about the singular cardinals problem it was quite unforeseen that the power of a singular cardinal can be so constrained. An analogous preservation result had been observed by Scott for measurable cardinals, and telling was that Silver used large-cardinal ideas connected with generic ultrapowers.

Silver's result spurred broad-ranging investigations both into the combinatorics and avenue of proof and into larger, structural implications. The basis of his argument was a ranking of ordinal-valued functions on $\mathrm{cf}(\kappa)$. Let $\langle \gamma_\alpha \mid \alpha < \mathrm{cf}(\kappa) \rangle$ be a sequence of ordinals unbounded in κ and for $\alpha < \mathrm{cf}(\kappa)$ let $\tau_\alpha \colon P(\gamma_\alpha) \to 2^{\gamma_\alpha}$ be a bijection. For $X \subseteq \kappa$ let f_X on $\mathrm{cf}(\kappa)$ be defined by: $f_X(\alpha) = \tau_\alpha(X \cap \gamma_\alpha)$, noting that $X_1 \neq X_2$ implies f_{X_1} and f_{X_2} differ for sufficiently large α. Then 2^κ is mirrored through these eventually different functions, which one can work to order according to an ideal over the uncountable $\mathrm{cf}(\kappa)$. The combinatorial possibilities of such rankings led to a series of limitative results on the powers of singular cardinals of uncountable cofinality, starting with the results of Fred Galvin and Hajnal, of which the paradigmatic example is that if $\aleph_{\omega_1}$ is a strong limit cardinal, then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

In the wake of Silver's proof, Thomas Jech and Prikry defined a κ-complete ideal over κ to be *precipitous iff* the corresponding generic ultrapower à la Solovay is well-founded. They thus put the focus on a structural property of saturated ideals that Silver had simulated to such good effect. Jech and Prikry pointed out that a proof of Kunen's for saturated ideals using iterated ultrapowers can be tailored to show: *If there is a precipitous ideal over κ, then κ is measurable in an inner model.* Then Mitchell showed: *If a measurable cardinal is Levy collapsed to ω_1, then there is a precipitous ideal over ω_1.* Hence, a first equi-consistency result was achieved for measurability and ω_1. With combinatorial characterizations of precipitousness soon in place, well-foundedness as thus modulated by forcing became a basic ingredient in a large-scale investigation of strong properties tailored to ideals and generic elementary embeddings.

The most dramatic and penetrating development from Silver's Theorem was Jensen's work on covering for L and its first extensions, the most prominent advances of the 1970s in set theory. Jensen had found Silver's result a "shocking discovery," and was stimulated to intense activity. By the end of 1974 he had made prodigious progress, solving the singular cardinals problem in the absence of $0^{\#}$ in three manuscripts, "Marginalia to a Theorem of Silver" and its two sequels. The culminating result featured an elegant and focal formulation of intuitive immediacy, the Covering Theorem (or "Lemma") for L: *If $0^{\#}$ does not exist, then for any uncountable set X of ordinals there is a $Y \in L$ with $|Y| = |X|$ such that $Y \supseteq X$.* (Without the "uncountable" there would be a counterexample using "Namba forcing".) This covering property expresses a global affinity between V and L, and its contrapositive provides a surprisingly simple condition sufficient for the existence of $0^{\#}$ and the ensuing indiscernible generation of L. As such, Jensen's theorem would find wide applications for implicating $0^{\#}$ and would provide a new initiative in inner model theory for encompassing stronger hypotheses.

The Covering Theorem gave the essence of Jensen's argument that in the absence of $0^{\#}$ the Singular Cardinals Hypotheses holds: Suppose that κ is singular and for reckoning with the powers of smaller cardinals consider $\lambda = \sup\{2^{\mu} \mid \mu < \kappa\}$. If there is a $\nu < \kappa$ such that $\lambda = 2^{\nu}$, then the functions f_X defined as above adapted to the present situation satisfy $f_X : \mathrm{cf}(\kappa) \to 2^{\nu}$, and so $\lambda \leq 2^{\kappa} \leq (2^{\nu})^{\mathrm{cf}(\kappa)} \leq \lambda$. If on the other hand λ is the strict supremum of increasing 2^{μ}'s, then $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$ and so the Zermelo-Kőnig inequality would dictate the least possibility for 2^{κ} to be λ^+. However, if for any $X \subseteq \kappa$ the range of f_X is covered by a $Y \subseteq \lambda$ with $Y \in L$ of cardinality $\mathrm{cf}(\kappa) \cdot \aleph_1$, then: there are $2^{\mathrm{cf}(\kappa) \cdot \aleph_1}$ subsets of each such Y and by the GCH in L, at most $|\lambda^{+L}|$ such Y. Hence, we would have $2^{\kappa} \leq 2^{\mathrm{cf}(\kappa) \cdot \aleph_1} \cdot |\lambda^{+L}| \leq \lambda^+$.

The Covering Theorem also provided another dividend that would grow in separate significance as having *weak covering property*: *Assume that $0^{\#}$ does* *not exist. If κ is singular, then $\kappa^{+L} = \kappa^+$.* If to the contrary $\kappa^{+L} < \kappa^+$, then $\mathrm{cf}(\kappa^{+L}) < \kappa$. Let $X \subseteq \kappa^{+L}$ be unbounded so that $|X| < \kappa$ and let $Y \in L$ cover X with $|Y| = |X| \cdot \aleph_1$. But then, the order-type of Y would be less than κ, contradicting the regularity of κ^{+L} in L.

A crucial consequence of weak covering is that *in the absence of $0^\#$, $\square_\kappa$ holds for singular κ*, since a $\square_\kappa$ sequence in the sense of L is then a $\square_\kappa$ sequence in V. The weak covering property would itself become pivotal in the study of inner models corresponding to stronger and stronger hypotheses, and the failure of $\square_\kappa$ for singular κ would become a delimitative proposition. Solovay had already established an upper bound on consistency by showing in the early 1970s that *if κ is λ^+-supercompact and $\lambda \geq \kappa$, then $\square_\lambda$ fails.*

Jensen's ingenious proof of the Covering Theorem for L proceeded by taking a counterexample X to covering with $\tau = \sup(X)$ and $|X|$ minimal; getting a certain Σ_1-elementary $j\colon J_\gamma \to J_\tau$ which contains X in its range through a Skolem hull construction so that $|\gamma| = |X|$ and, as X cannot be covered, γ is a cardinal in L; and extending j to an elementary embedding from L into L, so that $0^\#$ exists. The procedure for extending j up to some large J_δ was to consider a directed system of embeddings of structures generated by $\xi \cup p$ for some $\xi < \gamma$ and p a finite subset of J_δ, the transitized components of the system all being members of J_γ as γ is a cardinal in L, and to consider the corresponding directed system consisting of the j images. The choice of γ insured that the new directed system is also well-founded, and so isomorphic to some J_ζ. For effecting embedding extendibility, Jensen established the fine structural *Upward Extension of Embeddings Lemma*, according to which if N is the projectum structure for J_α and a Σ_1-elementary $e\colon N \to M$ is *strong* in that it preserves the well-foundedness of Σ_1 relations, then M itself is the projectum structure of some unique J_β and e can be extended uniquely to a Σ_1-elementary $\overline{e}\colon J_\alpha \to J_\beta$.

How can the proof of the Covering Theorem be adapted to establish a stronger result? The only possibility was to consider a larger inner model M and to establish that M has the *covering property*: for any uncountable set X of ordinals there is a $Y \in M$ with $|Y| = |X|$ such that $Y \supseteq X$. In groundbreaking work for inner model theory, Solovay in the early 1970s had developed a fine structure theory for inner models of measurability. Whilst a research student at Oxford University Anthony Dodd worked through this theory, and in early 1976 he and Jensen laid out the main ideas for extending the Covering Theorem to a new inner model, now known as the *Dodd-Jensen core model*, denoted K^{DJ}.

If $\langle L[U], \in, U\rangle$ is an inner model of measurability, say the κ-model, then there is a generic extension in which covering fails: If G is Prikry generic for U over $L[U]$, then G cannot be covered by any set in $L[U]$ of cardinality less than κ. Drawing back, there remains the possibility of "iterating out" the measurable cardinal: If $\langle L[U], \in, U\rangle$ is the κ-model, then $\langle L[W], \in, W\rangle$ is the λ-model for some $\lambda > \kappa$ exactly when it is an iterate of $\langle L[U], \in, U\rangle$, in which case $L[W] \subseteq L[U]$, $V_\kappa \cap L[U] = V_\kappa \cap L[W]$, and $U \notin L[W]$.

Thus, if $\langle L[U_\alpha] \mid \alpha \in \mathrm{On}\rangle$ enumerates the inner models of measurability, then starting with any one of them, the process of iterating it through the ordinals converges to a proper class $\bigcap_\alpha L[U_\alpha]$ which has no inner models of measurability, with the stabilizing feature that for any γ, $V_\gamma \cap \bigcap_\alpha L[U_\alpha] = V_\gamma \cap L[U_\beta]$ for sufficiently large β. Assuming that there are inner models of measurability, K^{DJ} is in fact characterizable as this residue class. Aspiring to this, but without making any such assumption, Dodd and Jensen provided a formulation of K^{DJ} in ZFC.

K^{DJ} was the first inner model of ZFC since Gödel's L developed using distinctly new generating principles. Dodd and Jensen's approach was to take K^{DJ} as the union of L together with "mice". Loosely speaking, a *mouse* is a set $L_\alpha[U|$ such that

$$\langle L_\alpha[U], \in, U\rangle \models U \text{ is a normal ultrafilter over } \kappa$$

satisfying: (i) there is a subset of κ in $L_{\alpha+1}[U] - L_\alpha[U]$, so that U is on the verge of not being an ultrafilter; (ii) $\langle L_\alpha[U], \in, U\rangle$ is iterable in that all the iterated ultrapowers are well-founded, and (iii) fine structure conditions about a projectum below κ leading to (i). Mice can be compared by taking iterated ultrapowers, so that there is a natural prewellordering of mice, and moreover, crucial elements about L can be lifted to the new situation because there is a generalization of condensation: Σ_1-elementary substructures of mice, when transitized, are again mice. This led to $K^{\mathrm{DJ}} \models \mathrm{GCH}$, and that K^{DJ} in the sense of K^{DJ} is again K^{DJ}.

Mice generate indiscernibles through iteration, and so if $0^\#$ does not exist, then $K^{\mathrm{DJ}} = L$; if $0^\#$ exists but $0^{\#\#}$ does not, then $K^{\mathrm{DJ}} = L[0^\#]$; and this continues through the transfinite by coding sequences of sharps. On the other hand, K^{DJ} has no simple constructive analysis from below and is rather like a maximal inner model on the brink of measurability: Its own "sharp", that there is an elementary embedding $j\colon K \to K$, is equivalent to the existence of an inner model of measurability. Indeed, this was Dodd and Jensen's primary motivation for the formulation of K^{DJ}. They used it in place of the elementary embedding characterization of the existence of $0^\#$, together with the L-like properties of K^{DJ}, to establish the *Covering Theorem for K^{DJ}: If there is no inner model of measurability, then K^{DJ} has the covering property.* This has the attendant consequences for the singular cardinals problem. Moreover, Dodd and Jensen were able to establish a covering result for inner models of measurability that accommodates Prikry forcing. Solovay had devised a set of integers $0^\dagger$ ("zero dagger"), analogous to $0^\#$, such that $0^\dagger$ exists exactly when for some κ-model $L[U]$ there is an elementary embedding $j\colon L[U] \to L[U]$ with critical point above κ. Dodd and Jensen established: *If $0^\dagger$ does not exist yet there is an inner model of measurability, then for the κ-model $L[U]$ with κ least, either (a) $L[U]$ has the covering property, or (b) there is a Prikry generic G for U over $L[U]$ such that $L[U][G]$ has the covering property.* Prikry forcing provides the only counterexample to covering! Hence, the inner models thus far considered were also "core models", models on the brink so that the lack of covering leads to the next large cardinal hypothesis.

In the light of the Dodd-Jensen work, Mitchell in the later 1970s developed the core model $K[\mathcal{U}]$ for coherent sequences $\mathcal{U}$ of ultrafilters, which corresponds to his $L[\mathcal{U}]$ as K^{DJ} does to $L[U]$. The mice are now sets of form $J_\alpha[W]$ with iterability and fine structure properties, where W is an ultrafilter sequence with $\mathcal{U}$ as an initial segment. Under the assumption that there is no κ satisfying $o(\kappa) = \kappa^{++}$, Mitchell established a covering theorem for $K[\mathcal{U}]$ setting out how the Dodd-Jensen covering proof for $L[U]$ involving Prikry generic sets can be cast in terms of having coherent systems of indiscernibles. With this result, Mitchell established that several propositions each have consistency strength at least that of $\exists\kappa(o(\kappa) = \kappa^{++})$. Two prominent propositions were that there is a singular cardinal κ such that $(\kappa^+)^{K[\mathcal{U}]} < \kappa^+$ for weak covering and that there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1, establishing a new lower bound in consistency strength for Kunen's consistency result from a huge cardinal.

4.4. Forcing Consistency Results

Through the 1970s a wide range of variegated forcing consistency results were established at a new level of sophistication that clarified relationships among combinatorial propositions and principles and often drew in large cardinal hypotheses and stimulated the development of method, especially in iterated forcing. A conspicuous series of results resolved questions of larger mathematics (Whitehead's Problem, Borel's Conjecture, Kaplansky's Conjecture, the Normal Moore Space Problem) in terms of relative consistency and set-theoretic principles, newly affirming the efficacy and adjudicatory character of set theory. In what follows, as we have begun to already, we pursue the larger longitudinal themes and results, necessarily saying less and less about matters of increasing complexity.

Much of the early formative work on strong large cardinal hypotheses and their integration into modern set theory through consistency results was carried out by Menachem Magidor, whose subsequent, broad-ranging initiatives have considerably advanced the entire subject. After completing his Hebrew University thesis in 1972 on supercompact cardinals, Magidor in the 1970s established a series of penetrating forcing consistency results involving strong hypotheses. In 1972-3 he illuminated the "identity crisis" issue of whether supercompactness and strong compactness are distinct concepts by establishing: (1) *It is consistent that the least supercompact cardinal is also the least strong compact cardinal*, and (2) *It is consistent that the least strong compact cardinal is the least measurable cardinal (and so much smaller than the least supercompact cardinal).* The proofs showed how changing many cofinalities with Prikry forcing to destroy measurable cardinals can be integrated into arguments about extending elementary embeddings.

In 1974 Magidor made a basic contribution to the theory of changing cofinalities, the first after Prikry. Magidor established: *If a measurable cardinal κ is of Mitchell order $o(\kappa) \geq \lambda$ for a regular $\lambda < \kappa$, then there is a forcing extension preserving cardinals in which* $\mathrm{cf}(\kappa) = \lambda$. Generalizing Prikry forcing, Magidor's conditions consisted of a finite sequence of ordinals and a sequence of sets drawn from normal ultrafilters in the Mitchell order, the sets providing for the possible ways of filling out the sequence. Like Prikry's forcing, Magidor's may at first have seemed a curious possibility for a new singularization. However, one of the subsequent discernments of Mitchell's core model for coherent sequences of measures is that, remarkably: *If a regular cardinal κ in V satisfies $\omega < \mathrm{cf}(\kappa) < \kappa$ in a generic extension, then V has an inner model in which $o(\kappa)$ is at least that cofinality.* Thus, the capability of changing cofinalities was exactly gauged; "Prikry-Magidor" generic sets as sequences of indiscernibles would become a basic component of Mitchell's covering work.

The most salient results of Magidor's of this period were two of 1976 that provided counterweight to Jensen's covering results on the singular cardinal problem. Magidor showed: (1) *If κ is supercompact, there is a forcing extension in which κ is $\aleph_\omega$ as a strong limit cardinal yet $2^{\aleph_\omega} > \aleph_{\omega+1}$*, and (2) *If κ is a huge cardinal, then there is a forcing extension in which $\kappa = \aleph_\omega$, $2^{\aleph_n} = \aleph_{n+1}$ for $n \in \omega$, yet $2^{\aleph_\omega} > \aleph_{\omega+1}$.* Thus, forcing arguments showed that the least singular cardinal can be a counterexample to the Singular Cardinals Hypothesis; the strong elementary embedding hypotheses allowed for an elaborated Prikry forcing interspersed with Levy collapses. The Prikry-Silver and the Magidor results showed through initial incursions of Prikry forcing how to arrange high powers for singular strong limit cardinals; it would be one of the great elaborations of method that equi-consistency results would eventually be achieved with weaker hypotheses.

With respect to the Jech-Prikry-Mitchell equi-consistency of measurability and precipitousness, Magidor showed that absorptive properties of the Levy collapse of a measurable cardinal to ω_1 can be exploited by subsequently "shooting" closed unbounded subsets of ω_1 through stationary sets to get: *If there is a measurable cardinal κ, then there is a forcing extension in which $\kappa = \omega_1$ and NS_{ω_1} is precipitous.* Thus a basic, definable ideal can be precipitous, and this naturally became a principal point of departure for the investigation of ideals.

The move of Saharon Shelah into set theory in the early 1970s brought in a new and exciting sense of personal initiative that swelled into an enhanced purposiveness across the subject, both through his solutions of major outstanding problems as well as through his development of new structural frameworks. A phenomenal mathematician, Shelah from his 1969 Hebrew University thesis on has worked in model theory and eventually infused it with a transformative, abstract classification theory for models. In both model theory and set theory he has remained eminent and has produced results at a furious pace, with nearly 1000 items currently in his bibliography (his papers are currently archived at http://shelah.logic.at/).

In set theory Shelah was initially stimulated by specific problems. He typically makes a direct, frontal attack, bringing to bear extraordinary powers of concentration, a remarkable ability for sustained effort, an enormous arsenal of accumulated techniques, and a fine, quick memory. When he is successful on the larger problems, it is often as if a resilient, broad-based edifice has been erected, the traditional serial constraints loosened in favor of a wide, fluid flow of ideas and the final result almost incidental to the larger structure. *What* has been achieved is more than a just succinctly stated theorem but rather the erection of a whole network of robust arguments.

Shelah's written accounts have acquired a certain notoriety that in large part has to do with his insistence that his edifices be regarded as autonomous conceptual constructions. Their life is to be captured in the most general forms, and this entails the introduction of many parameters. Often, the network of arguments is articulated by complicated combinatorial principles and transient hypotheses, and the forward directions of the flow are rendered as elaborate transfinite inductions carrying along many side conditions. The ostensible goal of the construction, that succinctly stated result that is to encapsulate it, is often lost in a swirl of conclusions.

Shelah's first and very conspicuous advance in set theory was his 1973, definitive results on Whitehead's Problem in abelian group theory: Is every Whitehead group, an abelian group G satisfying $\mathrm{Ext}^1(G, \mathbb{Z}) = 0$, free? Shelah established that $V = L$ implies that this is so, and that Martin's Axiom implies that there is a counterexample. Shelah thus established for the first time that a strong purely algebraic statement is undecidable in ZFC. With his L result specifically based on diamond-type principles, Shelah brought them into prominence with his further work on them, which were his first incursions into iterated forcing. As if to continue to get his combinatorial bearings, Shelah successfully attacked several problems on an Erdős-Hajnal list for partition relations, developing in particular a "canonization" theory for singular cardinals. By the late 1970s his increasing understanding of and work in iterated forcing would put a firm spine on much of the variegated forcing arguments about the continuum.

With an innovative argument pivotal for iterated forcing, Richard Laver in 1976 established the consistency of Borel's conjecture: *Every set of reals of strong measure zero is countable.* CH had provided a counterexample, and Laver established the consistency with $2^{\aleph_0} = \aleph_2$. His argument featured the adjunction of what are now called *Laver reals* in the first clearly set out *countable support iteration*, i.e. an iteration with non-trivial local conditions allowed only at countably many coordinates. The earlier Solovay-Tennenbaum argument for the consistency of MA had relied on finite support, and a Mitchell argument about Aronszajn trees, on an involved countable support with a "termspace" forcing, which would also find use. Laver's work showed that countable support iteration is both manageable and efficacious for preserving certain framing properties of the continuum to establish the consistency of propositions with $2^{\aleph_0} = \aleph_2$. Interestingly, a trade-off would develop however: while finite support iterations put all cardinals $\geq \aleph_2$ on an equal footing with respect to the continuum, countable support iterations restricted the continuum to be at most $\aleph_2$. With a range of new generic reals coming into play with the widening investigation of the continuum, James Baumgartner formulated a property common to the corresponding partial orders, *Axiom A*, which in particular ensured the preservation of ω_1. He showed that the countable support iteration of Axiom A forcings is Axiom A, thereby uniformizing the iterative adjunction of the known generic reals.

All this would retrospectively have a precursory air, as Shelah soon established a general, subsuming framework. Analyzing Jensen's consistency argument for SH + CH and coming to grips with forcing names in iterated forcing, Shelah came to the concept of *proper forcing* as a general property that preserves ω_1 and is preserved in countable support iterations. The instrumental formulation of properness is given in an appropriately broad setting:

First, for a regular cardinal λ, let $H(\lambda) = \{x \mid |\mathrm{tc}(\{x\})| < \lambda\}$, the sets hereditarily of cardinality less than λ. The $H(\lambda)$'s provide another cumulative hierarchy for V, one stratified into layers that each satisfy Replacement; whereas the V_α's for limit α satisfy every ZFC axiom except possibly Replacement, the $H(\lambda)$'s satisfy every ZFC axiom except possibly Power Set. A partial order $\langle P, < \rangle$ is *proper* if for any regular $\lambda > 2^{|P|}$ and countable $M \prec H(\lambda)$ with $P \in M$, every $p \in P \cap M$ has a $q \leq p$ such that $q \Vdash \dot{G} \cap M$ is M-generic. (Here, $\dot{G}$ a canonical name for a generic filter with respect to P, and q forcing this genericity assertion has various combinatorial equivalents.)

A general articulation of how all countable approximations are to have generic filters has been achieved, and its presentation under countable support iterations exhibited the efficacy of this remarkable move to a new plateau. Shelah soon devised variants and augmentations, and in a timely 1982 monograph *Proper Forcing* revamped forcing for combinatorics and the continuum with systemic proofs of new and old results. Proper forcing, presented in Chapter 5 of this Handbook, has become a staple part of the methods of modern set theory, with its applications wide-ranging and the development of its extended theory a fount of research.

In light of Shelah's work and Martin's Axiom, Baumgartner in the early 1980s established the consistency of a new encompassing forcing axiom, the *Proper Forcing Axiom* (PFA): *For any proper partial order P and collection $\mathcal{D}$ of $\aleph_1$ dense subsets of P, there is a filter $G \subseteq P$ meeting every member of $\mathcal{D}$.* Unlike MA, the consistency of PFA required large cardinal strength and moreover could not be achieved by iteratively taking care of the partial orders at issue, as new proper partial orders occur arbitrarily high in the cumulative hierarchy. Baumgartner established: *If there is a supercompact cardinal κ, then there is a forcing extension in which $\kappa = \omega_2$ and* PFA *holds.* In an early appeal to the full global reflection properties available at a supercompact cardinal Baumgartner iteratively took care of the emerging proper partial orders along a special diamond-like sequence that anticipates all possibilities. Laver first formulated this sequence, the “Laver diamond”, toward establishing what has become a useful result for forcing theory; in a forcing extension he made a supercompact cardinal “indestructible” by any further forcing from a substantial, useful class of forcings. PFA became a widely applied forcing axiom, showcasing Shelah’s concept, but beyond that, it would itself become a pivotal hypothesis in the large cardinal context.

Two points of mathematical practice should be mentioned in connection with Shelah’s move into set theory. First, through his work with proper forcing it has become routine to appeal in proofs to structures $\langle H(\lambda), \in, <^* , \ldots \rangle$ for regular λ sufficiently large, with $<^*$ some well-ordering of $H(\lambda)$ and $\ldots$ including all the sets concerned. One then develops systems of elementary substructures generated uniformly by Skolem functions defined via $<^*$. This technique, in providing some of the structure available in L-like inner models, has proved highly efficacious over a wide range from combinatorics to large cardinals.

Second, several of a developing Israeli school in set theory have followed Shelah in writing “$p > q$” for p being a stronger condition than q instead of “$p < q$”. The former is argued for as more natural, whereas the latter had been motivated structurally by Boolean algebras. This revisionism has no doubt led to confusion, until one realizes that it is a particular stamp of the Israeli school.

5. New Expansion

5.1. Into the 1980s

The 1980s featured a new and elaborating expansion in set theory significantly beyond the successes, already remarkable, of the previous decade. There were new methods and results of course, but more than that there were successful *maximizations* in several directions—definitive and evidently optimal results—and successful *articulations* at the interstices—new concepts and refinements that filled out the earlier explorations. A new wave of young researchers entered the fray, including the majority of the authors contributing to this Handbook, soon to become the prominent experts in their respective, newly variegated subfields. Our narrative now becomes even more episodic in increasingly inverse relation to the broad-ranging and penetrating developments, leaving accounts of details and some whole subjects to the chapter summaries at the end.

In 1977 Lon Radin toward his Berkeley thesis developed an ultimate generalization of the Prikry and Magidor forcings for changing cofinalities, a generalization that could in fact adjoin a closed unbounded subset, consisting of formerly regular cardinals, to a large cardinal κ while maintaining its regularity and further substantive properties. As graduate students at Berkeley, Hugh Woodin and Matthew Foreman saw the possibilities abounding in Radin forcing. While an undergraduate at Caltech Woodin did penetrating work on the consistency of Kaplansky's Conjecture (Is every homomorphism on the Banach algebra of continuous functions on the unit interval continuous?) and now with Radin forcing in hand would produce his first series of remarkable results. By 1979 Foreman and Woodin had the essentials for establishing: *If there is a supercompact cardinal κ, then there is forcing extension in which V_κ as a model of* ZFC *satisfies that the* GCH *fails everywhere, i.e.* $2^\lambda > \lambda$ *for every* λ. This conspicuously subsumed the Magidor result getting $\aleph_\omega$ a strong limit yet $2^{\aleph_\omega} > \aleph_{\omega+1}$ and put Radin forcing on the map for establishing global consistency results.

Shelah soon established two re-orienting results about powers of singular cardinals. Having come somewhat late into the game after Silver's Theorem, Shelah had nonetheless extended some of the limitative results about such powers, even to singular κ such that $\aleph_\kappa = \kappa$. Shelah subsequently established: *If there is a supercompact cardinal κ and α is a countable ordinal, then there is a forcing extension in which κ is $\aleph_\omega$ as a strong limit cardinal yet* $2^{\aleph_\omega} = \aleph_{\alpha+1}$. He thus extended Magidor's result by showing that the power of $\aleph_\omega$ can be made arbitrarily large below $\aleph_{\omega_1}$. In 1980 Shelah established the general result that *for any limit ordinal δ,* $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$*, so that in particular if $\aleph_\omega$ is a strong limit cardinal, then* $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$. Not only was he able to get an absolute upper bound in ZFC, but he had brought countable cofinality, the one cofinality unattended to by Silver's Theorem, into the scheme of things. Shelah's argument, based on the possible cofinalities of "reduced products" of a cofinal subset of $\aleph_\delta$, would evolve into a generally applicable method by the late 1980's, the remarkable *pcf theory*.

In 1978, Mitchell made a new breakthrough for the inner model theory of large large cardinals by developing such a model for "hypermeasurable cardinals", e.g. a measurable cardinal κ such that for some normal ultrafilter U over κ, $P(P(\kappa)) \subseteq \mathrm{Ult}(V, U)$, so that every ultrafilter over κ is in the ultrapower. This at least captured a substantial consequence of the 2^κ-supercompactness of κ, and so engendered the hope of getting L-like inner models for such strong hypotheses. Supercompactness, while increasingly relied on in relative consistency results owing to its reflection properties, was out of reach, but the Mitchell result suggested an appropriate weakening: A cardinal κ is *α-strong iff* there is an elementary embedding $j\colon V \to M$ for some inner model M, with critical point κ and $\alpha < j(\kappa)$ such that $V_\alpha \subseteq M$. (One can alternately require that the αth iterated power set $P^\alpha(\kappa)$ be a subset of M, which varies the definition only for small α like $\alpha = 2$ but makes the definition more germane for them.) κ is *strong iff* it is α-strong for every α.

Dodd and Jensen soon simplified Mitchell's presentation in what turned out to be a basic methodological advance for the development of inner model theory. While introducing certain redundancies, they formulated a general way of analyzing an elementary embedding in terms of *extenders*. The idea, anticipated in Jensen's proof of the Covering Theorem, is that elementary embeddings between inner models can be approximated arbitrarily closely as direct limits of ultrapowers with concrete features reminiscent of iterated ultrapowers.

Suppose that N and M are inner models of ZFC, $j\colon N \to M$ is elementary with a critical point κ, and $\beta > \kappa$. Let $\zeta \geq \kappa$ be the least ordinal satisfying $\beta \leq j(\zeta)$; the simple ("short") case is $\zeta = \kappa$, and the general case is for the study of stronger hypotheses. For each finite subset a of β, define E_a by:

$$X \in E_a \quad \textit{iff} \quad X \in P([\zeta]^{|a|}) \cap N \ \wedge \ a \in j(X) \ .$$

This is another version of the idea of generating ultrafilters from embeddings. E_a may not be in N, but $\langle N, \in, E_a\rangle \models$ "E_a is a κ-complete ultrafilter over $[\zeta]^{|a|}$". The (κ,β)-*extender derived from* j is $E = \langle E_a \mid a \text{ is a finite subset of } \beta\rangle$.

For each finite subset a of β, $\mathrm{Ult}(N, E_a)$ is seen to be elementarily embeddable into M, so that in particular $\mathrm{Ult}(N, E_a)$ is well-founded and hence identified with its transitive collapse, say M_a. Next, for $a \subseteq b$ both finite subsets of β, corresponding to how members of a sit in b there is a natural elementary embedding $i_{ab}\colon M_a \to M_b$. Finally,

$$\langle\langle M_a \mid a \text{ is a finite subset of } \beta\rangle, \langle i_{ab} \mid a \subseteq b\rangle\rangle$$

is seen to be a directed system of structures with commutative embeddings, so stipulate that $\langle M_E, \in_E\rangle$ is the direct limit, and let $j_E\colon N \to M_E$ be the corresponding elementary embedding. We thus have the *extender ultrapower* of N by E as a direct limit of ultrapowers. The crucial point is that the direct limit construction ensures that j_E and M_E approximate j and M "up to β", e.g. if $|V_\alpha|^M \leq \beta$, then $|V_\alpha|^M = |V_\alpha|^{M_E}$, i.e. the cumulative hierarchies of M and M_E agree up to α. Having formulated extenders derived from an embedding, a (κ,β)-*extender* is a sequence $E = \langle E_a \mid a \text{ is a finite subset of } \beta\rangle$ that satisfies various abstracted properties that enable the above construction.

In a manuscript circulated in 1980, Dodd and Jensen worked out inner models for strong cardinals. Building on the previous work of Mitchell, Dodd and Jensen formulated *coherent sequences of extenders*, built inner models relative to such, and established GCH in these models. The arguments were based on extending the established techniques of securing iterability and comparison through coiteration. The GCH result was significant as precursory for the further developments in inner model theory based on "iteration trees". Thus, with extenders the inner model theory was carried forward to encompass strong cardinals, newly arguing for the coherence and consistency of the concept. There would however be little further progress until 1985, for the aspiration to encompass stronger hypotheses had to overcome the problem of "overlapping extenders", having to carry out comparison through coiteration for local structures built on (κ_1,β_1)-extenders and (κ_2,β_2)-extenders with $\kappa_1 \leq \kappa_2 < \beta_1$. The difficulty here is one of "moving generators": if an extender ultrapower is taken with a (κ_1,β_1)-extender and then with a (κ_2,β_2)-extender, then $\kappa_2 < \beta_1$ implies that the generating features of the first extender ultrapower has been shifted by the second ultrapower and so one can no longer keep track of that ultrapower in the coiteration process. In any event, a crucial inheritance from this earlier work was the *Dodd-Jensen Lemma* about the minimality of iterations copied across embeddings, which would become crucial for all further work in inner model theory.

In the direction of combinatorics and the study of continuum, there was considerable elaboration in the 1970s and into the 1980s, particularly as these played into the burgeoning field of *set-theoretic topology.* Not only were there new elucidations and new transfinite topological examples, but large cardinals and even the Proper Forcing Axiom began to play substantial roles in new relative consistency results. The 1984 *Handbook of Set-Theoretic Topology* summed up the progress, and its many articles set the tone for further work.

In particular, Eric van Douwen's article provided an important service by standardizing notation for the cardinal characteristics, or invariants, of the continuum in terms of the lower case Fraktur letters. We have discussed the dominating number $\mathfrak{d}$, the least cardinality of a subset of Baire space cofinal under eventual dominance $<^*$. There is the *bounding number* $\mathfrak{b}$, the least cardinality of a subset of Baire space unbounded under eventual dominance $<^*$; there is the *almost disjoint number* $\mathfrak{a}$, the least cardinality of a subset of $P(\omega)$ consisting of infinite sets pairwise having finite intersection; there is a *splitting number* $\mathfrak{s}$, the least cardinality of a subset $S \subseteq P(\omega)$ such that any infinite subset of ω has infinite intersection with both a member of S and its complement; and, now, many more. The investigation of the possibilities for the cardinality characteristics and their ordering relations with each other would itself have sustained interest in the next decades, becoming a large theory to which both Chapters 6 and 7 of this Handbook are devoted.

Conspicuous in combinatorics and topology would be the work of Stevo Todorcevic. Starting with his doctoral work with Kurepa in 1979 he carried out an incisive analysis of uncountable trees—Suslin, Aronszajn, Kurepa trees and variants—and their linearizations and isomorphism types. In 1983 he dramatically re-oriented the sense of strength for the Proper Forcing Axiom by showing that PFA *implies that* $\square_\kappa$ *fails for every* $\kappa > \omega$. PFA had previously been shown consistent relative to the existence of a supercompact cardinal. With the failure of $\square_\kappa$ for singular κ having been seen as having quite substantial consistency strength, PFA was itself seen for the first time as a very strong proposition. Todorcevic would go from strength to strength, making substantial contributions to the theory of partition relations, eventually establishing definitive results about ω_1 as the archetypal uncountable order-structure. His chapter in this Handbook presents that single-handedly developed combinatorial theory of sequences and walks.

Starting in 1980 Foreman made penetrating inroads into the possibilities for very strong propositions holding low in the cumulative hierarchy based on the workings of generic elementary embeddings. Extending Kunen's work and deploying Silver's master condition idea, Foreman initially used 2-huge cardinals to get model-theoretic transfer principles to hold and saturated ideals to exist among the range of $\aleph_n$'s. He would soon focus on generic elementary embeddings and corresponding ideals themselves, even making them postulational for set theory. This general area of research has become fruitful, multi-faceted, and enormous, as detailed in Foreman's chapter on this subject in this Handbook.

In a major 1984 collaboration in Jerusalem, Foreman, Magidor, and Shelah established penetrating results that led to a new understanding of strong propositions and the possibilities with forcing. The focus was on a new, maximal forcing axiom: A partial order P *preserves stationary subsets of* ω_1 *iff* stationary subsets of ω_1 remain stationary in any forcing extension by P, and with this we have *Martin's Maximum* (MM): *For any P preserving stationary subsets of ω_1 and collection $\mathcal{D}$ of $\aleph_1$ dense subsets of P, there is a filter $G \subseteq P$ meeting every member of $\mathcal{D}$.* This subsumes PFA and is a maximally strong forcing axiom in that there is a P which does not preserve stationary subsets of ω_1 for which the conclusion fails. Foreman, Magidor, and Shelah established: *If there is a supercompact cardinal κ, then there is a forcing extension in which $\kappa = \omega_2$ and* MM *holds.*

Shelah had considered a weakening of properness called *semiproperness*, a notion for forcing that could well render uncountable cofinalities countable. To iterate such forcings, it had to be faced that the countable cofinality of limit stages cannot be ascertained in advance, and so he developed *revised countable support iteration* (RCS) based on names for the limit stage indexing. Foreman, Magidor, and Shelah actually carried out Baumgartner's PFA consistency proof for semiproper forcings with RCS iteration to establish the consistency of the analogous *Semiproper Forcing Axiom* (SPFA). Their main advance was that, although a partial order that preserves stationary subsets of ω_1 is not necessarily semiproper, it is in this supercompact collapsing context. (Eventually, Shelah did establish that MM and SPFA are equivalent.)

Foreman, Magidor, and Shelah then established the relative consistency of several propositions by deriving them directly from MM. One such proposition was that NS_{ω_1} is $\aleph_2$-saturated. Hence, not only was the upper bound for the consistency strength of having an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 considerably reduced from Kunen's huge cardinal, but for the first time the consistency of NS_{ω_1} itself being $\aleph_2$-saturated was established relative to large cardinals. Another formative result was simply that MM actually implies that $2^{\aleph_0} = \aleph_2$, starting a train of thought about forcing axioms actually determining the continuum. It would be by different and elegant means that Todorcevic would show in 1990 that PFA already implies that $2^{\aleph_0} = \aleph_2$.

With their work Foreman, Magidor, and Shelah had overturned a long-held view about the scaling down of large cardinal properties. In the first flush of new hypotheses and propositions, Kunen had naturally enough collapsed a large cardinal to ω_1 in order to transmute strong properties of the cardinal into an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1, and this sort of direct connection had become the rule. The new discovery was that a collapse of a large cardinal to ω_2 instead can provide enough structure to secure such an ideal. In fact, Foreman, Magidor, and Shelah showed that even the usual

Levy collapse of a supercompact cardinal to ω_2 engenders an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1. In terms of method, the central point is that such a collapse leads to substantial generic elementary embeddings with small critical points like ω_1. Woodin's later strengthenings and elaborations of these results would have far-reaching consequences.

5.2. Consistency of Determinacy

The developments of the 1980s which are the most far-reaching and presentable as sustained narrative have to do with the stalking of the consistency of determinacy. By the late 1970s a more or less complete structure theory for the projective sets was in place, a resilient edifice founded on determinacy with both strong buttresses and fine details. In 1976 the researchers had started the Cabal Seminar in the Los Angeles area, and in a few years, with John Steel and Woodin having joined the ranks, attention began to shift to sets of reals beyond the projective sets, to inner models, and to questions of overall consistency. Most of the work before the crowning achievements of the later 1980s appears in the several proceedings of the Cabal Seminar appearing in 1978, 1981, 1983, and 1988.

With the growing sophistication of methods, the inner model $L(\mathbb{R})$ increasingly became the stage for the play of determinacy, both as the domain to extend the structural consequences of AD and as the natural inner model for AD that can exhibit characterizations. Scales having held the key to the structure theory for the projective sets, Martin and Steel established a a limiting case for the scale property; with the $\mathbf{\Sigma}^2_1$ sets of reals being those definable with one existential third-order quantifier, they showed that AD and $V = L(\mathbb{R})$ imply that $\mathbf{\Sigma}^2_1$ is the largest class with the scale property. Steel moreover developed a fine structure theory for $L(\mathbb{R})$, and analyzing the minimal complexity of scales there, he extended some of the structure theory under AD to sets of reals in $L(\mathbb{R})$. As for characterizations, Kechris and Woodin showed that in $L(\mathbb{R})$, AD is equivalent to the existence of many ("Suslin") cardinals that have strong partition properties. Woodin also established that in $L(\mathbb{R})$, AD is equivalent to *Turing Determinacy*, determinacy for only sets of reals closed under Turing equivalence.

The question of the overall consistency of determinacy came increasingly to the fore. Is AD consistent relative to some large cardinal hypothesis? Or, with its strong consequences, can AD subsume large cardinals in some substantial way or be somehow orthogonal? Almost a decade after his initial result that the existence of a measurable cardinal implies $\mathbf{\Pi}^1_1$-determinacy, Martin and others showed that determinacy for sets in the "difference hierarchy" built on the $\mathbf{\Pi}^1_1$ sets implies the existence of corresponding inner models with many measurable cardinals. Then in 1978 Martin, returning to the homogeneity idea of his early $\mathbf{\Pi}^1_1$ result, applied it with the Martin-Solovay tree representation for $\mathbf{\Pi}^1_2$ sets together with algebraic properties of elementary embeddings posited close to Kunen's large cardinal inconsistency to establish $\mathbf{\Pi}^1_2$-determinacy. A direction was set but generality only came in 1984, when Woodin showed that an even stronger large cardinal hypothesis implies $\mathrm{AD}^{L(\mathbb{R})}$. So, a mooring was secured for AD after all in the large cardinal hierarchy. With Woodin's hypothesis apparently too remote it would now be a question of scaling it down according to the methods becoming available for proofs of determinacy, perhaps even achieving an equi-consistency result.

The rich 1984 Foreman-Magidor-Shelah work would have crucial consequences for the stalking of consistency also for determinacy. Shelah carried out a version of their collapsing argument that does not add any new reals but nonetheless gets an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1. Woodin then pointed out that with no new reals adjoined the generic elementary embedding induced by such an ideal can be used to establish that the ground model $L(\mathbb{R})$ reals are actually Lebesgue measurable. Thus Shelah and Woodin had established an outright result: *If there is a supercompact cardinal, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.* This result not only portended the possibility of getting $\mathrm{AD}^{L(\mathbb{R})}$ from a supercompact cardinal, but through the specifics of the argument stimulated the reducing of the hypothesis. While Woodin was visiting Jerusalem in June 1984, he came up with what is now known as a *Woodin cardinal*. The hypothesis was then reduced as follows: *If there are infinitely many Woodin cardinals with a measurable cardinal above them, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.* An early suggestion of optimality of hypothesis was that if the "infinitely" is replaced by "n" for some $n \in \omega$, then one can conclude that every $\mathbf{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable. The measurable cardinal hovering above would be a recurring theme, the purpose loosely speaking to maintain a stable environment with the existence of sharps.

Especially because of its subsequent centrality, it is incumbent to give an operative definition of Wooding cardinal: For a set A, κ is *α-A-strong iff* there is an elementary embedding $j\colon V \to M$ witnessing that κ is α-strong which moreover preserves A: $A \cap V_\alpha = j(A) \cap V_\alpha$. A cardinal δ is *Woodin iff* for any $A \subseteq V_\delta$, there is a $\kappa < \delta$ which is α-A-strong for every $\alpha < \delta$.

A Woodin cardinal, evidently a technical, consistency-wise strengthening of a strong cardinal, is an important example of concept formation through method. The initial air of contrivance gives way to seeing that Woodin cardinal seemed to encapsulate just wanted is needed to carry out the argument for Lebesgue measurability. That argument having been based on first collapsing a large cardinal to get a saturated ideal and then applying the corresponding generic elementary embedding, Woodin later in 1984 stalked the essence of method and formulated *stationary tower forcing*. An outgrowth of the Foreman-Magidor-Shelah work, this notion of forcing streamlines their forcing arguments to show that a Woodin cardinal suffices to get a generic elementary embedding $j\colon V \to M$ with critical point ω_1 and ${}^{\omega}M \subseteq M$. With a new, minimizing large cardinal concept isolated, there would now be dramatic new developments both in determinacy and inner model theory. One important scaling down result was the early 1985 result of Shelah: *If κ is*

Woodin, then in a forcing extension $\kappa = \omega_1$ *and* NS_{ω_1} *is* $\aleph_2$*-saturated.* The large cardinal strength now seemed minimal for getting such an ideal, and there was anticipation of achieving an equi-consistency.

Steel in notes of Spring 1985 developed an inner model for a weak version of Woodin cardinal. While inner models for strong cardinals had only required linear iterations for comparison, the new possibility of overlapping extenders and moving generators had led Mitchell in 1979 to develop *iteration trees* of iterated ultrapowers for searching for possible well-founded limits of models along branches. A particularly simple example of an iteration tree is an *alternating chain*, a tree consisting of two ω-length branches with each model in the tree an extender ultrapower of the one preceding it on its branch, via an extender taken from a correspoding model in the other branch. Initially, Steel tried to avoid alternating chains, but the Foreman-Magidor-Shelah work showed that for dealing with Woodin cardinals they would be a necessary part. Their use soon led to a major breakthrough in the investigation of determinacy.

In the Fall of 1985 Martin and Steel showed that Woodin cardinals imply the existence of alternating chains in which both branches have well-founded direct limits, and used this to establish: *If there are infinitely many Woodin cardinals, then* PD *holds.* This was a culmination of method in several respects. In the earlier Martin results getting $\mathbf{\Pi}^1_1$-Determinacy and $\mathbf{\Pi}^1_2$-Determinacy, trees on $\omega \times \kappa$ for some cardinal κ had been used, to each node of which were attached ultrafilters in a coherent way that governed extensions. Kechris and Martin isolated the relevant concept of *homogeneous tree*, the point being that sets of reals which are the projections $p[T]$ of such trees T—the *homogeneously Suslin sets*—are determined. With PD, the scale property had been propagated through the projective hierarchy. Now with Woodin cardinals, having representations via homogeneous trees was propagated, getting determinacy itself. In particular, Martin and Steel established: *If* $n \in \omega$ *and there are* n *Woodin cardinals with a measurable cardinal above them, then* $\mathbf{\Pi}^1_{n+1}$*-determinacy holds.*

Within weeks after the Martin-Steel breakthrough, Woodin used it together with stationary towers to investigate tree representations in $L(\mathbb{R})$ to establish: *If there are infinitely Woodin cardinals with a measurable cardinal above them, then* $\mathrm{AD}^{L(\mathbb{R})}$ *holds.* With the consistency strength of AD having been gauged by this result, Woodin soon established the crowning equi-consistency result: *The existence of infinitely many Woodin cardinals is equi-consistent with the Axiom of Determinacy.* Both directions of this result, worked out with hindsight in Chapters 22 and 23 of this Handbook, involve substantial new arguments.

This was a remarkable achievement of the concerted effort to establish the consistency strength of AD along the large cardinal hierarchy. But even this would just be a beginning for Woodin, who would go from strength to strength in establish many structural results involving AD and stronger principles, to become preeminent with Shelah in set theory.

5.3. Later Developments

In the later 1980s set theory continued to expand apace in various directions, and we conclude our historical survey by mentioning here a few of the most prominent developments, each of a different character but all being decisive advances.

In inner model theory, Martin and Steel in 1986 took the analysis of iteration trees beyond their determinacy work to develop inner models of Woodin cardinals. In order to effect comparison, they for the first time came to grips with the central iterability problem of the existence and uniqueness of iteration trees extending a given iteration tree. They were thus able to establish that "the measurable cardinal above" cannot be eliminated from their determinacy result by showing: *If $n \in \omega$ and there are n Woodin cardinals, then there is an inner model with n Woodin cardinals and a Δ^1_{n+2} well-ordering of the reals.* (The existence of such a well-ordering precludes $\mathbf{\Pi}^1_{n+1}$-determinacy.) These models were of form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders, but the comparison process used did not involve the models themselves, but rather a large model constructed from a sequence of *background extenders*, extenders in the sense of V whose restrictions to $L[\vec{E}]$ led to the sequence $\vec{E}$. With the comparison process thus external to the models, their structure remained largely veiled, and for example only CH, not GCH, could be established.

In 1987 Stewart Baldwin made a suggestion, one which Mitchell then newly forwarded, which led to a crucial methodological advance. Up to then, the extender models $L[\vec{E}]$ constructed relative to a coherent sequence of extenders $\vec{E}$ had each extender E in the sequence "measure" all the subsets in $L[\vec{E}]$ of the critical point. The Baldwin-Mitchell idea was to construct only with "partial" extenders E which if indexed at γ only measures the sets in $L_\gamma[\vec{E}\restriction\gamma]$. This together with a previous Mitchell strategy of carrying out the comparison process using finely calibrated partial ultrapowers ("dropping to a mouse") led to a comparison process internal to $L[\vec{E}]$ based on the use of fine structure. The infusion of fine structure made the development of the new extender models more complex, but with this came the important dividends of a more uniform presentation, a much stronger condensation, and a more systematic comparison process. During 1987-9, Mitchell and Steel worked out the details and showed that if there is a Woodin cardinal then there is an inner model $L[\vec{E}]$, L-like in satisfying GCH and so forth, in which there is a Woodin cardinal. The process involved the correlating of iteration trees for $L[\vec{E}]$ with iteration trees in V and applying the former Martin-Steel results. A canonical, fine structural inner model of a Woodin cardinal newly argued for the consistency of the concept, as well as provided a great deal of understanding about as set in a finely tuned, layer-by-layer hierarchy.

What about a core model "up to" a Woodin cardinal, in analogy to K^{DJ} for $L[U]$? In 1990, Steel solved the "core model iterability problem" by showing that large cardinals in V are not necessary for showing that certain models

$L[\vec{E}]$ have sufficient iterability properties. With this, he constructed a new core model, first building a "background certified" K^c based on extenders in V and then the "true" core model K. Steel was thus able to extend the previous work of Mitchell on the core model $K[\mathcal{U}]$ up to $\exists\kappa(o(\kappa) = \kappa^{++})$ to establish e.g.: *If there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 and a measurable cardinal, then there is an inner model with a Woodin cardinal.* Thus, Shelah's 1985 forcing result and Steel's, except for the artifact of "the measurable cardinal above", had calibrated an important consistency strength, and what had become a central goal of forcing and inner model theory was handily achieved.

In the early 1990s, Steel, Mitchell, and Ernest Schimmerling pushed the Jensen covering argument over the hurdles of the new fine structural Steel core model K to establish a covering lemma up to a Woodin cardinal. Schimmerling both established combinatorial principles in K as well established new consistency strengths, e.g. PFA *implies that there is an inner model with a Woodin cardinal*.

The many successes would continue in inner model theory, but we bring our narrative to a close at a fitting point. Mitchell's Chapter 18 in this Handbook is given over to the concerted study of covering over various models; Steel's Chapter 19 provides the outlines of inner model theory in general terms as well as an important application to HOD; and Schimmerling's Chapter 20 develops Steel's core model K up to a Woodin cardinal as well as provide applications across set theory.

The later 1980s featured a distinctive development that led to a new conceptual framework of applicability to singular cardinals, new incisive results in cardinal arithmetic, and a re-orienting of set theory to new possibilities for outright theorems of ZFC. Starting in late 1987 Shelah returned to the work on bounds for powers of singular cardinals and drew out an extensive underlying structure of possible cofinalities of reduced products, soon codified as *pcf theory*. With this emerged new work in *singular cardinal combinatorics*, with Shelah himself initially providing applications to model theory, partition relations, Jónsson algebras, Boolean algebras, and cardinal arithmetic. This last was epitomized by a dramatic result that exhibited how the newly seen structural constraints impose a tight bound: *If δ is a limit ordinal with $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$ then $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{+4})}$, so that in particular if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Suppose that A is an infinite set of cardinals and F is a filter over A. The *product* ΠA consists of functions f with domain A such that $f(a) \in a$ for every $a \in A$. For $f, g \in \Pi A$, the relation $=_F$ defined by $f =_F g$ *iff* $\{a \in A \mid f(a) = g(a)\} \in F$ is an equivalence relation on ΠA, and the *reduced product* $\Pi A/F$ consists of the equivalence classes. We can impose order, officially on $\Pi A/F$ but still working with functions themselves, by: $f <_F g$ *iff* $\{a \in A \mid f(a) < g(a)\} \in F$.

Shelah's new theory took as central the investigation of the possible cofinalities function:

$$\mathrm{pcf}(A) = \{\mathrm{cf}(\Pi A/D) \mid D \text{ is an ultrafilter over } A\}$$

as calibrated by the ideals

$$J_{<\lambda}[A] = \{b \subseteq A \mid \mathrm{cf}(\Pi A/D) < \lambda \text{ whenever } D \text{ is an ultrafilter over } A \text{ such that } b \in D\}.$$

These concepts had appeared before in Shelah's work, notably in his 1980 result $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$, but now they became autonomous and were propelled forward by the discovery of unexpectedly rich structure.

With an eye to substantive cofinal subsets A of a singular cardinal, the abiding assumption was that A is a set of regular cardinals satisfying $|A| < \min(A)$. With this one gets that for any ultrafilter D over A, $\mathrm{cf}(\Pi A/D) < \lambda$ *iff* $D \cap J_{<\lambda}[A] \neq \emptyset$, and further, that $\mathrm{pcf}(A)$ has a maximum element. At the heart is the striking result that $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda}[A]$ together with a single set $B_\lambda \subseteq A$. Shelah in fact got "nice" generators B_λ derived from imposing the structure of elementary substructures of a sufficiently large $H(\Psi)$. This careful control on the possible cofinalities then led, when A consists of all the regular cardinals in an interval of cardinals, to $|\mathrm{pcf}(A)| \leq |A|^{+++}$, and in particular to the $\aleph_{\omega_4}$ bound mentioned above.

Shelah's work on pcf theory to 1993 appeared in his 1994 book *Cardinal Arithmetic*, and since then he has further developed the theory and provided wide-ranging applications. Through its applicability pcf theory has to a significant extent been woven into modern set theory as part of the ZFC facts of singular cardinal combinatorics. Chapter 14 of this Handbook presents a version of pcf theory and its applications to cardinal arithmetic, and the theory makes it appearance elsewhere as well, most significantly in Chapter 15.

The Singular Cardinal Hypothesis (SCH) and the train of results starting with the Prikry-Silver result of the early 1970s were to be decisively informed by results of Moti Gitik. Gitik's work exhibits a steady engagement with central and difficult issues of set theory and a masterful virtuosity in the application of sophisticated techniques over a broad range. Gitik by 1980 had established through an iterated Prikry forcing the conspicuous singularization result that: *If there is a proper class of strongly compact cardinals, then in a* ZF *inner model of a class forcing extension every infinite cardinal has cofinality* ω. Mentioned earlier was the mid-1970s result that that NS_{ω_1} is precipitous is equi-consistent with having a measurable cardinal. In 1983, Gitik established: *The precipitousness of* NS_{ω_2} *is equi-consistent with having a measurable cardinal* κ *such that* $o(\kappa) = 2$ *in the Mitchell order.* The difficult, forcing direction required considerable ingenuity because of inherent technical obstructions.

Turning to the work on SCH, in 1988 Woodin dramatically weakened the large cardinal hypothesis needed to get a measurable cardinal κ satisfying

$2^\kappa > \kappa^+$, and hence the failure of SCH with the subsequent use of Prikry forcing, to a proposition technically strengthening measurability. He also showed that one can in fact get Magidor's conclusion that $\aleph_\omega$ could be the least cardinal at which GCH fails. Soon afterwards Gitik established both directions of an equi-consistency: First, he established that one can get the consistency of Woodin's proposition from just $\exists\kappa(o(\kappa) = \kappa^{++})$. Then, he applied a result from Shelah's pcf theory to Mitchell's $K[\mathcal{U}]$ analysis to establish, bettering a previous result of Mitchell, that $\exists\kappa(o(\kappa) = \kappa^{++})$ is actually necessary to get the failure of SCH. Hence, *The failure of* SCH *is equi-consistent with* $\exists\kappa(o(\kappa) = \kappa^{++})$.

Woodin's model in which GCH first fails at $\aleph_\omega$ required a delicate construction to arrange GCH below and an ingenious idea to get $2^{\aleph_\omega} = \aleph_{\omega+2}$. How about getting $2^{\aleph_\omega} > \aleph_{\omega+2}$? In a signal advance of method, Gitik and Magidor in 1989 provided a new technique to handle the general singular cardinals problem with appropriately optimal hypotheses. The Prikry-Silver two-stage approach, first making 2^κ large and then singularizing κ without adding any new bounded subsets or collapsing cardinals, had been the basic model for attacking the singular cardinals problem. Gitik and Magidor showed how to add many subsets to a large cardinal κ while *simultaneously* singularizing it without adding any new bounded subsets or collapsing cardinals. Thus, it became much easier to arrange any particular continuum function behavior below κ, like achieving GCH below, while at the same time making 2^κ arbitrarily large. Moreover, the new method smacked of naturalness and optimality.

The new Gitik-Magidor idea was to add many new Prikry ω-sequences corresponding to κ-complete ultrafilters over κ while maintaining the basic properties of Prikry forcing. There is an evident danger that if these Prikry sequences are too independent, information can be read from them that corresponds to new reals being adjoined. The solution was to start from a sufficient strong large cardinal hypothesis and develop an extender-based Prikry forcing structured on a "nice system" of ultrafilters $\langle U_\alpha \mid \alpha < \lambda\rangle$, a system such that for many $\alpha \leq \beta < \lambda$ there is a ground model function $f\colon \kappa \to \kappa$ such that: For all $X \subseteq \kappa$, $X \in U_\alpha$ *iff* $f^{-1}(X) \in U_\beta$. (Having such a projection function is the classical way of connecting two ultrafilters together, and one writes that $U_\alpha \leq_{\mathrm{RK}} U_\beta$ under the Rudin-Keisler partial order.) By this means one has the possibility of adding new subsets of κ, corresponding to different Prikry sequences, which are still dependent on each other so that no new bounded subsets need necessarily be added in the process. Gitik and Magidor worked out how their new approach leads to what turns out to be optimal or near optimal consistency results, and incorporating collapsing maps as in previous arguments of Magidor and Shelah, they got models in which GCH holds below $\aleph_\omega$ yet $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any prescribed countable ordinal α.

In subsequent work Gitik, together with Magidor, Mitchell, and others, have considerably advanced the investigation of powers of singular cardinals.

Equi-consistency results have been achieved for large powers of singular cardinals along the Mitchell order and with α-strong cardinals, and uncountable cofinalities have been encompassed, the investigation ongoing and with dramatic successes. Much of this work is systematically presented in Gitik's Chapter 16 in this Handbook.

We now leave the overall narrative, having pursued several longitudinal themes to appropriate junctures. Stepping back to gaze at modern set theory, the thrust of mathematical research should deflate various possible metaphysical appropriations with an onrush of new models, hypotheses, and results. Shedding much of its foundational burden, set theory has become an intriguing field of mathematics where formalized versions of truth and consistency have become matters for manipulation as in algebra. As a study couched in well-foundedness ZFC together with the spectrum of large cardinals serves as a court of adjudication, in terms of relative consistency, for mathematical propositions that can be informatively contextualized in set theory by letting their variables range over the set-theoretic universe. Thus, set theory is more of an open-ended framework for mathematics rather than an elucidating foundation. It is as a field of mathematics proceeding with its own internal questions and capable of contextualizing over a broad range that set theory has become an intriguing and highly distinctive subject.

6. Summaries of the Handbook Chapters

This Handbook is divided into three volumes with the first devoted to Combinatorics, the Continuum, and Constructibility; the second devoted to Elementary Embeddings and Singular Cardinal Combinatorics, and the third devoted to Inner Models and Determinacy.

The following chapter summaries engage the larger historical contexts as they serve to introduce and summarize the contents. In many cases we build on our preceding survey as a framework and proceed to elaborate it in the directions at hand, and in some cases we introduce the topics as new offshoots and draw them in. Consequently, some summaries are shorter on account of the leads from the survey and others longer because of the new lengths to which we go.

VOLUME I

1. Stationary Sets. The veteran set theorist Thomas Jech is the author of *Set Theory* (third millennium edition, 2002), a massive and impressive text that comprehensively covers the full range of the subject up to the elaborations of this Handbook. In this first chapter, Jech surveys the work directly involving stationary sets, a subject to which he has made important contributions. In charting out the ramifications of a basic concept buttressing the uncountable, the chapter serves, appropriately, as an anticipatory guide to techniques and results detailed in subsequent chapters.

The first section provides the basic theory of stationary subsets of a regular uncountable cardinal κ. The next describes the possibilities for *stationary set reflection*: For $S \subseteq \kappa$ stationary in κ, is there an $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α? With reflection having become an important heuristic in set theory, stationary set reflection commended itself as a specific, combinatorial possibility for investigation. Focusing on the non-stationary ideal, the third section surveys the possibilities for its saturation and precipitousness.

The later sections study these various issues as adapted to notions of closed unbounded and stationary for subsets of $P_\kappa\lambda = \{x \in P(\lambda) \mid |x| < \kappa\}$, a study that the author had pioneered in the early 1970s. The wide-ranging involvements in proper forcing, Boolean algebras and stationary tower forcing are described. Of particular interest are reflection principles based on $P_{\aleph_1}\lambda$. Foreman, Magidor, and Shelah in their major 1984 work had shown that Martin's Maximum implies that a substantial reflection principle holds for stationary subsets of $P_{\aleph_1}\lambda$ for every $\lambda \geq \omega_2$. Todorcevic then showed that a stronger reflection principle SRP follows from MM, one from which substantial consequences of MM already follow, like the $\aleph_2$-saturation of NS_{ω_1}. Qi Feng and Jech subsequently formulated a streamlined principle PRS equivalent to SRP.

2. Partition Relations. In this chapter two prominent figures in the field of partition relations, András Hajnal and Jean Larson, team up to present the recent work, the first bringing to bear his expertise in relations for uncountable cardinals and the second her expertise in relations for countable ordinals. The investigation of partition relations has been a steady, rich, and concrete part of the combinatorial investigation of the transfinite, a source of intrinsically interesting problems that have stimulated the application of a variety of emerging techniques.

With the classical, 1956 Erdős-Rado Theorem $\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa$ having established the context as the transfinite generalization of Ramsey's Theorem, extensive use of the basic tree or "ramification" method had led by the mid-1960s to an elaborately parametrized theory. This theory was eventually presented in the 1984 Erdős-Hajnal-Rado-Máté book, which is initially reflected in the first two sections of the chapter.

The next sections emphasize new methods as leading not only to new results but also providing new proofs of old results, and in this spirit they develop a 1991 method of Baumgartner, Hajnal, and Todorcevic and establish their generalizations of the Erdős-Rado Theorem. This method involves taking chains of elementary substructures of a sufficiently rich structure $\langle H(\lambda), \in, <^*, \ldots \rangle$ and associating ideals along the way. Next, the enhanced method of the recent, 1998 Foreman-Hajnal result on successors of measurable cardinals is used establish a watershed, 1972 Baumgartner-Hajnal Theorem in the special case $\omega_1 \longrightarrow (\alpha)^2_m$ for any $\alpha < \omega_1$ and $m \in \omega$. Shelah, with his considerable combinatorial prowess, has steadily made important contributions to the theory of partition relations, and several are presented, among them a recent result involving strongly compact cardinals and another invoking his pcf theory.

The investigation of partition relations for small countable ordinals was a current from the beginnings of the general theory in the late 1950s and has focused, for natural reasons, on the relation $\alpha \longrightarrow (\alpha, m)^2$ for finite m, the assertion that if the pairs from α are assigned 0 or 1, then either there is an $H \subseteq \alpha$ of ordertype α all of whose pairs are assigned 0, or m elements in α all of whose pairs are assigned 1. A formative early 1970s result was Chen-Chung Chang's that with ordinal exponentiation, $\omega^\omega \to (\omega^\omega, 3)^2$, the proof considerably simplified by Larson. Remarkably, after the passing of more than two decades Carl Darby and Rene Schipperus working independently established the first new positive and negative results, the latter by way of the same counterexamples. In the last two sections, a negative result $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 6)$ and a positive result $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)$ are established, the careful combinatorial analysis in terms of blocks of ordinals and trees illustrative of some of the most detailed work with small ordertypes.

3. Coherent Sequences. This chapter is a systematic account by Stevo Todorcevic of his penetrating analysis of uncountable order structures, with ω_1 being both a particular and a paradigmatic case. The chapter is a short version of his recent monograph *Walks on Ordinals and Their Characteristics* (2007), but has separate value for being more directed and closer to the historical route of discovery.

The analysis for a regular cardinal θ begins with a *C-sequence* $\langle C_\alpha \mid \alpha < \theta \rangle$ where for successors $\alpha = \beta + 1$, $C_\alpha = \{\beta\}$, and for limits α, C_α is a closed unbounded subset of α. In the case $\theta = \omega_1$, one requires that for limits α, C_α has order-type ω, so that we have a "ladder system". One can climb up, but also walk down: Given $\alpha < \beta < \theta$, let β_1 be the least member of $C_\beta - \alpha$, let β_2 the least member of $C_{\beta_1} - \alpha$, and so forth, yielding the *walk* $\beta > \beta_1 > \ldots > \beta_n = \alpha$. Through a sustained analysis Todorcevic has shown that these walks have a great deal of structure as conveyed by various "distance functions" or "characteristics" ρ on $[\theta]^2$, where $\rho(\alpha, \beta)$ packages information about the walk from β to α.

Initially, Todorcevic in 1985 used such a function to settle the main partition problem about the complexity of ω_1, by establishing the negative "square brackets partition relation" $\omega_1 \not\to [\omega_1]^2_{\omega_1}$: There is a function $f\colon [\omega_1]^2 \to \omega_1$ such that for any unbounded $X \subseteq \omega_1$, $f``[X]^2 = \omega_1$, i.e. for any $\zeta < \omega_1$ there are $\alpha < \beta$ both in X such that $f(\alpha, \beta) = \zeta$. Todorcevic's f was based on the property that if $S \subseteq \omega_1$ is stationary, then for any unbounded $X \subseteq \omega_1$ there are $\alpha < \beta$ both in X such that the walk from β to α has a member of S. More generally, Todorcevic introduced the *oscillation map* to effect a version of this property for regular $\theta > \omega_1$ to show that if there is a stationary $S \subseteq \theta$ which does not reflect, i.e. there is no $\alpha < \theta$ such that $S \cap \alpha$ is stationary in α, then the analogous $\theta \not\to [\theta]^2_\theta$ holds.

The first sections of the chapter develops several distance functions for the case $\theta = \omega_1$ as paradigmatic. Systematic versions of "special" Aronszajn trees and the (Shelah) result that adding a Cohen real adds a Suslin tree are presented, as well as a range of applications to Hausdorff gaps, Banach spaces, model theory, graph theory and partition relations.

The later sections encompass general θ, with initial attention given to systematic characterizations of Mahlo and weakly compact cardinals. There is soon a focus on *square* (or *coherent*) sequences, those C-sequences $\langle C_\alpha \mid \alpha < \theta \rangle$ such that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit of C_β. With these a range of applications is provided involving the principle $\square_\kappa$, higher Kurepa trees, and Jensen matrices. The oscillation map is latterly introduced, and with it the proof of the general negative square brackets partition relation as stated above. Finally, elegant characterizations for Chang's Conjecture are presented. Throughout, there is the impression that one has gotten at the immanent structure of the uncountable from which a wide range of combinatorial consequences flow.

4. Borel Equivalence Relations. Descriptive set theory as fueled by the incentive for generalization is appropriately construed as the investigation of definable sets in *Polish spaces*, i.e. separable, completely metrizable spaces. For such spaces one can define the Borel and projective sets and the corresponding hierarchies through definability. In the 1990s fresh incentives came into play that expanded the investigation into quotient spaces X/E for a Polish space X and a definable equivalence relation E on X, such quotients capturing various important structures in mathematics. New methods had to be developed, in what amounts to the investigation of definable equivalence relations on Polish spaces.

In this short chapter Greg Hjorth provides a crisp survey of Borel equivalence relations on Polish spaces as organized around the Borel reducibility ordering $\leq_B$. In an initial disclaimer, he points out how he is leaving aside several other approaches, but in any case his account provides a worthy look at a modern, burgeoning subject.

For Polish spaces X and Y, a function $f\colon X \to Y$ is *Borel* if the preimage of any Borel set is Borel. An equivalence relation on X is *Borel* if it is Borel as a subset of $X \times X$. If E is a Borel equivalence relation on X and F is a Borel equivalence relation on Y, then $E \leq_B F$ asserts that there is a Borel $f\colon X \to Y$ such that $x_1 E x_2 \leftrightarrow f(x_1) F f(x_2)$. The emphasis here is on the equivalence relations, with only the Borel sets of the underlying spaces being at issue. There is the correlative $E <_B F$, and with $\mathrm{id}(X)$ indicating the identity relation on X, an example is $\mathrm{id}(\mathbb{R}) <_B E_0$, where E_0 is the equivalence relation of eventual agreement on ${}^\omega 2$. E_0 is a reconstrual of Vitali's classical equivalence relation, with which he established that with AC there is a non-Lebesgue measurable set. The seminal Harrington-Kechris-Louveau "Glimm-Effros dichotomy" result is: *For any Borel equivalence relation E, exactly one of $E \leq_B \mathrm{id}(\mathbb{R})$ or $E_0 \leq_B E$ holds.*

Starting with this seminal result the author discusses various structure theorems, concluding with his work on *turbulence*. Next is the work on countable Borel equivalence relations, i.e. those whose equivalence classes are all countable. This topic has notable interactions across diverse fields of mathematics, and an enduring problem is how to characterize the *hyperfinite* Borel equivalence relations. The author next discusses $\leq_B$ as effective cardinality, bringing in his results with determinacy. The final topic is classification problems, problems of locating variously given Borel equivalence relations in the structure given by $\leq_B$. The range of issues here speaks to the importance and relevance of Borel equivalence relations in larger mathematics.

5. Proper Forcing. Uri Abraham provides a lucid exposition of Shelah's proper forcing. In a timely monograph *Proper Forcing* (1982) and a book *Proper and Improper Forcing* (1998), Shelah had set out his penetrating, wide-ranging work on and with proper forcing. Striking a nice balance, Abraham presents the basic theory of proper forcing and then some of the variants and their uses that illustrate its wide applicability. This chapter is commended to the reader conversant even with only the basics of forcing to assimilate what has become a staple part of the theory and practice of forcing. To be noted is that being of the Israeli school, Abraham writes "$p > q$" for p being a stronger condition than q.

In the first two sections, basic forcing notions are reviewed, and proper forcing is motivated and formulated. The basic lemma that properness is preserved in countable support iterations is carefully presented, as well as the basic fact that under CH a length $\leq \omega_2$ iteration of $\aleph_1$ size proper forcings satisfies the $\aleph_2$-chain condition and so preserves all cardinals.

A forcing partial order P is *${}^\omega\omega$-bounding iff* the ground model reals are cofinal under eventual dominance $<^*$ in the reals of any generic extension by P. The third section presents the preservation of ${}^\omega\omega$-bounding properness in countable support iterations. With this is established a finely wrought result of Shelah's, answering a question of classical model theory, that it is consistent that there are two countable elementarily equivalent structures having no isomorphic ultrapowers by any ultrafilter over ω.

A forcing partial order P is *weakly ${}^\omega\omega$-bounding iff* the ground model reals are unbounded under eventual dominance $<^*$ in the reals of any generic extension by P. The fourth section presents the preservation of weakly ${}^\omega\omega$-bounding properness, one that deftly and necessarily has to assume a stronger property at successor stages. With this is established another finely wrought result of Shelah's, answering a question in the theory of cardinal characteristics, that it is consistent with $2^{\aleph_0} = \aleph_2$ that the bounding number $\mathfrak{b}$ is less than the splitting number $\mathfrak{s}$.

The final section develops iterated proper forcing that adjoins no new reals. A relatively complex task, this has been a prominent theme in Shelah's work, and to this purpose he has come up with several workable conditions. Abraham motivates one condition, *Dee-completeness*, with his first result in set theory, and then establishes an involved preservation theorem. As pointed out, through this approach one can provide a new proof of Jensen's result that CH + SH is consistent, which for Shelah was an important stimulus in his initial development of proper forcing.

6. Combinatorial Cardinal Characteristics of the Continuum. This and the next chapters cover the recent, increasingly systematic, work across the wide swath having to do with cardinal characteristics, or invariants, of the continuum and their possible order relationships. In this chapter, the broad-ranging Andreas Blass provides a perspicuous account of combinatorial cardinal characteristics through to some of his own work. He deftly introduces characteristics in turn together with more and more techniques for their analysis, and at the end surveys the extensive forcing consistency results.

There is initially a discussion of the dominating number $\mathfrak{d}$ and the bounding number $\mathfrak{b}$, one that introduces several generalizing characteristics corresponding to an ideal $\mathcal{I}$: $\mathbf{add}(\mathcal{I})$, $\mathbf{cov}(\mathcal{I})$, $\mathbf{non}(\mathcal{I})$, $\mathbf{cof}(\mathcal{I})$. The next topic is the splitting number $\mathfrak{s}$ and related characteristics having to do with Ramsey-type partition theorems.

A systematic approach, first brought out by Peter Vojtáš, is then presented for describing many of the characteristics and the relationships among them. A triple $\mathbf{A} = \langle A_-, A_+, A\rangle$ such that $A \subseteq A_- \times A_+$ is simply a *relation*, and its *norm* $\|\mathbf{A}\|$ is the smallest cardinality of any $Y \subseteq A_+$ such that $\forall x \in A_- \exists y \in Y(\langle x, y\rangle \in A)$. The *dual* of $\mathbf{A} = \langle A_-, A_+, A\rangle$ is $\mathbf{A}^\perp = \langle A_+, A_-, \neg\breve{A}\rangle$, where $\neg\breve{A}$ is the complement of the converse $\breve{A}$ of A, i.e. $\langle x, y\rangle \in \neg\breve{A}$ *iff* $\langle y, x\rangle \notin A$. In these terms, for example, if $\mathfrak{D} = \langle {}^\omega\omega, {}^\omega\omega, <^*\rangle$, then $\|\mathfrak{D}\| = \mathfrak{d}$ and $\|\mathfrak{D}^\perp\| = \mathfrak{b}$. A *morphism* for a relation $\mathbf{A} = \langle A_-, A_+, A\rangle$ to another $\mathbf{B} = \langle B_-, B_+, B\rangle$ is a pair $\phi = (\phi_-, \phi_+)$ of functions such that $\phi_- : B_- \to A_-$; $\phi_+ : A_+ \to B_+$; and

$$\forall b \in B_- \forall a \in A_+(\langle \phi_-(b), a\rangle \in A \to \langle b, \phi_+(a)\rangle \in B)\,.$$

It is seen that having such a morphism implies that $\|\mathbf{A}\| \geq \|\mathbf{B}\|$ and $\|\mathbf{A}^\perp\| \leq \|\mathbf{B}^\perp\|$. Through this overlay of relations and morphisms one can efficiently incorporate both categorical combinations of relations as well as conditions on morphisms, like being Borel or continuous, into the study of characteristics.

The author proceeds to discuss characteristics corresponding to the ideal $\mathcal{B}$ of meager sets and to the ideal $\mathcal{L}$ of null sets: $\mathbf{add}(\mathcal{B})$, $\mathbf{cov}(\mathcal{B})$, $\mathbf{non}(\mathcal{B})$, $\mathbf{cof}(\mathcal{B})$, $\mathbf{add}(\mathcal{L})$, $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{L})$, $\mathbf{cof}(\mathcal{L})$. The main results are established in terms of relations and morphisms, and one gets to the inequalities among these characteristics and $\mathfrak{b}$ and $\mathfrak{d}$ as given by what is known as *Cichoń's diagram*. The characteristics of measure and category are further pursued in the next chapter.

The succeeding topics have to do with cardinalities of families $F \subseteq P(\omega)$ as mediated by $\subseteq^*$, where $X \subseteq^* Y$ *iff* $X - Y$ is finite. Forcing axioms are brought into play as now particularly informative for drawing ordering conclusions. Then characteristics corresponding to maximal almost disjoint (MAD) families and independent families are investigated.

The author finally discusses characteristics related to or developed through his own work. Discussing filters and ultrafilters over ω, he gets to his principle of Near Coherence of Filters (NCF), a principle proved consistent by Shelah, and results about ultrafilters generated from filters in terms of characteristics. He then discusses his *evasion* and *prediction*, which initially had an algebraic motivation but became broadened into a combinatorial framework that provides a unifying approach to many of the characteristics.

The concluding section is largely a survey of what happens to the characteristics when one iteratively adjoins many generic reals of one kind, dealing in turn with the following reals: Cohen, random, Sacks, Hechler, Laver, Mathias, Miller. As such, this is an informative account of these various generic reals and how they mediate the continuum.

7. Invariants of Measure and Category. Tomek Bartoszynski presents the recent work on measure and category as viewed through their cardinal invariants, or characteristics. The account updates the theory presented in the substantive *Set Theory: On the Structure of the Real Line* (1995) by Bartoszynski and Haim Judah, which had stood as a standard reference for this general area for quite some time.

After putting the language of relations and morphisms (see the previous summary) in place, the author pursues an approach, one advocated by Ireneusz Recław, of emphasizing classes of sets "small" according to various criteria corresponding to the ideal invariants. One develops *Borel* morphisms that lead to inclusion relations among the classes and thence to the inequalities of Cichon's diagram. Combinatorial characterizations of membership in these classes and thus of the invariants are given, as well as a new understanding of the ideal of null sets as maximal, in terms of embedding, among analytic P-ideals.

Turning to cofinality, the author establishes Shelah's remarkable and unexpected 1999 result that it is consistent that $\mathrm{cf}(\mathbf{cov}(\mathcal{L})) = \omega$. The author then provides a systematic way of associating to each of the invariants in Cichon's diagram a generic real so that iteration with countable support increases that invariant and none of the others. Corresponding issues about the classes of small sets further draw in proper forcing techniques.

8. Constructibility and Class Forcing. In this chapter Sy Friedman presents work on the limits of possibilities for reals in terms of forcing and constructibility, the supporting technique being *Jensen coding*. In the mid-1960s Solovay, when investigating the remarkable properties of $0^{\#}$, raised several questions about the scope of the recently devised forcing method. For sets x, y let $x \leq_L y$ denote that x is constructible from y, i.e. $x \in L[y]$, and let $x <_L y$ be correlative. $0^{\#}$ cannot be adjoined to L by forcing because of its global consequences for L, but $0^{\#}$ was plausibly considered minimal in this respect. A (weak form of a) question of Solovay's was: If r is a real satisfying $r <_L 0^{\#}$, does r belong to some generic extension of L?

In 1975-6 Jensen devised his impressive "coding the universe in a real" technique and with it established (a strong form of): *If* GCH *holds, then there is a class partial order P such that if G is P-generic, then $V[G]$ has the same cardinals and cofinalities yet for some real r there, $V[G] \models$ "$V = L[r]$".* The intricately woven P here was built using fine structure theory in L-like situations and provided a means of coding up more and more layers of the cumulative hierarchy while crucially maintaining its cardinal structure. Not only cofinalities but those properties compatible with models of form $L[r]$ all continue to hold, so that this real r veritably codes the entire universe. Jensen showed that assuming $0^\#$ exists it is consistent that there is such a real $r <_L 0^\#$, answering Solovay's question in the negative, the intention there having been to address forcing with *set* partial orders. Not only did Jensen bring class forcing into prominence for establishing new consistency results about sets, but also for establishing outright theorems of ZFC + "$0^\#$ exists".

Starting in the mid-1980s Friedman reworked and extended the Jensen theory and established some notable results about $0^\#$ and class forcing, and this work eventually appeared in his book *Fine Structure and Class Forcing* (2000). This chapter is a short version of the book, appropriate to the task of working more directly toward several problems of Solovay and developing techniques where needed. After stating three problems of Solovay as motivation, Friedman develops the criterion of *tameness* for class partial orders for preserving ZFC and gets at the property of *relevance*, having a generic definable in $L[0^\#]$. He then provides his proof of Jensen's coding theorem assuming that $0^\#$ does not exist, this assumption allowing a comparatively simple argument free of fine structure but making appeals to the Jensen Covering Theorem. With this the Solovay problems are addressed in turn. To conclude, wide-ranging applications are given as well as a nice list of open problems.

9. Fine Structure. This and the next chapter deal with fine structure and are complementary in that they present different versions, both due initially to Jensen, as well as applications in different directions. In this chapter Ralf Schindler and Martin Zeman provide an incisive, self-contained account of Jensen's original fine structure theory for the J_α hierarchy relativized to a predicate A. Much is drawn from Zeman's book *Inner Models and Large Cardinals* (2002), but diverging from it Schindler and Zeman steer to the use of the Mitchell-Steel $r\Sigma_n$ formulas for discussing iterated projecta and embeddings. With A being a sequence of extenders this was the approach that had been taken for the use of fine structure in inner model theory. The chapter thus provides the fine structure groundwork for Chapters 18, 19, and 20 of this Handbook.

After the preliminaries about J-structures, the chapter focuses on the *acceptable* ones, those that satisfy GCH in a strong form. The projecta of these J-stuctures are described, and then the Downward and Upward Extensions of Embeddings Lemmas are established. Iterated projecta are then formulated and $r\Sigma_n$ introduced for expressing preservation through embeddings using *very good* parameters. Next, standard parameters are fully analyzed and all the considerations about *soundness* and *solidity witnesses* necessary for inner model theory are given.

A later section analyzes *fine ultrapowers*, fine structure preserving ultrapowers by extenders, treating the "short" and "long" cases uniformly, and draws out the connections with the Upward Extensions of Embeddings Lemma. Finally, two illustrative applications to L are presented, with generalizable arguments: a proof, in the absence of $0^{\#}$, of the "countably closed" weak covering property for L and a proof of $\Box_\kappa$ for $\kappa > \omega$.

10. Σ^* Fine Structure. Philip Welch considerably rounds out the discussion of fine structure by presenting the Σ^* version and the extensive work on square principles and morasses, providing commentary throughout about the interactions with inner model theory.

Σ^* fine structure is due to Jensen and detailed in Zeman's book *Inner Models and Large Cardinals* (2002). The theory is a notable advance in that it isolated the "right" classes of formulas for the articulation of fine structure results. The classes form a certain ramified version of the Levy hierarchy, the $\Sigma_k^{(n)}$ formulas for $n, k \in \omega$, which level-by-level are able to capture syntactically the semantic role of standard parameters. In particular, $\Sigma_1^{(n)}(J_\alpha)$ relations can be uniformized by $\Sigma_1^{(n)}(J_\alpha)$ relations defined uniformly for all α. And the $\Sigma_1^{(n)}$ formulas are exactly the formulas preserved by the $r\Sigma_{n+1}$ embeddings involving very good parameters.

The first section of the chapter establishes the Σ^* theory, with the treatment much as in Zeman's book. The Σ^* approach is shown to advantage in the development of the Σ^* *ultrapower*, Σ^* fine structure preserving extender ultrapowers. Then the more general pseudo-ultrapower (which corresponds to the use of "long" extenders) is developed, with a refinement toward coming applications.

The second section is devoted to square principles. Jensen had established that if $V = L$, then in addition to the principles $\Box_\kappa$ a global, class version $\Box$ holds. Most of the section is taken up by a Σ^* pseudo-ultrapower proof of this result, one that provides a global $\Box$ sequence with uniform features.

The section concludes with an extensive and detailed description of the recent investigation of square principles in inner models. Of particular interest is the failure of $\Box_\kappa$, this for singular κ precluding covering properties for inner models. Around 2000 an elucidating systemic characterization was achieved. Solovay's initial 1970s result—that if κ is λ^+-supercompact and $\lambda \geq \kappa$, then $\Box_\lambda$ fails—had led to refinements, and Jensen had extracted a streamlined large cardinal concept, later dubbed *subcompactness*, still sufficient so that: *If κ is subcompact, then $\Box_\kappa$ fails.* Then in a remarkable analysis, Zeman and Schimmerling established: *In "Jensen-style" extender models $L[\vec{E}]$, if $\Box_\kappa$ fails, then κ is subcompact.* These results established the reach of $\Box_\kappa$ well beyond current inner model theory, in that subcompact cardinals, far stronger than Woodin cardinals, are not known to have canonical inner models. By 2005 Steel established: *If $\Box_\kappa$ fails for some singular strong limit cardinal κ, then* $\mathrm{AD}^{L(\mathbb{R})}$ *holds.*

The chapter is brought in an end with a survey of the extensive work on morasses. A $(\kappa, 1)$ morass is a system approximating the L_α's for $\kappa < \alpha \leq \kappa^+$

by means of L_β's for $\beta < \kappa$ and maps $f_{\beta,\beta'}$ between them as regulated by a series of conditions. Just after his development of fine structure Jensen formulated morasses and established their existence in L in order to establish model-theoretic "cardinal transfer" theorems there. A great deal of work has since been carried out on morass structures as providing approximations to large structures in terms of indexed arrays of small structures, and morasses have come to carry the weight of the extent of combinatorial structure in the constructible universe.

VOLUME II

11. Elementary Embeddings and Algebra. In this chapter Patrick Dehornoy presents a notable development arising out of the investigation of algebraic features of very strong elementary embeddings. After Kunen established his result that a strong large cardinal postulation is inconsistent, it was natural to investigate remaining possibilities just weaker and so still of great consistency strength. One was that there exists a (non-identity) elementary embedding $j\colon V_\lambda \to V_\lambda$ for some limit λ. There is a collective structure here, for letting $\mathcal{E}_\lambda$ be the set of such embeddings, $\mathcal{E}_\lambda$ is closed under functional composition $\circ$, as well as *application*: For $j, k \in \mathcal{E}_\lambda$, let $j[k] = \bigcup_{\gamma<\lambda} j(k \cap V_\gamma)$, regarding k of course as a set of ordered pairs; then $j[k]$ is in $\mathcal{E}_\lambda$ as well. Composition $\circ$ and application [ ] together satisfy a handful of laws, and the latter satisfies the left distributive law $j[k[l]] = j[k][j[l]]$. Martin's 1978 result, that if there is an "iterable" elementary $j\colon V_\lambda \to V_\lambda$ then $\mathbf{\Pi}^1_2$-Determinacy holds, first used application and these laws for j self-applied.

Laver saw that application provided a wealth of elementary embeddings and a proliferation of critical points, and initiated a systematic investigation into the structure of $\mathcal{E}_\lambda$ for its own sake. In 1989 he established the freeness of the subalgebra generated by one j in $\langle \mathcal{E}_\lambda, [\,]\rangle$ subject to the left distributive law and the analogous result for $\langle \mathcal{E}, [\,], \circ\rangle$. Moreover, with his analysis Laver established that the corresponding word problem for the left distributive law is solvable, i.e. it is recursively decidable whether two given expressions in the language of one generator and [ ] are equivalent according to the left distributive law. This elicited considerable interest, with a hypothesis near the limits of consistency entailing solvability in finitary mathematics. In 1992 Dehornoy eliminated the large cardinal assumption from the solvability result with an elegant argument that led to unexpected results about the Artin braid group.

Dehornoy in this chapter effectively presents the body of work on $\mathcal{E}_\lambda$ and the left distributive law. Beyond the solvability of the word problem, he also presents the Laver-Steel theorem about the set of critical points of members of $\mathcal{E}_\lambda$ having ordertype ω, a result that initially applied results about the Mitchell ordering in inner model theory; Randall Dougherty's result that the growth rate of the critical points is faster than Ackermann's function; and results on the finite "Laver tables" using $\mathcal{E}_\lambda \neq \emptyset$ that thus far have not been established in ZFC alone.

12. Iterated Forcing and Elementary Embeddings. James Cummings provides a lucid exposition of that core part of the mainstream of forcing and large cardinals having to do with iterated forcing and extensions of elementary embeddings. Forcing and large cardinals are elaborated in the directions of ideals and generic elementary embeddings in the next chapter and in the direction of Prikry-type forcings in Chapter 16. Drawing on his wide-ranging knowledge, Cummings provides a well-organized account, in mainly short, crisp sections, starting from the basics and proceeding through a series of techniques, with historical progression a rough guide and conceptual complexity a steady one. This chapter is commended to the reader conversant even with only the basics of forcing and large cardinals to assimilate what have become important techniques of modern set theory.

The early sections proceed leisurely through the basics of elementary embeddings, ultrapowers and extenders, large cardinal axioms, forcing, some forcing partial orders, and iterated forcing. The first ascent is to building generic objects to extend ("lift") elementary embeddings in forcing extensions. Describing Silver's Easton support iteration and the key idea of master condition, his 1971 result is established: *If κ is κ^{++}-supercompact, then there is a forcing extension in which κ is measurable and $2^\kappa = \kappa^{++}$.* Next, Magidor's important technique of making do with an "increasingly masterful" sequence of conditions is presented. Then, the general idea of *absorption*, embedding a complex partial order into a simple one, is discussed. This is illustrated with Magidor's 1982 result (also highlighted in Chapter 15): *If there are infinitely many supercompact cardinals, then in a forcing extension in which they become the $\aleph_n$'s, every subset of $\aleph_{\omega+1}$ reflects.*

Precipitiousness is the subject of the two longer sections of the chapter. In the first, the Jech-Prikry-Mitchell-Magidor mid-1970s result is established, building on the previous work: *If there is a measurable cardinal κ, then there is a forcing extension in which $\kappa = \omega_1$ and NS_{ω_1} is precipitous.* This involves exploiting the absorptive properties of the initial Levy collapse with iterated "club shooting". In the second, and longest, section a Gitik 1983 result is established: *The precipitousness of NS_{ω_2} is equi-consistent with having a measurable cardinal κ such that $o(\kappa) = 2$ in the Mitchell order.* The difficult, forcing direction exhibited Gitik's virtuosity of technique, and all the features of a "preparation forcing" before the iterated club shooting are carefully laid out: Namba forcing, RCS iteration, the S and $\mathbb{I}$ conditions.

The rest of the chapter reverts to short sections that describe a wide range of techniques and results, of which we mention the more conspicuous. Presenting Kunen's universal collapse and Silver's collapse, Kunen's focal 1972 result is established: *If κ is huge, then there is forcing extension in which $\kappa = \omega_1$ and there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1.* Laver's termspace forcing for introducing a universal generic object by forcing with a partial order of terms is described and applied to establish Magidor's 1973 result: *It is consistent that the least strong compact cardinal is the least measurable cardinal.* The "Laver diamond" and its original use to make supercompact cardinals "indestructible" is presented, and with this Baumgartner's 1983 consistency result is established: *If there is a supercompact cardinal* κ*, then there is a forcing extension in which* $\kappa = \omega_2$ *and* PFA *holds.* Finally, Woodin's technique of "altering generic objects" is used to establish his 1988 consistency result of getting a measurable cardinal κ satisfying $2^\kappa > \kappa^+$ from what turned out, by later work of Gitik, to be the optimal hypothesis. The encompassing of these various, historically important results in one chapter speak to how iterated forcing methods have been comprehensively assimilated in modern set theory.

13. Ideals and Generic Elementary Embeddings. In this the longest chapter of this Handbook, Matthew Foreman provides a wealth of methods and results surrounding the general theme of ideals and generic elementary embeddings. What is at play is the basic synthesis of forcing and ultrapowers whereby one starts with an ideal I over a cardinal κ; forces with $P(\kappa) - I$ where p is stronger than q if $p - q \in I$; thus produces an ultrafilter over the ground model $P(\kappa)$; and then gets a generic elementary embedding of the ground model into the corresponding ultrapower. With the possibilities of ideals occurring low in the cumulative hierarchy, so that large cardinal ideas can be brought to bear on classical problems of set theory, an enormous subject has grown as attested to by this chapter. Indeed, in it a very wide range and variety of material have been marshalled, and this comes together with an informal and inviting engagement that provides if not proofs, sketches of proofs, and if not sketches, outlines that "show".

Not just a compendium, the chapter has been organized in terms of overall guiding themes. At the broadest level are the "three parameters" describing the strength of a generic elementary embedding $j\colon V \to M$: how j moves the ordinals; how large and closed M is; and the nature of the forcing that provided j. This last is the new parameter at play beyond the "conventional" large cardinal hypotheses. Ideals through their forcing properties thus assuming a crucial role, another guiding theme is the distinction between "natural" ideals that have intrinsic definitions and ideals "induced" by elementary embeddings. As the chapter progresses, strong ideal properties gain an autonomy as "generic large cardinal hypotheses" in their own right, and the chapter is further delineated according to consequences of generic large cardinals and consistency results about them.

Section 2 introduces the basics of generic ultrapowers and begins the study of the correspondence between combinatorial properties of ideals and structural properties of generic ultrapowers. Topics include criteria for precipitousness, the disjointing property, normality, limitations on closure, canonical functions, selectivity and the use of generic embeddings for reflection.

Section 3 provides a range of examples of natural and induced ideals. Among the natural ideals considered are the nonstationary ideals NS_λ, their important generalizations to nonstationary subsets over power sets $P(X)$, *Chang ideals*, Shelah's $I[\lambda]$ and *club guessing ideals*, *non-diamond ideals*, and *uniformization ideals*. How induced ideals arise is taken up next, with an important example being the *master condition ideals*, with their connections to proper forcing. In a general setting, *goodness* and *self-genericity* are explored for making natural ideals also induced. Self-genericity can be secured through semiproper forcing and can secure the saturation or precipitousness of natural ideals.

Section 4 takes a closer look at combinatorial properties of ideals and structural properties of generic ultrapowers. Topics include a range of saturation properties, *layered ideals*, Rudin-Keisler projections, where the ordinals go under generic elementary embeddings, and the sizes of sets in dual filters. Iterations of generic elementary embeddings are also developed as well as generic elementary embeddings arising from *towers* of ideals, i.e. sequences of ideals interrelated by projection maps.

Section 5 considers consequences of positing strong ideals, or generic large cardinals, low in the cumulative hierarchy. The wide-ranging topics include graphs and groups; Chang's Conjecture, Jónsson cardinals, and $\square_\kappa$; CH, GCH, and SCH; stationary set reflection; Suslin and Kurepa trees; partition properties; descriptive set theory; and non-regular ultrafilters. As emphasized, NS_{ω_1} being $\aleph_2$-saturated importantly has countervailing consequences.

Section 6 discusses limitative results on the possibilites for generic large cardinals. These play a role analogous to the Kunen limitation on conventional large cardinals, and indeed, argumentation for it is initially applied. A range of restrictions on ideal properties is subsequently presented, among them results that stand as remarkable successes: the Gitik-Shelah result that if κ is regular and $\delta^+ < \kappa$, then the ideal generated by NS_κ and $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \delta\}$ is not κ^+-saturated; their result that there is no $\aleph_1$-complete $\aleph_0$-dense nowhere prime ideal; the Matsubara-Shioya result that for $\omega < \kappa \leq \lambda$ with κ regular, $I_{\kappa\lambda}$ is not precipitous; and the Foreman-Magidor result that for $\omega < \kappa \leq \lambda$ with κ regular, $\mathrm{NS}_{\kappa\lambda}$ is not λ^+-saturated unless $\kappa = \lambda = \omega_1$.

Having progressed to the middle of the chapter, one sees that the chapter naturally divides into halves, the latter having to do with consistency results for strong ideal assumptions, or generic large cardinals. The long Section 7 attends to the main consistency results for induced ideals having strong properties. After developing the basic master condition theory for extending elementary embeddings, a general theorem—the Duality Theorem—is established for characterizing the forcing necessary for constructing the elementary embedding coming from an induced ideal. With this in place, a systematic account of various forcing techniques for getting precipitous and saturated ideals is provided. Highlights are Kunen's technique for getting an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 from a huge cardinal; Magidor's variation for which an "almost huge" cardinal suffices; Foreman's iteration to get κ-complete κ^+-saturated ideals over κ for every regular $\kappa > \omega$; Woodin's $\aleph_1$-complete $\aleph_1$-dense ideal over ω_1 from an almost huge cardianl; and Foreman's $\aleph_1$-complete $\aleph_1$-dense uniform ideal over ω_2 from two coordinated almost huge cardinals.

Section 8 in turns attends to consistency results for natural ideals having strong properties. In the first of two main approaches, one starts with an induced ideal with strong properties and forces that ideal to be a natural ideal while retaining substantial properties. Important examples are the Magidor and Woodin arguments for getting the nonstationary ideal to be precipitous and (somewhere) saturated respectively, and the Foreman-Komjáth argument for getting the tail club guessing filter to be saturated. In the second approach, one starts with a natural ideal and manipulates its antichain structure to make the generic ultrapower have strong properties. The important example is the "catching antichain" technique of the 1984 Foreman-Magidor-Shelah work for getting the nonstationary ideal to be saturated.

Section 9 broaches the extension of the context to towers of ideals. First brought into prominence by Woodin with his stationary tower forcing, this extension allows for more flexibility in minimizing assumptions and in drawing conclusions. After considering "induced" towers, techniques based on antichain catching are presented for getting nice generic ultrapowers. The stationary towers are the "natural" towers, and examples of Woodin and Douglas Burke are described. Finally, examples of stationary tower forcing are provided.

Section 10 briefly discusses the consistency strength of ideal assumptions. How inner model theory has successfully established lower bounds complementing forcing consistency results is quickly summarized. The focus, however, is on how knowing the image of just a few sets under a generic elementary embedding suffices to show that there is a conventional large cardinal in an inner model whose embedding agrees with the generic embedding. Notably, equi-consistency results for very large cardinals like the n-huge cardinals are derived by this means.

Section 11 is a speculative discussion of the possibility of adopting generic large cardinals along with their conventional cousins as additional axioms for mathematics. There is summarizing, comparisons, and prediction, and the reader could profitably read this section before surmounting all the others. Section 11 is an extensive, detailed list of open problems. These two last sections indicate the wealth of possibilities at this general confluence of the methods of forcing and ultrapowers.

14. Cardinal Arithmetic. In this chapter Uri Abraham and Menachem Magidor provide a broad-based account of Shelah's pcf theory and its applications to cardinal arithmetic, an account that exhibits the gains of considerable experience.

A beginning section sets out a general theory of ordinal-valued functions modulo ideals and cofinal sequences thereof, through to the existence of *exact* upper bounds as derived from a diamond-like *club guessing* principle. Delineating consequences, Silver's Theorem and a covering result of Magidor are established forthwith.

The next sections develop the basic theory of the central pcf function as calibrated by the crucial ideals $J_{<\lambda}[A]$. The various aspects of an unexpectedly rich structure are presented, the surround of the focal result that $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda}[A]$ together with a single set $B_\lambda \subseteq A$.

The latter sections make the ascent to the applications in cardinal arithmetic. First, the general Shelah study of the cofinality of $[\mu]^\kappa = \{x \subseteq \mu \mid |x| = \kappa\}$ under $\subseteq$ is presented. One takes a sufficiently large $H_\Psi(= H(\Psi))$ and structured chains of elementary substructures to get specifically related generators B_λ. With this the 1980 Shelah result $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ is secured. Proceeding through a finer analysis leading to "transitive" generators B_λ, the now famous result, instantiated by $2^{\aleph_\omega} < \aleph_{\omega_4}$ when $\aleph_\omega$ is a strong limit, is established.

The last section is devoted to Shelah's remarkable "revised GCH" result established in the early 1990s. With his investigation of cofinalities leading to "covering" sets Shelah advocated the consideration of

$$\lambda^{[\kappa]} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \ \wedge \ \forall u \in [\lambda]^\kappa \exists x \in [\mathcal{P}]^{<\kappa}(u = \bigcup x)\}.$$

as a "revised" power set operation. GCH is equivalent to the assertion that for all regular $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$. Using a variant of the pcf function, Shelah established that $\lambda^{[\kappa]} = \lambda$ for every $\lambda \geq \beth_\omega$ (where $\beth_\omega = \sup\{\beth_n \mid n \in \omega\}$ with $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$) and with $\kappa < \lambda$ sufficiently large. Thus, pcf theory provided a viable, substantive version of the GCH provable in ZFC.

15. Successors of Singular Cardinals. The investigation of combinatorial properties at successors of singular cardinals, with $\aleph_{\omega+1}$ being paradigmatic, has emerged as a distinctive subject in modern set theory. Historically, the early forcing arguments to secure substantial propositions low in the cumulative hierarchy by collapsing large cardinals to $\aleph_1$ or $\aleph_2$ did not adapt to $\aleph_{\omega+1}$. The situation became accentuated when the 1970s work on covering properties for inner models showed that the failure of $\square_\kappa$ for singular κ would require strong large cardinal hypotheses. In the 1980s expansion, the relative consistency of strong propositions about $\aleph_{\omega+1}$ entailing the failure of $\square_{\aleph_\omega}$ were duly achieved, and with the emergence of pcf theory a new combinatorially elaborated setting was established as well. In recent years, the conceptual space between $\square_\kappa$-like properties and their antithetical reflection properties has become clarified through methods and principles that have particular applicability at successors of singular cardinals.

Todd Eisworth in this chapter provides a well-organized account of the modern theory for successors of singular cardinals, an account that covers the full range from consistency results to combinatorics. After a first section setting out three illustrative problems about $\aleph_{\omega+1}$ the second section takes on one, stationary set reflection, as its theme. Let $\mathrm{Refl}(\kappa)$ be the assertion that every stationary $S \subseteq \kappa$ reflects, i.e. there is an $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α. A central tension is brought to the foreground with the discussion of how $\square_\kappa$ denies $\mathrm{Refl}(\kappa^+)$ in a strong sense and how supercompact cardinals, and even strong compact cardinals through indecomposable ultrafilters, imply versions of stationary set reflection. The rest of the section is devoted to establishing, as an entrée into the issues, Magidor's 1982 result: *If there are infinitely many supercompact cardinals, then in a forcing extension in which they become the* $\aleph_n$*'s,* $\mathrm{Refl}(\aleph_{\omega+1})$ *holds.*

The third section is given over to a detailed exegesis of the ideal $I[\lambda]$. Part of his deep combinatorial analysis, Shelah isolated $I[\lambda]$ after strands had appeared in his work as early as 1978, and $I[\lambda]$ has grown in importance to become a central concept. In accessible terms, $S \subseteq \lambda$ is in $I[\lambda]$ *iff* there is a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ of bounded subsets of λ and a closed unbounded $C \subseteq \lambda$ such that every $\delta \in S \cap C$ is singular and has a cofinal $A \subseteq \delta$ of ordertype $\mathrm{cf}(\delta)$, each of whose initial segments appears in $\{a_\beta \mid \beta < \delta\}$. This articulates a subtle sense of fast approachability, and for singular μ, AP_μ asserts that $I[\mu^+]$ is an improper ideal, i.e. $\mu^+ \in I[\mu^+]$. $\square_\mu$ implies AP_μ, and through Shelah's incisive analysis of $I[\lambda]$, one gets to the consistency of the failure of $\mathrm{AP}_{\aleph_\omega}$ from a supercompact cardinal. The section is brought to a close with Shelah's result, a bulwark of his pcf theory, on the existence of scales: With μ singular let $A \subseteq \mu$ be a set of regular cardinals cofinal in μ of ordertype $\mathrm{cf}(\mu)$ such that $\mathrm{cf}(\mu) < \min(A)$ as in pcf theory. Consider ΠA with respect to the filter $F = \{X \subseteq \mathrm{cf}(\mu) \mid |\mathrm{cf}(\mu) - X| < \mathrm{cf}(\mu)\}$ of co-bounded sets. Then Shelah showed that $\langle \Pi A, <^*_F \rangle$ has a linearly ordered, cofinal sequence of length μ^+—a *scale* for μ. (In terms of pcf theory, $\Pi A/F$ has true cofinality μ^+.)

The fourth section provides an extensive exploration of applications of scales and weak square principles. Attention soon focuses on the Foreman-Magidor Very Weak Square at μ (VWS_μ), particularly its close relationship to $I[\mu^+]$. VWS_μ is a square principle so weak that AP_μ implies it, and moreover, it is consistent to have a supercompact cardinal together with VWS_μ holding for every singular μ. The rest of the section is devoted to how scales with additional properties get us further across the divide between weak square principles and reflection properties. A family consisting of non-empty sets is *free iff* it has an injective choice function, and is κ*-free iff* every subfamily of cardinality less that κ is free. $\mathrm{NPT}(\kappa, \theta)$ is the assertion that there is a κ-free, non-free family of κ non-empty sets each of cardinality less than θ. That $\mathrm{NPT}(\kappa, \aleph_1)$ fails for any singular cardinal κ is part of Shelah's work on singular compactness. The existence of "good" scales leads to $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$, a central result of important work of Magidor and Shelah on the freeness of abelian groups. The notions of "very good" and even "better" scales provide avenues for further combinatorial elucidation.

The last section discusses square-brackets partition relations, with the focus on Jónsson algebras. The existence of such algebras was an important motivation of Shelah's development of pcf theory, and early on Shelah established that $\aleph_{\omega+1}$ carries a Jónsson algebra. The general question of whether every successor of a singular cardinal carries a Jónsson algebra remains unsolved, and the section sketches the expanse of Shelah's work here.

16. Prikry-Type Forcings. In this chapter Moti Gitik presents the full range of forcing techniques that have been developed to investigate powers of singular cardinals and the Singular Cardinal Hypothesis. With his technical virtuosity and persistence Gitik has been the main contributor to the subject, and to the organization and presentation of this chapter he brings to bear his extensive knowledge, providing several simplifications of the previously published work. To be noted is that being of the Israeli school, Gitik writes "$p > q$" for p being a stronger condition than q.

The first half deals with the work on countable cofinality. An initial section presents the basic Prikry forcing and its variants through to a strongly compact version, all having the characteristic property of adjoining new cofinal subsets without adjoining bounded subsets or collapsing cardinals. The next several sections then present the Gitik-Magidor extender-based forcing for adjoining many Prikry sequences with optimal hypotheses. As a warm-up, the simpler case when κ is already singular, $\kappa = \sup\{\kappa_n \mid n \in \omega\}$, is presented. One posits extenders on each κ_n and uses the embeddings to develop a system of ultrafilters $U_{n\alpha}$ on κ_n for adjoining Prikry sequences t_α. The forcing itself relies on getting Cohen subsets of κ^+ to guide the construction. Then the main case of an extender-based Prikry forcing with a single extender on a regular κ is presented. This forcing elaborates the previous by singularizing κ and confronts the added difficulty that the support of a condition may have cardinality κ. Finally, the forcing that additionally brings the whole situation down to render $\kappa = \aleph_\omega$ with interwoven Levy collapses is presented.

The latter half of the chapter begins with the work on uncountable cofinality. First, the basics of Radin forcing for adjoining a closed unbounded subset to a large cardinal consisting of formerly regular cardinals is carefully presented in an extensive section. This forcing had originally been given in terms of an elementary embedding $j\colon V \to M$, and next, a presentation based on a coherent sequence of ultrafilters is given, this providing a treatment also encompassing Magidor forcing for changing to uncountable cofinality. Then Carmi Merimovich's extender-based Radin forcing is broached.

The last section handles iterations of general "Prikry-type forcings". Such an iteration had first occurred in Magidor's 1973 result that *it is consistent that the least strongly compact cardinal is the least measurable cardinal*, and here Magidor's proof is simplified. After discussing an interesting forcing due to Jeffrey Leaning, the section turns to Easton support iterations of Prikry-type forcings. It is observed that this provides another way of establishing the consistency of the failure of SCH from the optimal hypothesis $\exists\kappa(o(\kappa) = \kappa^{++})$. The chapter ends with five open problems about powers of singular cardinals.

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17. Beginning Inner Model Theory. In this first of several chapters on inner model theory, William Mitchell authoritatively sets out the theory from $L[U]$ and K^{DJ} through to inner models of strong cardinals, the "coarse theory" not requiring fine structure. He thus performs the service of laying out the larger features and strategies of inner model theory that will frame the later chapters. There is iteration, comparison, coherence, and coiteration, and at one end sharps and mice and the other end coherent sequences of (non-overlapping) extenders. Beyond this, he provides two illuminating discussions about the further developments that involve fine structure. One is on the advantages of the modern Baldwin-Mitchell presentation with partial extenders even for the cases that he considers. The other is about what in general the core model should be in set theory, separate from any specific large cardinal assumptions.

18. The Covering Lemma. Mitchell here draws on his experience and expertise to provide an incisive account of the covering leitmotiv for inner models, which has been central to the development of inner model theory. The Jensen argument for the Covering Lemma for L has not only stimulated the formulation of new inner models in which the argument can be applied but has proven to be robust through these models to establish various results about the global affinity between inner models and the universe.

The first two sections discuss variants of the covering lemma and their applications. What is brought out is that the basic Jensen argument as a conceptual construction can be implemented in a range of inner models, but that the conclusions that one can draw depends on the large cardinal hypotheses involved and the complexity that one wants to sustain.

The third section outlines a proof, complete except for some fine structure details, of the Jensen and Dodd-Jensen covering results for L and $L[U]$. Although proofs for these cases have been devised that do not appeal to fine structure, it is deployed here in order to maintain generalizability. In fact, the Baldwin-Mitchell approach with partial extenders is already adopted for the technical advantages of local uniformity that it provides. One significant feature of the $L[U]$ case is that a weak covering property is established first and used to study ultrapower-generated indiscernibles leading to Prikry generic sequences.

The last section is devoted largely to a proof of covering for Mitchell's core model $K[\mathcal{U}]$ for coherent sequences $\mathcal{U}$ of ultrafilters. The previous proof has now to be further elaborated on account of the possible generation of complicated systems of indiscernibles, including possibly those leading to generic sequences for the Magidor forcing for changing to uncountable cofinalities. Drawing out what is possible from the covering argument, an elaborate conclusion is articulated and established. Gitik, for one direction of his culminating equi-consistency result on the Singular Cardinals Hypothesis, had applied this covering conclusion together with elements of Shelah's pcf theory to establish that if SCH fails, then in an inner model $\exists\kappa(o(\kappa) \geq \kappa^{++})$ holds. This synthetic result is next presented as a crucial application. The section, and chapter, concludes with a discussion of how the covering proof and conclusion can be extended to a strong cardinal, and the progress made with weaker versions of covering up to a Woodin cardinal and beyond.

19. An Outline of Inner Model Theory. In this chapter John Steel provides a general theory of extender models, the canonical inner models for large cardinals, getting to his model K^c. Moreover, he provides a remarkable application, to the effect that under $\mathrm{AD}^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})}$ up to a high rank V_δ is an extender model. Since it was Steel who in the mid-1990s provided the framework and made the crucial, final advances in this inner model theory, this chapter carries the stamp of experience and authority. The next chapter provides the construction of Steel's core model K up to a Woodin cardinal, a construction based on K^c, and a range of combinatorial applications. Chapter 22 describes how iteration trees, a basic component of the K^c construction, found their first substantial use in determinacy.

After covering the basics of extenders, an early section sets out the carefully wrought definition of a *fine extender sequence* $\vec{E}$. These are coherent sequences enhanced with acceptability for $J^{\vec{E}}$ and the Baldwin-Mitchell idea of having E_α be only an extender for subsets in $J_\alpha^{\vec{E}\restriction\alpha}$. A *potential premouse* is then a structure $J_\alpha^{\vec{E}}$ where $\vec{E}$ is a fine extender sequence. With the Chapter 9 preliminaries, fine structure considerations are imposed on potential premice and fine structure preserving ultrapowers are schematically described.

The next section engages the project of comparing two potential premice through coiteration. Iteration trees become central for handling overlapping extenders, and iterability for comparison is articulated in terms of games and iteration strategies for securing well-founded limits of models along branches. Fine structural considerations have become crucial to carrying out the process internally in extender models.

The succeeding section establishes the Dodd-Jensen Lemma about the minimality of iterations copied across fine structure preserving maps, as well as a weak Neeman-Steel version sufficient for present purposes. A further section deals with crucial results about solidity and condensation. These sections, elaborating the analysis starting with iterations trees, carve a fine path through a thicket of detail.

With these preparations, a culminating section provides the K^c construction and the resulting Steel *background certified core model* K^c. The model is an extender model $L[\vec{E}]$ for a fine extender sequence $\vec{E}$ defined according to the following stratagem: Given $\vec{E}\restriction\alpha$, an F is next adjoined if it is "certified" by a "background extender" F^*, in that F is the restriction to $J_\alpha^{\vec{E}\restriction\alpha}$ of F^*, an extender in V with sufficiently strong properties to guarantee iterability of $\vec{E}\restriction\alpha^\frown\langle F\rangle$. That such an $\vec{E}$ can be defined canonically is at the heart of the construction.

The concluding two sections bring inner model theory and determinacy together for the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$. Both sections proceed under the assumption that there are infinitely many Woodin cardinals with a measurable cardinal above them, so that in particular $\mathrm{AD}^{L(\mathbb{R})}$ holds. The main thrust of the first section is that the reals in the minimal iterable inner model M_ω satisfying "There are infinitely many Woodin cardinals" are exactly the reals in $\mathrm{OD}^{L(\mathbb{R})}$. The last section builds on this work to establish, using the (full) Dodd-Jensen

Lemma, that $\mathrm{HOD}^{L(\mathbb{R})}$ is "almost" an iterate M_∞ of M_ω. Specifically, for δ the large projective ordinal $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta = M_\infty \cap V_\delta$. This suffices in particular to establish under $\mathrm{AD}^{L(\mathbb{R})}$ that $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}$. It is remarkable that an inner model incipiently based on global definability can be shown to have structure as given by local definability and extender analysis.

20. A Core Model Tool Box and Guide. Building on the general theory of the previous chapter, Ernest Schimmerling develops its historical source, Steel's core model K up to a Woodin cardinal, and discusses combinatorial applications of it across set theory. Having been one of the contributors to the covering lemma theory for K and the initiating investigator of combinatorial principles there, Schimmerling is centrally placed to provide a measured, wide-ranging account.

The first half of the chapter is devoted to the basic theory of K. Going "up to" a Woodin cardinal, the "anti-large cardinal hypothesis" that there is no inner model with a Woodin cardinal is assumed. But moreover as Steel initially did, an additional "technical hypothesis" that there is a measurable cardinal Ω is assumed. Ω becomes regulative for the construction of K, schematically playing the role of the class On of ordinals. Regarding K^c as now a set premouse of height Ω, one works with *weasels*, other such premice, and uses the crucial simplifying property that if they have no Woodin cardinals, then their iteration trees have at most one cofinal well-founded branch. A definition of K second-order on $H(\Omega)$ is first developed, and then a first-order, recursive definition.

With K in hand, a useful "tools" section provides, without proof, a range of properties of K, from covering, forcing absoluteness and rigidity to combinatorial principles.

The next section outlines a proof of the "countably closed" weak covering property for K. The proof assumes familiarity with that of analogous results as given e.g. in Chapters 9 and 18 and very much depends on the first-order definition of K.

The final section provides, without proof, applications of K and generally, core models at the level that involves iteration trees. One sees at a glance how central this inner model theory has become, with the involvements described in determinacy, trees, ideals, forcing axioms, and pcf theory.

21. Structural Consequences of AD. In this first of several chapters on determinacy, Steve Jackson surveys the structural consequences of determinacy for sets of reals. The chapter thus serves as a fitting sequel to Moschovakis's book *Descriptive Set Theory* (1980). The advances have been in two directions, the extension of the scale theory beyond the projective sets into a substantial class of sets of reals in $L(\mathbb{R})$ and the analysis of the fine combinatorial structure of cardinals provided by the computation of the projective ordinals. With both directions calibrated by the analysis of definable sets in terms of definable well-ordered stratifications, the structure theory has remarkable richness and complexity as well as overall coherence.

An early section lays the basis with a review of basic notions: scales and periodicity, the Coding Lemma, projective ordinals, Wadge reducibility—and with some topics already going beyond the scope of the Moschovakis book—$\boldsymbol{\Sigma}^2_1$ sets of reals and infinite-exponent partition relations.

The next section develops the scale theory provided by Suslin cardinals under AD, the arguments mainly due to Martin. Let $S(\kappa)$ denote the class of κ-Suslin sets. A cardinal κ is *Suslin iff* $S(\kappa) - \bigcup_{\kappa'<\kappa} S(\kappa') \neq \emptyset$. That $\aleph_1$ is a Suslin cardinal is a classical result, and PD implies that the projective ordinals $\boldsymbol{\delta}^1_n$ for odd $n \in \omega$ are Suslin. The late 1970s Martin-Steel result that AD + $V = L(\mathbb{R})$ implies that $\boldsymbol{\Sigma}^2_1$ is the largest class with the scale property and $\boldsymbol{\Sigma}^2_1 = \bigcup_\kappa S(\kappa)$ provides the new, broad context. With $S(\kappa)$ taken as the analogue of the analytic sets, corresponding analogues of the projective hierarchy and projective ordinals are formulated. The scale property is then inductively propagated using Wadge reducibility and the weakly homogeneous trees available. Thus, the scale theory of the projective sets has been successfully abstracted, with the arguments applied in a suitably articulated setting.

The succeeding two sections present a schematic approach to the computation of the projective ordinals, which had been carried out by the author in a *tour de force* in the latter 1980s. $\kappa \longrightarrow (\kappa)^\lambda$ asserts that if the increasing functions from λ into κ are partitioned into two cells, then there is an $H \subseteq \kappa$ of cardinality κ such that all the increasing functions from λ into H are in one cell. The *strong partition property for* κ is the assertion $\kappa \longrightarrow (\kappa)^\kappa$ and the *weak partition property for* κ is the assertion $\forall\lambda < \kappa((\kappa \longrightarrow (\kappa)^\lambda)$. In the early 1970s Martin established under AD the strong partition property for ω_1, a striking result at the time. Kunen then carried out a detailed analysis of ultrapowers that led to the weak partition property for $\boldsymbol{\delta}^1_3$, which Martin had previously shown under AD to be $\aleph_{\omega+1}$, the third uncountable regular cardinal. In the section on "a theory of ω_1", this work is reorganized by starting with the weak partition property for ω_1 and establishing in turn the upper bound $\boldsymbol{\delta}^1_3 \leq \aleph_{\omega+1}$; the strong partition property for ω_1; the lower bound $\aleph_{\omega+1} \leq \boldsymbol{\delta}^1_3$; and the weak partition property for $\boldsymbol{\delta}^1_3$. This is done in terms of generalizable "descriptions", and the section on higher descriptions starts with the weak partition property for $\boldsymbol{\delta}^1_3$ and proceeds analogously to establish the upper bound $\boldsymbol{\delta}^1_5 \leq \aleph_{\omega^{\omega^\omega}+1}$; the strong partition property for $\boldsymbol{\delta}^1_3$; the lower bound $\aleph_{\omega^{\omega^\omega}+1} \leq \boldsymbol{\delta}^1_3$; and the weak partition property for $\boldsymbol{\delta}^1_5$. In this indicated propagation with descriptions, the author's computation of $\boldsymbol{\delta}^1_5$ and larger projective ordinals has been given a fortunate perspicuity and surveyability.

The final section explores the possibilities for extending throughout $L(\mathbb{R})$ the sort of fine analysis given by the computation of the projective ordinals. A weak square principle $\boxminus_{\kappa,\lambda}$ is established toward the goal of getting at global principles that might help propagate the inductive analysis via the Suslin cardinals.

22. Determinacy in $L(\mathbb{R})$. Woodin's culminating result that AD is equi-consistent with the existence of infinitely many Woodin cardinals figures centrally in this and the next chapters, which establish each direction of the equi-consistency in turn. In this chapter Itay Neeman develops the theme of getting determinacy from large cardinals. In getting technically optimal such results through the use of "long" games, Neeman's book *The Determinacy of Long Games* (2004) was an important contribution along these lines. In this chapter Neeman ultimately provides a complete, tailored proof of Woodin's result that if there are infinitely Woodin cardinals with a measurable cardinal above them, then $\mathrm{AD}^{L(\mathbb{R})}$. He first provides the historical and mathematical lines of approach in terms of concepts and methods of wider applicability and then proceeds with his own, well-crafted trajectory to the final conclusion.

The first several sections presents the basic, Martin-Steel theory of iteration trees. Iterability for the needed case of linear compositions of trees of length ω is articulated in terms of games and strategies and then established. The importance of Woodin cardinals is then brought out for creating complex iteration trees, the complexity discussed in terms of the author's notion of *type* for a set of formulas in place of the former Martin-Steel alternating chains.

The next sections start the ascent to the determinacy of sets in $L(\mathbb{R})$. The first vehicle is the concept of a homogeneously Suslin set of reals, a projection of a homogeneous tree and hence determined. After recasting Martin's classical $\mathbf{\Pi}^1_1$-Determinacy result, the 1985 Martin-Steel breakthrough result is presented, with its propagation of determinacy through the projective hierarchy with Woodin cardinals and iteration trees.

The last several sections make the final ascent with the author's specific approach, one based on getting determinacy by making Woodin cardinals countable with forcing rather than using stationary tower forcing as in Woodin's original proof. First, Woodin cardinals, through forcing and absoluteness, are shown to establish the determinacy of an important class of sets of reals wider than the homogeneously Suslin sets, the *universally Baire sets* of Qi Feng, Magidor, and Woodin. Second, getting at the technical heart of the matter, it is shown that given any real, models with many Woodin cardinals can be iterated to absorb the real in a further generic extension. Finally, with a least counter-example argument, AD is established in a "derived model" assuming the existence of infinitely many Woodin cardinals—getting one direction of Woodin's equi-consistency result—and assuming further the existence of a measurable cardinal above, AD is established in $\mathrm{L}(\mathbb{R})$ itself.

23. Large Cardinals from Determinacy. In this extensive, well-rounded, and sophisticated chapter Peter Koellner and Hugh Woodin set out the latter's work on getting large cardinals from determinacy hypotheses. The focal results were in place by the early 1990s, but this is the first venue where a full-fledged, systematic account is provided. With hindsight the authors are able to present a well-motivated, self-contained development organized around structural themes buttressing the extensive results.

The first two thirds of the chapter are framed as making an ascent to the Generation Theorem, an abstract theorem that provides a template for generating Woodin cardinals from refined determinacy hypotheses. In fact, the early sections add layer upon layer of complexity in an informative, well-motivated manner to get at more and more large cardinal conclusions.

Section 2 casts Solovay's seminal 1967 result that ω_1 is measurable under ZF + AD in a generalizable manner that draws out boundedness and coding techniques for getting normal ultrafilters. The generalizability is then illustrated by showing that under ZF + AD the projective ordinal $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$, "the least stable in $L(\mathbb{R})$", is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. Gearing up, Section 3 reviews the Moschovakis Coding Lemma and provides a strong, uniform version that will become crucial. Section 4 then establishes, as a precursor to the Generation Theorem, that under ZF + DC + AD a pivotal ordinal $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. First, reflection properties are developed that will play the role played earlier by boundedness. Then the notion of *strong normality* is used to establish that $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is λ-strong for cofinally many $\lambda < \Theta^{L(\mathbb{R})}$. Reflection properties and uniform coding are then worked to secure strong normality. Finally, with crucial appeals to AD and special properties of $\mathrm{HOD}^{L(\mathbb{R})}$, the strongness properties established for $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ are shown to relativize for $T \subseteq \Theta$ in $\mathrm{HOD}^{L(\mathbb{R})}$ to provide corresponding λ-T-strong cardinals δ_T, thus leapfrogging up to get that $\Theta^{L(\mathbb{R})}$ is Woodin in $\mathrm{HOD}^{L(\mathbb{R})}$.

The heights are reached in Section 5 where the work of the previous section is abstracted to establish two theorems on Woodin cardinals in a general setting. The first shows that in certain strong determinacy contexts HOD can contain many Woodin cardinals, and the second is the central Generation Theorem. The aim of this theorem is to show that the construction of Section 4 can be driven by lightface determinacy alone. To simulate the previous use of real parameters, the notion of *strategic determinacy* is introduced, a notion that resembles boldface determinacy but can nonetheless hold in settings with AC. Indeed, this notion is motivated by showing that it can hold in $L[S, x]$, where S is a class of ordinals and x is a real. With this in hand the Generation Theorem is finally established, and a number of instantial cases are presented.

Section 6 applies the Generation Theorem to derive the optimal amount of large cardinal strength from both lightface and boldface determinacy. The main lightface result is that ZF + DC + Δ^1_2-determinacy implies that there is a Turing cone of reals x such that $\omega_2^{L[x]}$ is a Woodin cardinal in $\mathrm{HOD}^{L[x]}$. The task here is to show that Δ^1_2-determinacy secures strategic determinacy. The main boldface result is that ZF + AD implies that in a generalized Prikry forcing extension, there are infinitely many Woodin cardinals in the corresponding HOD. The task here is to show that the Generation Theorem can be iteratively applied to generate infinitely many Woodin cardinals.

Section 7 attends to a reduction to second-order Peano Arithmetic. A first localization of the Generation Theorem shows that Δ^1_2-determinacy implies that for a Turing cone of reals x, $\omega_1^{L[x]}$ is a Woodin cardinal in $L[x]$. A second localization then shows that the proof can in fact be carried out in second-order Peano Arithmetic, to establish that if that theory plus Δ^1_2-determinacy is consistent, then so is ZFC + "On is Woodin", the latter assertion to be understood schematically.

The synthetic final Section 8 describes the remarkable confluences, seen in the later 1990s, of definable determinacy and inner model theory. First, actual equivalences between propositions of definable determinacy and propositions about the existence of inner models with Woodin cardinals are described. Then, the earlier HOD analysis is revisited in light of the Steel work on $\mathrm{HOD}^{L(\mathbb{R})}$, described in his Chapter 19. The full $\mathrm{HOD}^{L(\mathbb{R})}$ is not itself an extender model, but can nonetheless be comprehended as a fine-structural inner model of a new sort.

24. Forcing over Models of Determinacy. In this chapter Paul Larson describes work of Woodin on forcing over models of determinacy, and we take the opportunity to first describe that broad reach of that work. After his great successes culminating in his synthetic equi-consistency results about AD and large cardinals, Woodin in the mid-1990s entered a new, middle period of his research with the investigation of $\mathbb{P}_{\max}$ forcing extensions of models of AD. Quickly becoming a far-reaching theory of maximal and canonical forcing extensions that model ZFC, the subject shed new light on the inner workings of determinacy at the level of $P(\omega_1)$ and the extent of structure in ZFC extensions, even to the possible failure of the Continuum Hypothesis.

Woodin's remarkable *The Axiom of Determinacy, Forcing Axioms, and the Non-Stationary Ideal* (1999) in nearly one-thousand pages sets out of his work into his middle period. The book's Chapter 4 provides its main thrust, the specification of a canonical, maximal model of ZFC in the following sense: *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and that there is a Woodin cardinal with a measurable cardinal above it. Then there is in* $L(\mathbb{R})$ *a (countably closed and homogeneous) partial order* $\mathbb{P}_{\max}$ *so that for* G $\mathbb{P}_{\max}$*-generic over* $L(\mathbb{R})$, $L(\mathbb{R})[G]$ *models* ZFC*, and: for any* Π_2 *(i.e.* $\forall x \exists y$*) sentence satisfied in the structure* $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$*, that sentence is already satisfied in* $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle^{L(\mathbb{R})[G]}$*, the structure relativized to the generic extension.*

With $H(\omega_2)$ suitably accommodating $P(\omega_1)$ and the intrinsic ideal NS_{ω_1} participating, $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$ is arguably the next natural generalization of second-order arithmetic, which is identifiable with $\langle H(\omega_1), \in\rangle$. A pivotal, historical point about $\mathbb{P}_{\max}$ is that since $\neg$CH is equivalent to a Π_2 sentence of $\langle H(\omega_2), \in\rangle$ and there is a generic extension satisfying $\neg$CH yet preserving the hypotheses of the above result, CH actually fails in $L(\mathbb{R})[G]$. Generally, various combinatorial propositions about ω_1 are similarly consistent via "mild" forcing and are expressible as Π_2 assertions about $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$, and hence, these propositions hold in $L(\mathbb{R})[G]$. In this very substantial sense, $L(\mathbb{R})[G]$ is a canonical generic extension of $L(\mathbb{R})$.

Woodin's next chapters have to do with variants of $\mathbb{P}_{\max}$ that arrange the consistency of various combinatorial propositions about ω_1. He starts with applications of an axiom that codifies his motivation for formulating $\mathbb{P}_{\max}$:

(*) $\mathrm{AD}^{L(\mathbb{R})}$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-generic extension of $L(\mathbb{R})$.

Woodin then develops a variant $\mathbb{Q}_{\max}$ of $\mathbb{P}_{\max}$ that provides extensions in which NS_{ω_1} is $\aleph_1$-dense. Woodin had famously shown that NS_{ω_1} being $\aleph_1$-dense is equivalent in ZF to AD, and with $\mathbb{Q}_{\max}$ he provides a systematic treatment of this result.

Pushing the limits in another direction, Woodin in the penultimate chapter investigates $\mathbb{P}_{\max}$ extensions of AD models larger than $L(\mathbb{R})$. This enterprise is fueled by a corresponding strong form AD^+ of AD, and with it Woodin is able to starting scaling combinatorial propositions about ω_2 and even forms of Chang's Conjecture.

In his final chapter Woodin casts a light into the horizon with the formulation of his Ω-logic. With this new logic and AD^+, a more pristine approach can be taken to $\neg$CH, one that can subsume $\mathbb{P}_{\max}$ extensions in a more direct, albeit abstract, formulation. In work of the 21st century, Woodin will argue for the negation of the Continuum Hypothesis on the basis of his Ω-logic and a corresponding Ω Conjecture.

Larson in this final chapter of this Handbook offers a preparatory guide to Woodin's $\mathbb{P}_{\max}$, one that is to be highly appreciated for providing a patient, accessible approach. The first seven sections present a complete, self-contained analysis of the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ in an illuminating manner, proceeding incrementally by introducing hypotheses and methods as needed. After setting out the theory of iterated generic elementary embeddings fundamental to $\mathbb{P}_{\max}$, the partial order is formulated and its countable closure is established. After developing *A-iterability*, a generalized iterability property, it is applied to establish crucial structural results about the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$. Then the heralded Π_2 maximality with respect to $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$ is established, and assuming Woodin's axiom (*), the notable minimality result that any subset of ω_1 added by a generic filter generates the entire extension.

The last several sections briefly consider $\mathbb{P}_{\max}$ extensions of larger models under AD^+; Woodin's Ω-logic and Ω Conjecture; and several variations of $\mathbb{P}_{\max}$, starting with Woodin's $\mathbb{P}_{\max}$. This sampling reflects on the accomplishments with $\mathbb{P}_{\max}$ and suggests the expansive possibilities to be explored.

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