THE JOURNAL OF SYMBOLIC LOGIC
Volume 51, Number 2, June 1986

FINEST PARTITIONS FOR ULTRAFILTERS

AKIHIRO KANAMORI

If a uniform ultrafilter U over an uncountable cardinal κ is not outright countably complete, probably the next best thing is that it have a *finest partition*: a master function $f: \kappa \to \omega$ with $f^{-1}(\{n\}) \notin U$ for each $n \in \omega$ such that for any $g: \kappa \to \kappa$, either (a) it is one-to-one on a set in U, or (b) it factors through f (mod U), i.e. for some function h, $\{\alpha < \kappa \mid h(f(\alpha)) = g(\alpha)\} \in U$. In this paper, it is shown that recent contructions of irregular ultrafilters over ω_1 can be amplified to incorporate a finest partition.

Henceforth, let us assume that *all ultrafilters are uniform*.

There has been an extensive study of substantial hypotheses, which are nonetheless weaker than countable completeness, on ultrafilters over uncountable cardinals. To survey some results and to establish a context, let us first recall the *Rudin-Keisler* (RK) *ordering* on ultrafilters: If U_i is an ultrafilter over I_i for $i = 1, 2$, then $U_1 \leq_{\mathrm{RK}} U_2$ iff there is a projecting function $\psi: I_2 \to I_1$ such that $U_1 = \psi_*(U_2) = \{X \subseteq I_1 \mid \psi^{-1}(X) \in U_2\}$. $U_1 =_{\mathrm{RK}} U_2$ iff $U_1 \leq_{\mathrm{RK}} U_2$ and $U_2 \leq_{\mathrm{RK}} U_1$; and $U_1 <_{\mathrm{RK}} U_2$ iff $U_1 \leq_{\mathrm{RK}} U_2$ yet $U_1 \neq_{\mathrm{RK}} U_2$. In terms of this ordering, if an ultrafilter U has a finest partition f, then $f_*(U)$ over ω is maximum amongst all RK predecessors of U: for any $g: \kappa \to \kappa$, if $g_*(U) <_{\mathrm{RK}} U$, then g is not one-to-one on a set in U, so since g factors through f with some h, $g_*(U) = h_*(f_*(U))$. Say now that an ultrafilter U over $\kappa > \omega$ is *indecomposable* iff whenever $\omega < \lambda < \kappa$, there is no $V \leq_{\mathrm{RK}} U$ such that V is a (uniform) ultrafilter over λ. In other words, whenever $\psi: \kappa \to \lambda$, there is a set $X \subseteq \lambda$ with $|X| < \lambda$ such that $\psi^{-1}(X) \in U$. By setting $X_\beta = \{\alpha < \kappa \mid \psi(\alpha) \geq \beta\}$, we have for regular λ the equivalent notion of U being *λ-descendingly complete*: whenever $\{X_\beta \mid \beta < \lambda\} \subseteq U$ is a descending sequence, i.e. $\beta < \bar{\beta}$ implies $X_\beta \supseteq X_{\bar{\beta}}$, then $\bigcap_{\beta < \lambda} X_\beta \in U$. Since we are requiring $\lambda > \omega$, this is rather like having a κ-complete ultrafilter over a measurable cardinal κ, except that countable completeness has been left out.

Indecomposable ultrafilters were investigated by Prikry [P3], and Silver [Si] showed that if U is an indecomposable ultrafilter over some $\kappa > 2^{\omega_1}$, then U has a finest partition. He used this result to establish that if there is an indecomposable ultrafilter over an inaccessible cardinal, then $0^{\#}$ exists, and Ketonen [Ke2] provided further consequences. Since then, Donder, Jensen, and Koppelberg [DJK] used the

Received February 13, 1985; revised April 25, 1985.

core model K to weaken Silver's hypothesis in various ways and to strengthen his conclusion to the existence of an inner model with a measurable cardinal.

That indecomposability is not a vacuous concept was already known by Prikry [P1]: he showed that if a κ-complete ultrafilter U over a measurable cardinal κ is used in his well-known forcing, then in the generic extension any ultrafilter extending U is indecomposable over κ, now a strong limit cardinal of cofinality ω. Recently, Woodin (unpublished) showed how to collapse a measurable cardinal to $\aleph_\omega$, a strong limit cardinal, in such a way that there is an indecomposable ultrafilter over it in the generic extension. Also, Sheard [Sh] showed how to change a measurable cardinal to an inaccessible, non-weakly-compact cardinal in such a way that there is an indecomposable ultrafilter over it in the generic extension. In all these cases, by Silver's result there is a finest partition.

Another substantial hypothesis on ultrafilters is irregularity. An ultrafilter U is (λ, κ)-*regular* iff there are κ sets in U such that any λ of them have empty intersection. An ultrafilter over κ is simply *regular* iff it is (ω, κ)-regular. Regularity of ultrafilters was first studied by Keisler, who asked whether every ultrafilter over ω_1 was regular. Prikry [P2] established that if $V = L$, then this is so; Ketonen [Ke1] weakened the hypothesis to $0^\#$ does not exist; and then Jensen (see [DJK]) used the core model K to show that if CH holds and there is an irregular ultrafilter over ω_1, then there is an inner model with a measurable cardinal.

The first example of an irregular ultrafilter over ω_1 was provided by Laver [L], using a model of Woodin. First, let us reaffirm some notation. By an *ideal over* κ is meant a nontrivial, κ-complete ideal over κ. If I is such an ideal, then $I^+ = \{X \subseteq \kappa \mid X \notin I\}$ is the collection of I-positive measure sets, and $I^* = \{X \subseteq \kappa \mid \kappa - X \in I\}$ is the filter over κ dual to I. An ideal is λ-*dense* iff there is a set $\Delta \subseteq I^+$ with $|\Delta| = \lambda$ such that the following condition holds: for every $X \in I^+$, there is a $Y \in \Delta$ so that $Y \subseteq X \pmod I$, i.e. $Y - X \in I$. Woodin (unpublished) showed that starting from strong determinacy hypotheses (e.g. $AD_{\mathbf{R}}$ and Θ is regular), a model of ZFC can be constructed in which the following assertion holds:

$$(*) \qquad \diamond \text{ and there is a normal, } \omega_1\text{-dense ideal } I \text{ over } \omega_1.$$

Laver [L] then established that a consequence of $(*)$ is the existence of an irregular ultrafilter over ω_1. Very recently, relying only on ZFC plus a large cardinal axiom, Foreman, Magidor and Shelah [FMS] showed how to generically change a huge cardinal to ω_1 in such a way that there is an irregular ultrafilter over ω_1 in the extension. This evolved from their work on strong forcing axioms, and involves an elaboration of Laver's proof.

Finally turning to the task at hand, let us show how to amplify these constructions of irregular ultrafilters over ω_1 to incorporate a finest partition. This is quite a strong property in the presence of CH (which holds in the above models). For instance, with CH any ultrafilter U over ω_1 with a finest partition f is easily seen to satisfy $|\omega^{\omega_1}/U| = |\omega^\omega/f_*(U)| = 2^\omega = \omega_1$, since if $g_1: \omega_1 \to \omega$ has a factoring function $h_i: \omega \to \omega$ with $\{\alpha < \omega_1 \mid h_i(f(\alpha)) = g_i(\alpha)\} \in U$ for $i = 0, 1$, then

$$\{\alpha < \omega_1 \mid g_0(\alpha) \neq g_1(\alpha)\} \in U \quad \text{iff} \quad \{n \in \omega \mid h_0(n) \neq h_1(n)\} \in f_*(U).$$

However, it is well known that for any regular ultrafilter V over ω_1, $|\omega^{\omega_1}/V| = 2^{\omega_1}$.

We shall provide a detailed construction from the Woodin hypothesis (*) above, and, at the end of the paper, describe the corresponding modifications for the Foreman et al. construction. This work also shows how to procure a *separating* ultrafilter over ω_1 (see [KT1] and [KT2]), answering a question of Taylor; Baumgartner independently provided such an ultrafilter by similar means. We are indebted to Alan Taylor for asking the question that ultimately resulted in this paper.

THEOREM. *Assume (*). Suppose that D is any (nonprincipal) ultrafilter over ω, and $f\colon \omega_1 \to \omega$ any map such that $f^{-1}(\{n\}) \in I^+$ for every $n \in \omega$. Then there is an ultrafilter U over ω_1 extending I^* such that $f_*(U) = D$, and moreover f is a finest partition for U.*

COROLLARY. *Assume (*). Then there are ultrafilters U over ω_1 and D over ω such that $E <_{\mathrm{RK}} U$ iff $E =_{\mathrm{RK}} D$.*

The corollary is immediate if we take D to be RK-minimal in the theorem, and such ultrafilters, called *Ramsey* ultrafilters, can be easily constructed using CH, a consequence of $\diamondsuit$. Also a U as in the corollary must be *separating* in the sense of [KT1] and [KT2], by Rudin's lemma: whenever $\psi_*(D) = D$, ψ must be the identity on a set in D. Finally, notice that the situation of the corollary stands in stark contrast to the rich RK structure below regular ultrafilters: If CH and V is a regular ultrafilter over ω_1, then every ultrafilter E over ω is $<_{\mathrm{RK}} V$, and moreover there are $(2^\omega)^+$ functions $\omega_1 \to \omega$ distinct (mod V) all of which project V to E (see Lemma 2.2 of [KT2]).

PROOF OF THE THEOREM. The overall structure of the argument follows [L]. Let us set up the necessary notational scaffolding. First of all, $\diamondsuit$ will be used in the following version, easily seen to be equivalent to the usual formulation: There are $\{f_\alpha \mid \alpha < \omega_1\}$ such that $f_\alpha\colon \alpha \times \omega \times \omega \to \alpha$ and whenever $g\colon \omega_1 \times \omega \times \omega \to \omega_1$, $\{\alpha \mid g \upharpoonright \alpha \times \omega \times \omega = f_\alpha\}$ is stationary in ω_1.

Now set $X_n = f^{-1}(\{n\}) \in I^+$. Let Δ be a dense set of size ω_1 for I, and $\langle Y_\xi \mid \xi < \omega_1 \rangle$ an enumeration of $\Delta \cup \{\varnothing\}$, where $Y_0 = \varnothing$. Set $\Gamma = \{\bigcup Z \mid Z$ is a countable subset of $\Delta\}$. Then $|\Gamma| = \omega_1$ by CH, which is a consequence of $\diamondsuit$, and $\varnothing$ is the only element of Γ not in I^+.

Next, define $M = \{A_{in} \mid i, n \in \omega\}$ to be an *array* iff (i) $M \subseteq \Gamma$, and (ii) $\bigcup_i A_{in} \subseteq X_n$ (mod I). With CH, enumerate as $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ all arrays, so that each array occurs cofinally often.

Finally, let J be the ideal generated by $I \cup \{f^{-1}(X) \mid X \notin D\}$, so that

$$A \in J^+ \quad \text{iff} \quad \{n \mid X_n \cap A \in I^+\} \in D.$$

We will now construct a sequence $\langle F_\alpha \mid \alpha < \omega_1 \rangle$ by induction, so that the following conditions hold: each F_α is a countable family of subsets of ω_1; $\alpha < \beta$ implies $F_\alpha \subseteq F_\beta$; and $F_\alpha \cup J^*$ has the finite intersection property (f.i.p.). (Idea: $\bigcup_{\alpha < \omega_1} F_\alpha \cup J^*$ will generate the desired ultrafilter.)

At limit stages $\delta < \omega_1$, set $F_\delta = \bigcup_{\alpha < \delta} F_\alpha$. Also, if we set $W_{in} = \bigcup_{\alpha < \delta} Y_{f_\delta(\alpha, i, n)}$, where f_δ is from our $\diamondsuit$ sequence, and $\bigcup_i \bigcup_n W_{in}$ has the f.i.p. with respect to $F_\delta \cup J^*$, then set $F_{\delta+1} = F_\delta \cup \{\bigcup_i \bigcup_n W_{in}\}$. Otherwise, set $F_{\delta+1} = F_\delta$. (Idea: We incorporate g: $\omega_1 \to \omega$, where $W_{in} = X_n \cap g^{-1}(\{i\})$.)

Suppose now that we already have $F_{\alpha+1}$. Consider the array $M_\alpha = \{A_{in} \mid i, n \in \omega\}$. If $\bigcup_i \bigcup_n A_{in}$ is *not* in the filter generated by $F_{\alpha+1} \cup J^*$, set $F_{\alpha+2} = F_{\alpha+1}$. Otherwise, we will add $\bigcup_n A_{h(n)n}$ to $F_{\alpha+1}$ to get $F_{\alpha+2}$ for a carefully chosen $h: \omega \to \omega$. (Idea: We take care of factoring $g: \omega_1 \to \omega$, where $A_{in} = X_n \cap g^{-1}(\{i\})$.)

First, let $F_{\alpha+1} = \{\bar{H}_j \mid j \in \omega\}$, and set $H_j = \bigcap_{i \le j} \bar{H}_i$. Now set

$$K_j = \left\{ m \mid X_m \cap H_j \cap \bigcup_i \bigcup_n A_{in} \in I^+ \right\}.$$

Then $K_j \in D$ since by assumption $\bigcup_i \bigcup_n A_{in}$ is in the filter generated by $F_{\alpha+1} \cup J^*$. Also, by definition of array, $X_m \cap H_j \cap \bigcup_i \bigcup_n A_{in} = H_j \cap \bigcup_i A_{im}$ except on a set in I. For each $m \in K_0$, let $k(m) =$ the largest $j \le m$ such that $m \in K_j$. Then $H_{k(m)} \cap \bigcup_i A_{im} \in I^+$, so by countable completeness of I there is an $h(m)$ such that $H_{k(m)} \cap A_{h(m)m} \in I^+$. Finally, set $B = \bigcup_m A_{h(m)m}$.

Then, for every j, $B \cap H_j \in J^+$: Consider $m \in K_j - j$. Then $j \le k(m)$, so $H_j \cap A_{h(m)m} \in I^+$, i.e. $\{m \mid X_m \cap B \cap H_j \in I^+\} \supseteq K_j - j$. But $K_j - j \in D$. (This is the only place where we use the fact that D is nonprincipal.) Thus, all conditions continue to be satisfied if we set $F_{\alpha+2} = F_{\alpha+1} \cup \{B\}$.

This completes the construction. Let U be the filter generated by $\bigcup_{\alpha<\omega_1} F_\alpha \cup J^*$.

We took care of all arrays. The following lemma says that any $g: \omega_1 \to \omega$ can be "reduced" to an array, by invoking the $\diamondsuit$ sequence.

LEMMA 1. *Suppose that $\{S_i \mid i \in \omega\}$ is such that $\bigcup_i S_i$ has the f.i.p. with respect to U. Then there is an array $\{A_{in} \mid i, n \in \omega\}$ such that, for every $n \in \omega$, $A_{in} \subseteq S_i \cap X_n$* (mod I), *and $\bigcup_i \bigcup_n A_{in} \in U$.*

PROOF. Let $\langle C_\alpha \mid \alpha < \omega_1 \rangle$ be an enumeration of all finite intersections from $\bigcup_{\alpha<\omega_1} F_\alpha$, and if $C_\alpha \cap (S_i \cap X_n) \in I^+$, choose $g(\alpha, i, n)$ so that $Y_{g(\alpha,i,n)} \subseteq C_\alpha \cap (S_i \cap X_n)$ (mod I), and set $g(\alpha, i, n) = 0$ otherwise. For closed unboundedly many $\beta < \omega_1$, $\{C_\alpha \mid \alpha < \beta\}$ is the set of finite intersections from F_β. So using $\diamondsuit$, find such a β, a limit ordinal, such that $g \restriction \beta \times \omega \times \omega = f_\beta$.

Set $A_{in} = \bigcup_{\alpha<\beta} Y_{f_\beta(\alpha,i,n)} \in \Gamma$. Thus, $A_{in} \subseteq S_i \cap X_n$ (mod I). Since $\bigcup_i S_i$ has the f.i.p. with respect to U, for any $\alpha < \beta$,

$$\left\{ n \mid X_n \cap C_\alpha \cap \bigcup_i S_i \in I^+ \right\} \in D.$$

For any n in this set, there is an i such that $C_\alpha \cap S_i \cap X_n \in I^+$. Hence, $Y_{g(\alpha,i,n)} \in I^+$, and so $C_\alpha \cap A_{in} \in I^+$. Thus,

$$\left\{ m \mid X_m \cap C_\alpha \cap \bigcup_i \bigcup_n A_{in} \in I^+ \right\} \in D.$$

$\bigcup_i \bigcup_n A_{in}$ thus has the f.i.p. with respect to $F_\beta \cup J^*$, and by the construction at stage β, $\bigcup_i \bigcup_n A_{in}$ was added, and hence is in U.

The next two lemmas are the important consequences of Lemma 1.

LEMMA 2. *U is an ultrafilter over ω_1.*

PROOF. Suppose that $X \subseteq \omega_1$ has the f.i.p. with respect to U. We will show that $X \in U$. In Lemma 1, take each $S_i = X$. Then for the corresponding array $\{A_{in} \mid i, n \in \omega\}$ we have $\bigcup_i \bigcup_n A_{in} \subseteq X$ (mod I), and $\bigcup_i \bigcup_n A_{in} \in U$.

LEMMA 3. *If $g: \omega_1 \to \omega$, then g factors through f* (mod U).

PROOF. By Lemma 1, find an array $\{A_{in} \mid i, n \in \omega\}$ such that $A_{in} \subseteq g^{-1}(\{i\}) \cap X_n$

(mod I), and $\bigcup_i \bigcup_n A_{in} \in F_\beta \cup J^*$ for some $\beta < \omega_1$. This array is some M_α for $\alpha \geq \beta$, and a corresponding $T = \bigcup_n A_{h(n)n}$ was added to $F_{\alpha+1}$ in the construction. Now

$$Z = \bigcup_i \bigcup_n (A_{in} - (g^{-1}(\{i\}) \cap X_n)) \in I,$$

so

$$\{\alpha < \omega_1 \mid h(f(\alpha)) = g(\alpha)\} \supseteq T \cap (\omega_1 - Z) \in U.$$

To complete the proof, we need the following result from Laver [L]:

LEMMA 4 (LAVER). *If an ultrafilter V over a regular cardinal $\kappa > \omega$ is κ-generated over the dual of a normal ideal H over κ (i.e. V is generated by $H^* \cup T$, where $|T| = \kappa$), then V must be **weakly normal**: whenever $\{\alpha < \kappa \mid \psi(\alpha) < \alpha\} \in V$ for $\psi: \kappa \to \kappa$, there is a $\delta < \kappa$ such that $\{\alpha < \kappa \mid \psi(\alpha) \leq \delta\} \in V$.*

The following lemma about weakly normal ultrafilters is simple to see:

LEMMA 5. *If V is a weakly normal ultrafilter over a regular cardinal $\kappa > \omega$, then for any $\psi: \kappa \to \kappa$, either ψ is strictly increasing on a set in V, or else is bounded* (mod V), *i.e. there is an $\eta < \kappa$ such that $\{\alpha \mid \psi(\alpha) \leq \eta\} \in V$.*

PROOF. Given the ψ, define ϕ by $\phi(\alpha) =$ least β such that $\psi(\alpha) \leq \psi(\beta)$. Thus, $\phi(\alpha) \leq \alpha$ for every $\alpha < \kappa$. If $\{\alpha \mid \phi(\alpha) = \alpha\} \in V$, then clearly ψ is strictly increasing on this set. If $\{\alpha \mid \phi(\alpha) < \alpha\} \in V$, by weak normality there is a $\delta < \kappa$ such that $\{\alpha \mid \phi(\alpha) \leq \delta\} \in V$. Then if we set $\eta = \sup\{\psi(\beta) \mid \beta \leq \delta\}$, then $\{\alpha \mid \psi(\alpha) \leq \eta\} \in V$.

Now the proof is complete. The U constructed above must be weakly normal by Lemma 4, and hence by Lemma 5 any $g: \omega_1 \to \omega$ is either one-to-one on a set in U, or else is bounded (mod U). In the latter case, we can bijectively identify the range of g with a subset of ω, and hence by Lemma 3 see that g factors through f (mod U).

Let us now turn to the modification for the Foreman et al. [FMS] construction. Say that an ideal over κ is *layered* iff there is a continuous, increasing chain of Boolean algebras $\langle B_\alpha \mid \alpha < \kappa^+ \rangle$ such that $P(\kappa)/I = \bigcup_{\alpha < \kappa^+} B_\alpha$, $|B_\alpha| = \kappa$ for each α, and there is a stationary $S \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \kappa\}$ such that for each $\alpha \in S$, B_α is a $<\kappa$-complete algebra and a regular subalgebra of $P(\kappa)/I$. Foreman et al. first show how to use a huge cardinal to construct a layered ideal over ω_1. Then they show that the conditions on a layered ideal are enough to further force without adding any new subsets of ω_1 so as to procure a generic sequence $\langle U_\alpha \mid \alpha < \omega_2 \rangle$, where each U_α is an ultrafilter on B_α, so that $U = \bigcup_{\alpha < \omega_2} U_\alpha$ is a weakly normal ultrafilter over ω_1 extending I^*.

They also develop a careful version of the forcing which ensures that for any $g: \omega_1 \to \omega$, there is a $\bar{g}: \omega_1 \to \omega$ such that the equivalence class mod I of $\bar{g}^{-1}(\{n\})$ $\in B_0$ for each $n \in \omega$, and $\{\alpha < \omega_1 \mid g(\alpha) = \bar{g}(\alpha)\} \in U$. (This corresponds to the $\diamond$ argument of our Lemmas 1 and 3.) Thus, $|\omega^{\omega_1}/U| = \omega_1$ can be arranged, since $|B_0|$ $= \omega_1$ and CH holds in the model. To further arrange a finest partition for U, we can first dovetail the construction of our theorem into the Foreman et al. construction of the initial U_0 on B_0 in the ultrafilter sequence, so that U_0 has a finest partition for functions $g: \omega_1 \to \omega$ such that the equivalence class mod I of $g^{-1}(\{n\}) \in B_0$ for every $n \in \omega$. (To avoid conflicts, we can incorporate the construction of our theorem using $\diamond$ restricted to the stationary set $\omega_1 - S$, where S is as in the definition of layered ideal.) Now, the careful version of the Foreman et al. construction applied to this B_0 will result in a finest partition for U. Notice that $(*)$ is still a comparatively strong assumption, since our theorem provides a direct construction from (*), whereas the argument just outlined is a forcing consistency proof.

Actually, the Foreman et al. argument works for any regular λ less than the huge cardinal, to procure a non-(λ, λ^+)-regular ultrafilter over λ^+. However, our incorporation of a finest partition seems to be limited to the case $\lambda = \omega$, if one examines the details of going from $F_{\alpha+1}$ to $F_{\alpha+2}$ in the construction. Thus, we can ask:

Question 1. If $\lambda > \omega$, can there be ultrafilters U over λ^+ and D over λ such that $E <_{RK} U$ iff $E \leq_{RK} D$?

Question 2. Can there be ultrafilters U_i over ω_i for $i = 0, 1, 2$ such that $E <_{RK} U_2$ iff $E =_{RK} U_0$ or $E =_{RK} U_1$?

REFERENCES

[DJK] H. D. Donder, R. B. Jensen and B. Koppelberg, *Some applications of the core model*, ***Set theory and model theory*** (R. B. Jensen and A. Prestel, editors), Lecture Notes in Mathematics, vol. 872, Springer-Verlag, Berlin, 1981, pp. 55–97.

[FMS] M. Foreman, M. Magidor and S. Shelah, *Martin's axiom, saturated ideals, and nonregular ultrafilters. Part II*, preprint.

[KT1] A. Kanamori and A. Taylor, *Negative partition relations for ultrafilters on uncountable cardinals*, ***Proceedings of the American Mathematical Society***, vol. 92 (1984), pp. 83–89.

[KT2] ———, *Separating ultrafilters on uncountable cardinals*, ***Israel Journal of Mathematics***, vol. 47 (1984), pp. 131–138.

[Ke1] J. Ketonen, *Nonregular ultrafilters and large cardinals*, ***Transactions of the American Mathematical Society***, vol. 224 (1976), pp. 61–73.

[Ke2] ———, *Some cardinal properties of ultrafilters*, ***Fundamenta Mathematicae***, vol. 107 (1980), pp. 225–235.

[L] R. Laver, *Saturated ideals and nonregular ultrafilters*, ***Patras logic symposion*** (G. Metakides, editor), North-Holland, Amsterdam, 1982, pp. 297–305.

[P1] K. Prikry, *Changing measurable into accessible cardinals*, ***Dissertationes Mathematicae (Rozprawy Matematyczne)***, vol. 68 (1970).

[P2] ———, *On a problem of Gillman and Keisler*, ***Annals of Mathematical Logic***, vol. 2 (1971), pp. 179–187.

[P3] ———, *On descendingly complete ultrafilters*, ***Cambridge summer school in mathematical logic*** (A. Mathias and H. Rogers, Jr., editors), Lecture Notes in Mathematics, vol. 337, Springer-Verlag, Berlin, 1973, pp. 459–488.

[Sh] M. Sheard, *Indecomposable ultrafilters over small large cardinals*, this Journal, vol. 48 (1983), pp. 1000–1007.

[Si] J. Silver, *Indecomposable ultrafilters and $0^\#$*, ***Proceedings of the Tarski symposium*** (L. Henkin et al., editors), Proceedings of Symposia in Pure Mathematics, vol. 25, American Mathematical Society, Providence, Rhode Island, 1974, pp. 357–364.

DEPARTMENT OF MATHEMATICS
BOSTON UNIVERSITY
BOSTON, MASSACHUSETTS 02215