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ON GÖDEL INCOMPLETENESS AND FINITE COMBINATORICS

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Gödel's paper on formally undecidable propositions [3] raised the possibility that finite combinatorial theorems could be discovered which are independent of powerful axiomatic systems such as first-order Peano Arithmetic. An important advance was made by J. Paris in the late 1970's; building on joint work with L. Kirby, he used model-theoretic techniques to investigate arithmetic incompleteness and proved theorems of finite combinatorics which were unprovable in Peano Arithmetic [11]. The Paris–Harrington paper [13] gives a self-contained presentation of the proof that a straightforward variant of the familiar finite Ramsey Theorem is independent of Peano Arithmetic. In this paper, we consider a simple finite corollary of a theorem of infinite combinatorics of Erdös and Rado [1] and show it to be independent of Peano Arithmetic. This formulation avoids the Paris–Harrington notion of *relatively large* finite set and deals with a generalized notion of partition. This shift of focus also provides for simplifications in the proofs and directly yields a level-by-level analysis for subsystems of Peano Arithmetic analogous to that in [12].

We have tried to provide a treatment of the proof whose organization and brevity make it suitable for expository purposes. These results were first discussed in 1982, and almost all the details worked out by a year later. We would like to thank Peter Clote for his later interest and involvement in this web of ideas.

1. Definitions and the main results

We begin by recalling Ramsey's Theorem. Let $[X]^n$ denote the collection of subsets of X of cardinality n. If X is a set of natural numbers and if f is a function with domain $[X]^n$, we write $f(x_1, \ldots, x_n)$ for $f(\{x_1, \ldots, x_n\})$ with the understanding that $x_1 < x_2 < \cdots < x_n$. In keeping with notation used in logic and in Ramsey theory, we identify each natural number n with the set of its predecessors: $n = \{0, 1, \ldots, n-1\}$. Also, we shall use N to denote the set of natural numbers as well as its cardinality. If n, k and r are either N or members of N, $X \to (k)^n_r$ means that whenever $f:[X]^n \to r$ there is $H \in [X]^k$ such that f is constant on $[H]^n$; in this case we say that H is *homogeneous* for f. Ramsey [14] established the Infinite Ramsey Theorem:

For any $n, r \in N$, $\quad N \to (N)^n_r$

as well as the Finite Ramsey Theorem:

For any n, r, $k \in N$, there is an $m \in N$ such that $m \rightarrow (k)^n_r$.

In 1950, Erdös and Rado [1] established a generalization of Ramsey's Theorem with no restrictions on the number of cells of the partition. If f has domain $[X]^n$, where $X \subseteq N$, we say that $H \subseteq X$ is *canonical for f* if there is a $v \subseteq n$ satisfying the following condition: if $s, t \in [H]^n$ are construed as increasing functions mapping n into X, then

$$f(s) = f(t) \leftrightarrow s \mid v = t \mid v.$$

In other words, $f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1}) \leftrightarrow x_i = y_i$ for all i in v. We shall write $v = v(H)$ when v makes H canonical. By way of example, if $v(H) = \emptyset$, then H is homogeneous for f in the usual sense; and if $v(H) = n$, then f is injective on $[H]^n$.

Using the Infinite Ramsey Theorem, Erdös and Rado established:

Theorem 1.1 [1]. *For any $n \in N$, if $f : [N]^n \rightarrow N$, then there is an infinite $H \subseteq N$ canonical for f.*

The Infinite Ramsey Theorem in turn can be seen to be an immediate corollary of Theorem 1.1: because of the '$\leftrightarrow$' requirement in the definition of 'canonical,' if the range of the partition f is finite, then $v = \emptyset$ is the only possible case.

A function $f : [X]^n \rightarrow N$, where $X \subseteq N$, is said to be *regressive* if $f(s) < \min(s)$ for all s such that $\min(s) > 0$. Of course, there cannot be large homogeneous sets for such functions in the usual sense, but there is a natural notion of homogeneity here: we say that a set $H \subseteq X$ is *min-homogeneous* for f if $\min(s) = \min(t)$ implies $f(s) = f(t)$; that is, if $f \mid [H]^n$ only depends on the minimum element. If $k, n \in N$, the notation $X \rightarrow (k)^n_{\text{reg}}$ means that whenever $f : [X]^n \rightarrow N$ is regressive, there is $H \in [X]^k$ min-homogeneous for f.

The following is a straightforward corollary to the Erdös–Rado result.

Corollary 1.2. *For any $n \in N$, $N \rightarrow (N)^n_{\text{reg}}$.*

Proof. The case $n = 1$ is trivial; so assume $n > 1$. Let $f : N \rightarrow N$ be regressive and let H be an infinite canonical set for f with $v = v(H)$. We claim that either $v = \emptyset$ or $v = \{0\}$. Suppose to the contrary that v contains some $i \neq 0$. If h is the least non-zero element of H, there would be arbitrarily many n-tuples from H with first element h which disagree on v and hence are mapped to different values less than h by f, a contradiction. Now, if $v = \emptyset$, f is homogeneous on $[H]^n$, and if $v = \{0\}$, f is min-homogeneous on $[H]^n$, because f must be injective according to the minimum element; that is, $f(s) = f(t) \leftrightarrow \min(s) = \min(t)$.

Now consider the proposition

For any n, $k \in N$, there is an $m \in N$ such that $m \rightarrow (k)^n_{\text{reg}}$. $(*)$

There is a simple argument by contradiction which establishes $(*)$ from Corollary 1.2: if for some $n, k \in N$, there are regressive counterexamples $f_m : [m]^n \to m$ for every $m \in N$, use $f : [N]^{n-1} \to N$ defined by

$$f(x_0, \ldots, x_n) = f_{x_n}(x_0, \ldots, x_{n-1})$$

to derive a contradiction.

It is the proposition $(*)$ that we will prove independent of first-order Peano Arithmetic. The notion of regressive function comes from Set Theory and the combinatorics of regular cardinals. We will remark further on this connection later but here let us finish with the preliminaries necessary to state our main results.

Peano Arithmetic, abbreviated PA, is the first-order theory in the language containing $0, 1, +, \cdot, <$ axiomatized by the defining properties of these primitive notions together with the induction scheme for all formulas (allowing parameters). A function $f : N \to N$ is *provably recursive* in PA if $f(m) = n$ just in case $\mathrm{PA} \vdash F(m, n)$ for some formula $F(x, y)$ which is Δ_1 in PA satisfying $\mathrm{PA} \vdash \forall x\, \exists y\, F(x, y)$. Thus if f is a provably recursive function in PA, the fact that f is total is a consequence of PA. A function f is said to *eventually dominate* the function g iff $f(n) > g(n)$ for all but at most a finite number of integers $n \in N$.

Our main results are

Theorem A. *The assertion* $(*)$ *is not provable in* PA.

Theorem B. *The function* $\nu(n) =$ *the least* m *such that* $m \to (2n)^n_{\mathrm{reg}}$ *is not provably total in* PA *and eventually dominates every provably recursive function of* PA.

Theorem C. *The function* $\nu_2(n) =$ *the least* m *such that* $m \to (n)^2_{\mathrm{reg}}$ *eventually dominates all primitive recursive functions; its rate of growth is approximately equal to that of the Ackermann function.*

These results are, respectively, Corollaries 2.3, 2.4 and 4.6 below.

For comparison and for future reference, let us recall the Paris–Harrington variant of the Finite Ramsey Theorem which is also independent of PA, [13]. We say that $H \subseteq N$ is *relatively large* if H has at least as many elements as its minimum element, that is $|H| \geq \min(H)$. The notation $X \to_* (k)^n_r$ requires that the appropriate partitions have relatively large homogeneous sets of cardinality $\geq k$. Paris and Harrington showed that the proposition

$$\text{For any } n, r, k \in N, \text{ there is an } m \in N \text{ such that } m \to_* (k)^n_r \tag{PH}$$

is true but not provable in PA.

Our paper owes a great deal to the Ketonen–Solovay paper [4]. There, a direct combinatorial proof of the Paris–Harrington result is given. This approach uses a level-by-level analysis of the rate of growth of the functions involved in terms of the Grzegorczyk–Wainer hierarchy for the provably recursive functions of PA. Regressive functions are actually an integral part of [4] although homogeneity is still tied there to the notion of relativey large finite set. In an earlier manuscript version of [4] the analogies with combinatorial notions from the study of large cardinals in Set Theory were more explicitly developed. In Set Theory the key result on regressive functions is Fodor's Lemma [2]: if $f:\kappa\to\kappa$ is regressive and κ is a regular uncountable cardinal, then f is constant on a stationary subset of κ. The result that $\kappa\to(\kappa)^m_{\text{reg}}$ for measurable cardinals κ due to Rowbottom, [15]. The combinatorial theorem $(*)$ is thus a true miniaturization of combinatorics on transfinite cardinal numbers.

We next give a direct, finitistic proof that (PH) implies $(*)$. This result appears as Lemma 1.8 in [4]. It can be formalized in PA or in *Primitive Recursive Arithmetic* (PRA); this is the first-order theory in the language containing $0, 1, +, \cdot, <$ and a function symbol for each primitive recursive function, axiomatized by the defining properties of the primitive notions, the recursion equations for the primitive recursive functions and the induction scheme only for quantifier-free formulas. Induction for Σ_0 formulas (i.e. allowing bounded quantifiers) is provable in PRA.

Theorem 1.3 [4]. (PH) *implies* $(*)$.

Proof. Given $n, k\in N$, first find $m\in N$ such that whenever $h:[m]^{n+1}\to 3$, there is $H\subseteq m$ homogeneous for h such that $|H|\geq \min(H)+n$ and $|H|\geqslant k+n$. This self-refinement of PH is a straightforward consequence of it, cf. [13, 2.14]. So given $f:[m]^n\to m$ regressive, define $g:[m]^{n+1}\to 3$ by

$$g(x_0,\dots,x_n)=\begin{cases}0 & \text{if } f(x_0,\dots,x_{n-1})=f(x_0,x_2,\dots,x_n),\\ 1 & \text{if } f(x_0,\dots,x_{n-1})<f(x_0,x_2,\dots,x_n),\\ 2 & \text{if } f(x_0,\dots,x_{n-1})>f(x_0,x_2,\dots,x_n).\end{cases}$$

Let $H\subseteq m$ be homogeneous for g and satisfy $|H|\geqslant\min(H)+n$ and $|H|\geqslant k+n$. As f is regressive, the first condition on H insures that g is constantly 0 on $[H]^{n+1}$, else there would be too many different values of f below $\min(H)$. By the second condition on H, we can take H' to consist of the first k elements of H and have $|H-H'|\geqslant n$. We now argue that H' is min-homogeneous for f as follows: given $x_0<x_1<\cdots<x_{n-1}$ and $x_0<y_1<\cdots<y_{n-1}$ all from H', let $z_1<\cdots<z_{n-1}$ be $n-1$ elements in H such that $z_1>\max(x_{n-1},y_{n-1})$. Then $f(x_0,x_1,\dots,x_{n-1})=f(x_0,x_2,\dots,x_{n-1},z_1)=f(x_0,x_3,\dots,z_1,z_2)=\cdots=f(x_0,z_1,\dots,z_{n-1})$, and similarly for $f(x_0,y_1,\dots,y_{n-1})$. Thus $f(x_0,x_1,\dots,x_{n-1})=f(x_0,y_1,\dots,y_{n-1})$.

The method of proof we will use in the next section to establish the independence of $(*)$ from PA is model-theoretic; it is thus in the tradition of [13], [11] and [5]. In [6], Kirby and Paris give an elegant independence result based on work of Goodstein and on [4]. Their result seems to require correlation with the function hierarchies and thus far has eluded a quick model-theoretic treatment.

In Section 3, we develop the combinatorics to carry through the [4] scheme for $(*)$, thereby eliminating the original bootstraps arguments from relatively large. Finally, in Section 4 we refine the previous arguments as well as discuss generalizations of $(*)$. Interestingly enough, the refinements provide a level-by-level analysis of subsystems of PA using indiscernibles.

2. Independence

The main purpose of this section is to provide a model-theoretic proof of the independence of $(*)$, which, stripped of exegesis, is remarkably brief. Harrington's idea of diagonal indiscernibles for reducing full induction to induction for Σ_0 formulas is still crucial, but by focusing on $(*)$ we can avoid the diverting combinatorics of [13] needed for procuring and spreading out the indiscernibles. The idea for the proof of the following proposition occurs in [9, p. 406] and was also noticed by Laver.

Proposition 2.1. *Assume* $(*)$. *For any* $e, k, n \in N$ *and formulas* $\psi_0, \ldots, \psi_e$ *in the language of arithmetic in at most* $n+1$ *free variables, there is a set* $H \in [N]^k$ *which constitute diagonal indiscernibles for these formulas, i.e. given* $c_0 < c_1 < \cdots < c_n$ *and* $c_0 < d_1 < \cdots < d_n$ *all in* H *and any* $p < c_0$, *then*

$$\psi_i(p, c_1, \ldots, c_n) \leftrightarrow \psi_i(p, d_1, \ldots, d_n)$$

holds for each $i \leqslant e$.

Proof. We shall assume $k \geqslant 2n+1$ for technical reasons. Let $m \to (w)^{2n+1}_{\text{reg}}$, where $w \to (k+n)^{2n+1}_{e+2}$. Given $x_0 < \cdots < x_{2n} < m$, if there is an $i \leqslant e$ and a $p < x_0$ such that $\psi_i(p, x_1, \ldots, x_n)$ and $\psi_i(p, x_{n+1}, \ldots, x_{2n})$ have different truth values, then let $f(x_0, \ldots, x_{2n})$ be the least such p and $i(x_0, \ldots, x_{2n})$ the least such i. Otherwise, set $f(x_0, \ldots, x_{2n}) = 0$ and $i(x_0, \ldots, x_{2n}) = e+1$.

By hypothesis on m, there is a set $H_0 \in [m]^w$ which is min-homogeneous for f. Next, by hypothesis on w, there is an $H_1 \in [H_0]^{k+n}$ and a fixed $i \leqslant e+1$ such that $i = i(x_0, \ldots, x_{2n})$ for every $\{x_0, \ldots, x_{2n}\} \in [H_1]^{2n+1}$.

Suppose first that $i = e+1$. Then let $z_1 < \cdots < z_n$ be the last n elements of H_1, and $H = H_1 - \{z_1, \ldots, z_n\}$. Given any $c_0 < c_1 < \cdots < c_n$ and $c_0 < d_1 < \cdots < d_n$ all from H, for each $i \leqslant e$ and each $p < c_0$, $\psi_i(p, c_1, \ldots, c_n)$ must have the same truth value as $\psi_i(p, z_1, \ldots, z_n)$ and so also must $\psi_i(p, d_1, \ldots, d_n)$.

Thus, the argument would be complete if we can derive a contradiction from the assumption that $i \leqslant e$. To this end, let $x_0 < \cdots < x_{3n}$ be all from H_1. (Remember that we are assuming that $k \geqslant 2n+1$.) By the min-homogeneity of H_0, there is a fixed value $p < x_0$ for f on any $2n+1$ sequence starting with x_0. However, at least two of $\psi_i(p, x_1, \ldots, x_n)$, $\psi_i(p, x_{n+1}, \ldots, x_{2n})$, and

$\psi_i(p, x_{2n+1}, \ldots, x_{3n})$ must have the same truth value, contradicting the choice of $i \leq e$.

Notice that this proposition is formalizable in any theory which has a truth predicate for the formulas concerned. We will only require it for Σ_0 formulas in what follows, so we can take PRA as the theory, since there is a primitive recursive truth predicate for the Σ_0 formulas. Of course, in a quick exposition. PRA can be replaced by PA.

Also, there is a proof of the proposition using only (*) for exponent $n+1$ rather than $2n+1$, and this has implications in the level-by-level analysis of Paris, [12] — see Section 4.

We now turn to the standard sort of model-theoretic result leading to independence. Recall that any non-standard model M of enough of arithmetic has a proper initial segment which we can identify with N, the set of natural numbers. In general, if I is a proper initial segment of M, we write $I < M$. Also, if $a, b \in M$ and $a \in I < M$ yet $b \notin I$, we write $a < I < b$. As our I's will be closed under + and $\cdot$, we will regard them as substructures of M. Finally, $[a, b]$ denotes as usual the closed interval $\{x \mid a \leq x \leq b\}$.

Theorem 2.2. *Suppose that* $M \models \mathrm{PRA}$ & $[a, b] \rightarrow (2c)^c_{\mathrm{reg}}$, *where* $c \in M - N$. *Then there is an* $I < M$ *with* $a < I < b$ *such that* $I \models \mathrm{PA}$.

Proof. We shall first get $< c_i \mid i \in N >$, with $a < c_i < b$ for each i, which constitute diagonal indiscernibles for all the Σ_0 formulas, i.e., whenever ψ is Σ_0 in say $n+1$ free variables, and $i_0 < i_1 < \cdots < i_n$ and $i_0 < j_1 < \cdots < j_n$, then

$$M \models \forall p < c_{i_0} (\psi(p, c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(p, c_{j_1}, \ldots, c_{j_n})).$$

One way to do this is as follows: Let $\sigma(k)$ assert that there are at least k diagonal indiscernibles in the interval between a and b for the first k Σ_0 formulas (in some standard coding). By the proposition and the succeeding comment about PRA, it is easy to see that for each $k \in N$, $M \models \sigma(k)$:

In the notation of the lemma, it suffices to find a min-homogeneous set of size $\geq w$ for some regressive f on $[[a, b]]^{2n+1}$, where $w, n \in N$. Remembering that $c \in M - N$, we can define $\bar{f}$ on $[[a, b]]^c$ by $\bar{f}(s) = f$ (first $2n+1$ elements of s). By $[a, b] \rightarrow (2c)^c_{\mathrm{reg}}$ there is a set $\bar{X}$ of size $2c$ min-homogeneous for $\bar{f}$. If we then let X consist of the first c elements of $\bar{X}$, then X is min-homogeneous for f — the last c members of $\bar{X}$ can be used to extend any $2n+1$-tuple from X to a c-tuple from $\bar{X}$.

Note that we can take $\sigma(k)$ to be primitive recursive, since it can be gotten from the primitive recursive truth predicate for Σ_0 formulas through bounded quantification. Hence, by 'overspill' (which in this case just asserts that N is not definable in $M \models \mathrm{PRA}$) there is a $t \in M - N$ such that $M \models \sigma(t)$. We can take $\langle c_i \mid i \in N \rangle$ to be the first N of the t indiscernibles provided.

We now let I be determined by $\langle c_i \mid i \in N \rangle$, i.e., $I = \{x \in M \mid \exists i\,(x < c_i)\}$, and establish that $I \models \mathrm{PA}$:

First, note that if $i_0 < i_1 < i_2$, then for any $p < c_{i_0}$, $p + c_{i_1} = c_{i_2}$ would imply $p + c_{i_1} = c_j$ for any $j > i_2$ by diagonal indiscernibility, contradicting the distinctness of the c_i's. Hence, $c_{i_0} + c_{i_1} \leq c_{i_2}$ and so *I is closed under addition.*

Next, suppose that $i_0 < i_1 < i_2$ and for some $p < c_{i_0}$,

$$p \cdot c_{i_1} < c_{i_2} \leq (p+1) \cdot c_{i_1}. \tag{\#}$$

Then by adding c_{i_1} to both sides of the first inequality we get $(p+1) \cdot c_{i_1} < c_{i_1} + c_{i_2}$, and for any $j > i_2$, $c_{i_1} + c_{i_2} \leq c_j$ by the previous paragraph, so that $(p+1) \cdot c_{i_1} < c_j$. But this would contradict the diagonal indiscernibility applied to the second inequality of (#). Thus, there is no such p, and so $c_{i_0} \cdot c_{i_1} \leq c_{i_2}$ and *I is closed under multiplication.*

It remains to establish that I satisfies the Induction schema. Notice that if $p < c_{i_0}$ and ψ is any Σ_0 formula, then since Σ_0 formulas are absolute between $I < M$ and the c_i's are diagonal indiscernibles,

$$
\begin{aligned}
& I \models \exists x_1 \forall x_2 \cdots \exists x_n \psi(p, x_1, \ldots, x_n) \\
\text{iff } & \exists i_1 > i_0 \forall i_2 > i_1 \cdots \exists i_n > i_{n-1} I \models \exists x_1 < c_{i_1} \forall x_2 < c_{i_2} \cdots \exists x_n < c_{i_n} \psi(p, x_1, \ldots, x_n) \\
\text{iff } & \exists i_1 > i_0 \forall i_2 > i_1 \cdots \exists i_n > i_{n-1} M \models \exists x_1 < c_{i_1} \forall x_2 < c_{i_2} \cdots \exists x_n < c_{i_n} \psi(p, x_1, \ldots, x_n) \\
\text{iff } & M \models \exists x_1 < c_{i_0+1} \forall x_2 < c_{i_0+2} \cdots \exists x_n < c_{i_0+n} \psi(p, x_1, \ldots, x_n) \\
\text{iff } & I \models \exists x_1 < c_{i_0+1} \forall x_2 < c_{i_0+2} \cdots \exists x_n < c_{i_0+n} \psi(p, x_1, \ldots, x_n).
\end{aligned}
$$

Now to verify Induction, suppose that ϕ is any formula in parameters $p_0, \ldots, p_e$ and variable of induction x. It suffices to establish that if $\exists x\ \phi(p_0, \ldots, p_e, x)$, then there is a $<$-least such x. But if $\phi(p_0, \ldots, p_e, \bar{x})$, we can slightly modify ϕ by using a primitive recursive pairing function to construe $p_0, \ldots, p_e, \bar{x}$ as one p and blocks of like quantifiers as one quantifier, find an i_0 such that $p < c_{i_0}$, and implement the above reduction to a Σ_0 formula. But M satisfies Induction for Σ_0 formulas (allowing parameters) since $M \models \mathrm{PRA}$, and since $I < M$, so does I. Hence, $I \models \mathrm{PA}$.

(In terms of the Indicator Theory of Paris and Kirby, the proof of 2.2 shows that

$$Y(a, b) = \max c([a, b] \to (2c)^c_{\mathrm{reg}})$$

is an Indicator for models of PA.)

Let $\Pi_n(N)$ be the true Π_n sentences of Arithmetic. We can include $\Pi_1(N)$ in the independence result:

Corollary 2.3. (*) *is not provable in* $\mathrm{PA} + \Pi_1(N)$.

Proof. Let $M \models \mathrm{PA} + \Pi_1(N)$, and $a \in M - N$. We can suppose that there is a least

b such that $M \models [a, b] \to (2a)^a_{\text{reg}}$. By the Theorem, there is an $I < M$ such that $a < I < b$ and $I \models \text{PA}$. but Π_1 sentences persist downward, so $I \models \Pi_1(N)$. Finally, $I \models \neg\exists x\,([a, x] \to (2a)^a_{\text{reg}})$, since the coding of subsets of any $x \in I$ is certainly absolute between I and M.

The above corollary establishes our Theorem A. We now turn to Theorem B. We first argue by contradiction to show that the map $\bar{v}(n) =$ the least $m([n, m] \to (2n)^n_{\text{reg}})$ eventually dominates every provably recursive function of PA. So suppose that $\bar{v}(n) \leq g(n)$ on an infinite set $D \subseteq N$ for some provably recursive g. By an ultrapower construction with D in the ultrafilter, there is a non-standard elementary extension $M > N$ with infinite a in M such that $M \models \bar{v}(a) \leq g(a)$. By the construction in the proof of Corollary 2.3, there is $I < M$ such that $a < I < g(a)$ and $I \models \text{PA}$. But then $I \models \neg\exists y\,[y = g(a)]$ contradicting the assumption that g is provably recursive in PA.

If we now define v by $v(n) =$ the least m such that $m \to (2n)^n_{\text{reg}}$, then it is easy to correlate $\bar{v}$ to v. For example, $\bar{v}(k) \leq v(2k)$ for any k: If $m \to (4k)^{2k}_{\text{reg}}$, then for any regressive $f : [[k, m]]^k \to m$, define regressive $\bar{f} : [m]^{2k} \to m$ by $\bar{f}(s) = f$ (first k elements of s) if $\min(s) \geq k$, and $= 0$ otherwise. If $\bar{H} \in [m]^{4k}$ is min-homogeneous for $\bar{f}$, since $[k, m] \cap \bar{H}$ has at least $3k$ elements, let H be the first $2k$ of these. Then H is min-homogeneous for f, since the last k members of $\bar{H}$ can be used to extend any k-tuple from H to a $2k$-tuple from $\bar{H}$.

It is not difficult to show that if $v(2n)$ dominates every provably recursive function of PA, so does $v(n)$. We have thus established:

Corollary 2.4. *The function $v(n) =$ the least m such that $m \to (2n)^n_{\text{reg}}$ is not provably total in* PA *and eventually dominates every provably recursive function of* PA.

We can now go on to establish as in [7] or [13] that, in PA (*) is actually equivalent to the 1-Consistency of PA, i.e., the statement "PA together with the Π_1-theory of the universe is consistent." Put yet another way, (*) is equivalent to the Gödel statement $\text{Con}(\text{PA} + T_1)$, where T_1 is the set of Π_1-sentences true according to some standard complete Π_1-formula. Since (PH) is equivalent to this principle, we have:

Corollary 2.5. (PH) *and* (*) *are equivalent* (*and this is formalizable in* PA).

Very recently (March, 1985), Paris has established this corollary directly by clever combinatorial means. (See also the remarks after 4.7.) Of the several mathematical propositions now known to be equivalent to the 1-Consistency of PA, our proofs might argue for (*) as leading most directly to independence.

3. Combinatorics

The emphasis of the previous section was on a short, global proof of the independence of (*). We now turn to further combinatorial consequences of (*), primarily in order to provide those adjustments to Ketonen and Solovey [4] needed when only (*) is assumed. Assuming a known functional hierarchy result about the functions provably recursive in PA, the elegant paper [4] establishes the independence of (PH) by entirely combinatorial means, that is to say by means directly formalizable in PRA without appeal to higher principles such as Compactness. In brief, they first recall the Grzegorczyk–Wainer Hierarchy $\langle F_\alpha \mid \alpha < \epsilon_0 \rangle$ (where ϵ_0 is the least ordinal ϵ such that $\epsilon^\omega = \epsilon$ in ordinal exponentiation), a hierarchy of functions $F_\alpha : N \to N$ which, under the ordering of eventual dominance, is increasing in α and cofinal in the class of provably recursive functions. They then establish from (PH) that for any F_α, there is an $f : [N]^n \to N$ for some $n \in N$ such that any relatively large homogeneous set H for f has the property that $x < y$ both in H implies $F_\alpha(x) \leq y$. Actually, they establish a level-by-level correlation: there is such an $f : [N]^n \to N$ for F_α just in case $\alpha < \gamma_{n-2}$ (where $\langle \gamma_i \mid i \in N \rangle$ is inductively defined by $\gamma_0 = \omega$ and $\gamma_{i+1} = \omega^{\gamma_i}$, so that $\epsilon_0 = \sup \gamma_i$). Finally, they show that this directly implies that Ramsey functions in the context of (PH), for example $\sigma(n) = \text{least } m(m \to * (n+1)^n_n)$, eventually dominate any F_α, and thus that (PH) is not provable in PA. (Going beyond the results of [13], they complete their *tour de force* by providing careful upper bounds for $\sigma(n)$. However, we will not deal with this aspect here.)

The following combinatorial propositions highlight the arguments needed to adopt the [4] scheme to (*). Routine applications of the Finite Ramsey Theorem are involved here as well as in [4], but recalling that that theorem is provable in PRA, everything will be formalizable in PRA.

(For the rest of the section, our notation implicitly assumes that our min-homogeneous sets H are finite, when we need to adjust them by eliminating a $\max(H)$. This anticipates the [4] application, and no such adjustments are necessary for infinite H.)

Proposition 3.1. *There is a (primitive recursive) function $p : N \to N$ such that: For any n, $e \in N$, whenever $f_i : [N]^n \to N$ is regressive for each $i \leq e$, there is a $\rho : [N]^{n+1} \to N$ regressive such that: If $\bar{H}$ is min-homogeneous for ρ, then $H = \bar{H} - (p(e) \cup \{\max(\bar{H})\})$ is min-homogeneous for each f_i.*

The proof is an adaptation of the Harrington idea in [13], Lemmas 2.7 and 2.8, to which the following lemmas correspond:

Lemma 3.2. *If $f : [N]^n \to N$ is regressive, then $H \subseteq N$ is min-homogeneous for f iff f every $u \subseteq H$ of cardinality $n+1$ is min-homogeneous for f.*

Proof. If $H \subseteq N$ is not min-homogeneous, let $x_0 < \cdots < x_{n-1}$ be the lexigraphically least sequence drawn from H such that there are $x_0 < y_1 < \cdots < y_{n-1}$ all from H with $f(x_0, x_1, \ldots, x_{n-1}) \neq f(x_0, y_1, \ldots, y_{n-1})$, where we can take $\langle y_1, \ldots, y_{n-1} \rangle$ to be the lexigraphically least with this property. If i is the least such that $x_i \neq y_i$, then it is not difficult to see that $f(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_i, y_i, y_{i+1}, \ldots, y_{n-2})$ and thus $u = \{x_0, \ldots, x_i, y_i, y_{i+1}, \ldots, y_{n-1}\}$ is not min-homogeneous for f.

For any $x \in N$, let $\log(x) =$ least $d \in N$ such that $2^d \geq x$. Then any $y < x$ can be represented as $(y_0, \ldots, y_{\log(x)-1})_2$ in binary notation, with each $y_i < 2$. Notice that $2\log(x) + 1 \leq x$ for every $x \geq 7$. (So 7 plays a peculiar role in the present context for a reason entirely different than in [13, p. 1137].) The following lemma allows us to press down the values of a regressive function further, at the cost of increasing the exponent.

Lemma 3.3. *If $f : [N]^n \to N$ is regressive, there is an $\bar{f} : [N]^{n+1} \to N$ regressive such that:*

(i) *$\bar{f}(s) < 2\log(\min(s)) + 1$, and*

(ii) *if $\bar{H}$ is min-homogeneous for $\bar{f}$, then $H = \bar{H} - (7 \cup \{\max(\bar{H})\})$ is min-homogeneous for f.*

Proof. Write $f(s) = (y_0(s), \ldots, y_{d-1}(s))_2$, where $d = \log(\min(s))$. Define $\bar{f}$ on $[N]^{n-1}$ by:

$$\bar{f}(x_0, \ldots, x_n) = \begin{cases} 0, & \text{if either } x_0 < 7, \text{ or } \{x_0, \ldots, x_n\} \\ & \text{is min-homogeneous for } f, \\ 2i + y_i(x_0, \ldots, x_{n-1}) + 1, & \text{otherwise, where } i < \log(x_0) \text{ is} \\ & \text{the least such that } \{x_0, \ldots, x_n\} \\ & \text{is not min-homogeneous for } y_i. \end{cases}$$

Then $\bar{f}$ is regressive and satisfies (i).

To verify (ii), suppose that $\bar{H}$ is min-homogeneous for $\bar{f}$ and H is as described. If $\bar{f} \mid [H]^{n+1} = \{0\}$, then we are done by the previous lemma. Suppose on the contrary that there are $x_0 < \cdots < x_n$ all in H such that $\bar{f}(x_0, \ldots, x_n) = 2i + y_i(x_0, \ldots, x_{n-1}) + 1$. Given any $s, t \in [\{x_0, \ldots, x_n\}]^n$ with $\min(s) = \min(t) = x_0$, note that $\bar{f}(s \cup \{\max(\bar{H})\}) = \bar{f}(x_0, \ldots, x_n) = \bar{f}(t \cup \{\max(\bar{H})\})$ by min-homogeneity. But then, $y_i(s) = y_i(t)$, so that $\{x_0, \ldots, x_n\}$ was min-homogeneous for y_i after all — a contradiction.

Proof of Proposition 3.1. Note that, given any $e \in N$, for sufficiently large $x \in N$, $(2\log(x) + 1)^{e+1} \leq x$; let $p(e)$ be the least such x. We shall verify the proposition using this p. So, suppose that n, e, and f_i for $i \leq e$ are as given. To each f_i

associate $\bar{f}_i$ as in the previous lemma, and define ρ on $[N]^{n+1}$ by:

$$\rho(s)=\begin{cases}\langle \bar{f}_0(s),\ldots,\bar{f}_e(s)\rangle, & \text{if } \min(s)\geqslant p(e),\\ 0 & \text{otherwise.}\end{cases}$$

By the definition of p, ρ can be coded as a regressive function. Since $p(e)\geqslant 7$ for any e, the proof is now complete because of the previous lemma.

The next important juncture in [4] is Lemma 1.8, the verification of (*) for relatively large homogeneous sets and some straightforward generalizations of it. We discussed in Section 1 how (PH) implies (*), and the generalizations are inductively derivable from (*) alone. Actually, [4] in Lemma 1.9 requires a self-refinement of (*) in which the values on the min-homogeneous set are non-decreasing; the following proposition accomplishes the task directly from (*):

Proposition 3.4. *If $f:[N]^n\to N$ is regressive, there are $\sigma_1:[N]^{n+1}\to N$ regressive and $\sigma_2:[N]^{n+1}\to 2$ such that: If $H\subseteq N$ of cardinality $>n+1$ is both min-homogeneous for σ_1 and homogeneous for σ_2, then $H-\{\max(H)\}$ is min-homogeneous for f and if $s,t\in[H]^n$ with $\min(s)<\min(t)$, then $f(s)\leqslant f(t)$.*

Proof. Define $\sigma_1:[N]^{n+1}\to N$ by:

$$\sigma_1(x_0,\ldots,x_n)=\min(f(x_0,\ldots,x_{n-1}),f(x_1,\ldots,x_n)),$$

and $\sigma_2:[N]^{n+1}\to 2$ by:

$$\sigma_2(x_0,\ldots,x_n)=\begin{cases}0 & \text{if } f(x_0,\ldots,x_{n-1})\leqslant f(x_1,\ldots,x_n),\\ 1 & \text{otherwise.}\end{cases}$$

Now let H be as hypothesized, and suppose first that σ_2 is constantly 0 on $[H]^{n+1}$. By using $\max(H)$ as the last argument in applications of σ_1, it is straightforward to see that the conclusions of the Proposition are satisfied.

Assume to the contrary that σ_2 is constantly 1 on $[H]^{n+1}$. Let $x_0<\cdots<x_{n+1}$ be $n+2$ elements from H. Then $f(x_0,\ldots,x_{n-1})>f(x_1,\ldots,x_n)>f(x_2,\ldots,x_{n-1})$ by two applications of σ_2, so that $\sigma_1(x_0,\ldots,x_n)=f(x_1,\ldots,x_n)$ and $\sigma_1(x_0,x_2,\ldots,x_{n+1})=f(x_2,\ldots,x_{n+1})$. But this contradicts the min-homogeneity of σ_1 on H, and the proof is complete.

[4] now proceeds to establish the results about the Grzegorczyk–Wainer Hierarchy alluded to earlier. They rely on an inductive bootstraps argument based on relatively large homogeneous sets, but we describe how this can be avoided. To be concrete, we establish the analogue to their prototype result for F_ω (their Theorem 1.9); the corresponding juncture in the general case (their Theorem 3.5) can be handled similarly.

For any function $f:N\to N$, let $F^1(x)=F(x)$, and inductively, $F^{n+1}(x)=F(F^n(x))$. The first $\omega+1$ functions in the Grzegorczyk–Wainer Hierarchy are:

$$F_0(x)=x+1;\qquad F_{n+1}(x)=F_n^{x+1}(x);\quad \text{and}\quad F_\omega(x)=F_x(x).$$

It can be shown by induction that if $m \leqslant n$ and $x \leqslant y$, then $F_m(x) \leqslant F_n(y)$, a fact assumed below.

Proposition 3.5. *There are functions $\tau_1:[N]^2 \to N$ regressive, $\tau_2:[N]^2 \to N$ regressive, and $\tau_3:[N]^2 \to 2$ so that the following is true: Suppose $H \subseteq N$ of cardinality >2 is:* (a) *min-homogeneous for τ_1 and has the property that if $s, t \in [H]^2$ with $\min(s) < \min(t)$, then $\tau_1(s) \leqslant \tau_1(t)$;* (b) *min-homogeneous for τ_2; and* (c) *homogeneous for τ_3. Then for any $x < y$ both in H, $F_\omega(x) \leqslant y$.*

Proof. Define $\tau_1:[N]^2 \to N$ by:

$$\tau_1(x, y) = \begin{cases} 0 & \text{if } F_\omega(x) \leqslant y, \\ e-1 & \text{otherwise, where } e \text{ is the least such that } y < F_e(x). \end{cases}$$

Notice that $0 < e \leqslant x$ here, since F_0 is the successor function and $F_\omega(x) = F_x(x)$. Thus, τ_1 is a well-defined regressive function.

Next, define $\tau_2:[N]^2 \to N$ by:

$$\tau_2(x, y) = \begin{cases} 0 & \text{if } F_\omega(x) \leqslant y, \\ k-1 & \text{otherwise, where } F^k_{\tau_1(x,y)}(x) \leqslant y < F^{k+1}_{\tau_1(x,y)}(x). \end{cases}$$

τ_2 is also a well-defined regressive function, since $0 < k \leqslant x$ by an appeal to the definition of the F_n's.

Finally, define $\tau_3:[N]^2 \to 2$ by:

$$\tau_3(x, y) = \begin{cases} 0 & \text{if } F_\omega(x) \leqslant y, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose now that H is as hypothesized. To conclude the argument, we shall establish that τ_3 is constantly 0 on H. Assume to the contrary, and let $x < y < z$ all be from H. If $e-1 = \tau_1(x, y)$ and $k-1 = \tau_2(x, y)$, then

$$F^k_{e-1}(x) \leqslant y < z < F^{k+1}_{e-1}(x)$$

by min-homogeneity. However, the leftmost inequality implies $F^{k+1}_{e-1}(x) \leqslant F_{e-1}(y)$ since the F_n's are non-decreasing. Also, $e-1 \leqslant \tau_1(y, z)$ by condition (a) of the Proposition on H, so that

$$F_{e-1}(y) \leqslant F_{\tau_1(y,z)}(y) \leqslant z.$$

Thus, we have arrived at the contradiction $z < z$.

With these arguments in hand, one can reorganize the [4] scheme in several ways to establish that functions like $v(n) = \text{least } m(m \to (2n)^n_{\text{reg}})$ eventually dominate every F_α with $\alpha < \epsilon_0$. We should point out a weakness of our Propositions as they now stand: If we trace the exponent n needed to procure one regressive function to combine the two functions τ_1 and τ_2 of Proposition 3.5, then first by Proposition 3.4 we need a $\sigma_1:[N]^3 \to N$ taking $\tau_1 = f$ in that proposition, and then we must combine τ_2 and σ_1 by Proposition 3.1, to finally get one function: $[N]^4 \to N$. But by analogy with the level-by-level correlation in [4] between n and $\alpha < \gamma_{n-2}$ alluded to at the beginning of the section, we ought to be able to get the exponent down from 4 to 3. The following proposition provides the necessary augmentation to Proposition 3.1, by showing that we can incorporate one function: $[N]^{n+1} \to N$ into the proceedings.

Proposition 3.6. *Let $p: N \to N$ be as in Proposition* 3.1. *For any n, $e \in N$, whenever $f_i: [N]^n \to N$ is regressive for each $i \leq e$, and $f: [N]^{n+1} \to N$ is regressive, there are $\rho_1: [N]^{n+1} \to N$ regressive and $\rho_2: [N]^{n+1} \to 2$ such that if $\bar{H}$ is both min-homogeneous for ρ_1 and homogeneous for ρ_2, then $H = \bar{H} - (p(e) \cup \{\max(\bar{H})\})$ is min-homogeneous for each f_i with $i \leq e$ and for f.*

Proof. The idea is extend the proof of Proposition 3.1 by taking advantage of the fact that in Lemma 3.3, if H is sufficiently large and min-homogeneous for $\bar{f}$, then $\bar{f}$ on $[H]^{n+1}$ is in fact constantly 0. So, for $i \leq e$, let $\bar{f}_i$ correspond to f_i as before, and define $\rho_2: [N]^{n+1} \to 2$ by:

$$\rho_2(s) = \begin{cases} 0 & \text{if } \bar{f}_i(s) \neq 0 \text{ for some } i \leq e, \\ 1 & \text{otherwise.} \end{cases}$$

Then define $\rho_1: [N]^{n+1} \to N$ regressive by

$$\rho_1(s) = \begin{cases} \langle \bar{f}_0(s), \ldots, \bar{f}_e(s) \rangle & \text{if } \rho_2(s) = 0 \text{ and } \min(s) \geq p(e), \\ f(s) & \text{otherwise.} \end{cases}$$

As before, ρ_1 can be coded as a regressive function.

Suppose now that $\bar{H}$ is as hypothesized, and H is as defined from $\bar{H}$. If ρ_2 on $[H]^{n+1}$ were constantly 0, we can derive a contradiction as in the proof of Lemma 3.3. Thus, ρ_1 on $[H]^{n+1}$ must be constantly 1, and so the proof is complete.

It can now be checked in detail that (the idea of the proof of) this proposition can be used to prove results for (*) fully analogous to [4].

4. Refinement and generalization

In this concluding section, we discuss first technical improvements for previous propositions which have a consequence about subsystems of PA, and then a generalization of (*) based on the growth rate of functions.

Let $\mathrm{I}\Sigma_n$ be the subsystem of PA consisting of the defining properties of the primitive notions together with the induction schema restricted to Σ_n formulas. Thus, $\mathrm{I}\Sigma_1$ already subsumes PRA. Let $(\mathrm{PH})_n$ denote the restriction of PH to fixed exponent n, and $(*)_n$ the restriction of (*) to fixed exponent n. Paris in his meticulous paper [12] ramifies his model-theoretic analysis of PA by providing a strong level-by-level correlation of $I\Sigma_n$ for $n > 0$ with several propositions, including $(\mathrm{PH})_{n+1}$. Although the idea of diagonal indiscernibles may seem tailored for a global proof, we show how it can be used in a level-by-level analysis with the focus on $(*)_n$. Such a possibility was considered by Hajek, and perhaps others.

First, we outline a result of Clote which will correlate the coming results with $(\mathrm{PH})_{n+1}$. The quick proof of Theorem 1.3 actually shows that $(\mathrm{PH})_{n+1}$ implies $(*)_n$. Clote noticed that by working through Mills' notion of arboricity [10], the result can be sharpened to what we shall later see is best possible.

Proposition 4.1 (Clote). *For $n > 0$, $(\mathrm{PH})_n$ implies $(*)_n$.*

Proof (outline). The following notions are due to Mills [10]. If $A \subseteq N$ and $f: N \to N$, an *$f(x)$-small-branching A-tree* is a tree T with field A such that aTb implies $a < b$, and any $a \in A$ has at most $f(a)$ immediate successors in T. Set $A^0 = A - \{\max(A)\}$. Then A is *0-fold c-$f(x)$-arboreal iff* $|A| \geqslant c+1$; and inductively, A is *$(n+1)$-fold c-$f(x)$-arboreal iff* for every $f(x)$-small-branching A-tree T, there is a path P through T such that P^0 is n-fold c-$f(x)$-arboreal. Finally, if $X \subseteq N$ and $F: [X]^{n+1} \to N$, say that $H \subseteq N$ is *pre-homogeneous for F iff* $F \mid [H]^{n+1}$ only depends on the first n elements, i.e., $F(x_0, \ldots, x_{n-1}, a) = F(x_0, \ldots, x_{n-1}, b)$ for any $x_0 < \cdots < x_{n-1} < a < b$ all from H.

The argument of [10, Lemma 3.5] establishes the following: Suppose $c, n \geqslant 1$ and A is n-fold c-x^{2^x}-arboreal with $\min(A) > 0$. Suppose also that f with domain $[A]^{n+1}$ is regressive. Then, there is an $H \subseteq A$ such that H is pre-homogeneous for f and H^0 is $(n-1)$-fold c-x^{2^x}-arboreal.

Thus, the following is an immediate corollary by induction, since pre-homogeneous and min-homogeneous coincide at exponent 2: For any n, suppose $c \geqslant 1$ and A is n-fold $(c-1)$-x^{2^x}-arboreal with $\min(A) > 0$. Then, $A \to (c)^{n+1}_{\mathrm{reg}}$.

Finally, [10, Theorem 3.6] provides careful lower and upper bounds on the sizes of A such that $A \to_* (n+2)^{n+1}_c$ in terms of arboricity. It is immediate from this theorem that for any n, if A is such that $\min(A) \geqslant \max\{n^{2n}, 2c\}$, $A \to_* (n+2)^{n+1}_c$ implies that A is n-fold $(c-n-1)$-x^{2^x}-arboreal, and thus that $A \to (c-n)^{n+1}_{\mathrm{reg}}$. Careful information is provided here which more than suffices to show that $(\mathrm{PH})_{n+1}$ implies $(*)_{n+1}$ for any n.

Paris [12, Theorem 3.6] established that in $I\Sigma_1$, $(\mathrm{PH})_{n-1}$ is equivalent to the 1-Consistency of $I\Sigma_n$ for $n \geqslant 1$. (Recalling the discussion at the end of Section 2, this means $\mathrm{Con}(I\Sigma_n + T_1)$, where T_1 is the set of Π_1 sentences true according to some standard complete Π_1 formula.) With Proposition 4.1 in hand, we shall focus on $(*)_{n+1}$ and show how it can be used to generate indiscernibles to establish the 1-Consistency of $I\Sigma_n$. To do this, we first establish the conclusion of Proposition 2.1 assuming only $(*)_{n+1}$, improving the $(*)_{2n+1}$ of the short proof given. The following lemma should have a familiar ring:

Lemma 4.2. *There are three regressive functions $\eta_1, \eta_2, \eta_3: [N]^2 \to N$ such that whenever $\bar{H}$ is min-homogeneous for all of them, then $H = \bar{H}$-(the last three elements of $\bar{H}$) has the property that $x < y$ both in H implies $x^x \leq y$.*

Proof. Define $\eta_1, \eta_2, \eta_3: [N]^2 \to N$ by:

$$\eta_1(x, y) = \begin{cases} 0 & \text{if } x + x \leq y \\ y - x & \text{otherwise,} \end{cases}$$

$$\eta_2(x, y) = \begin{cases} 0 & \text{if } x \cdot x \leq y \\ u & \text{otherwise, where } u \cdot x \leq y < (u+1) \cdot x, \end{cases}$$

and

$$\eta_3(x, y) = \begin{cases} 0 & \text{if } x^x \leq y \\ v & \text{otherwise, where } x^v \leq y < x^{v+1}. \end{cases}$$

Suppose that $\bar{H}$ is as hypothesized, and let $z_1 < z_2 < z_3$ be the last three elements of $\bar{H}$. If $x < y$ are both in $\bar{H} - \{z_3\}$, then since $\eta_1(x, y) = \eta_1(x, z_3)$, clearly we must have $\eta_1(x, y) = 0$. Hence, η_1 on $[\bar{H} - \{z_3\}]^2$ is constantly 0.

Next, assume that $x < y$ are both in $\bar{H} - \{z_2, z_3\}$ and $\eta_2(x, y) = u > 0$. Then

$$u \cdot x \leq y < z_2 < (u+1) \cdot x$$

by min-homogeneity, and $(u+1) \cdot x \leq x + y$ by adding x to both sides of the first inequality, which in turn is $< y + y \leq z_2$ by the previous paragraph. But this leads to the contradiction $z_2 < z_2$. Hence, on $[\bar{H} - \{z_2, z_3\}]^2$ is constantly 0.

Finally, we can iterate the argument to show that η_3 on $[\bar{H} - \{z_1, z_2, z_3\}]^2$ is constantly 0, and so the proof is complete.

This is all that we will need of a clearly inductive argument which proceeds through the classical Grzegorczyk Hierarchy, or equivalently, through $\langle F_n \mid n \in \omega \rangle$ of Section 3.

Here is the heralded improvement:

Proposition 4.3. *The conclusion of Proposition* 2.1 *already follows from* $(*)_{n+1}$.

Proof. Given the $e, k, n \in N$ and formulas $\psi_0, \ldots, \psi_e$ in at most $n+1$ free variables, first define $q: N \to N$ by $q(x) =$ largest $d \in N$ such that $2^{e+1} \cdot d \leq x$, and then define $f: [N]^{n+1} \to N$ by:

$$f(x_0, \ldots, x_n) = \langle \delta_{ip} \mid i \leq e \ \& \ p < q(x_0) \rangle$$

where

$$\delta_{ip} = \begin{cases} 0 & \text{if } \psi_i(p, x_1, \ldots, x_n) \text{ is true,} \\ 1 & \text{otherwise.} \end{cases}$$

By definition of q, we can code f as a regressive function.

The idea now is to combine the functions of the previous lemma with f to get a min-homogeneous set spread out enough to accommodate q. However, a direct application of Proposition 3.6 would be restricted to $n \geqslant 2$, so we use an idea similar to the proof of that proposition. First define $g:[N]^2 \to 4$ from the functions of the previous lemma as follows:

$$g(x, y) = \begin{cases} 0 & \text{if } \eta_j(x, y) = 0 \text{ for } j = 1, 2, 3, \\ j & \text{otherwise, where } j \text{ is the least such that } \eta_j(x, y) \neq 0. \end{cases}$$

Then define $h:[N]^{n+1} \to N$ regressive by:

$$h(x_0, \ldots, x_n) = \begin{cases} \eta_j(x_0, x_1) & \text{if } g(x_0, x_1) = j > 0, \\ f(x_0, \ldots, x_n) & \text{otherwise.} \end{cases}$$

(We assume from here that $n > 0$, else the Proposition is trivial.)

By $(*)_{n+1}$, let H_0 of cardinality $2^{e+1} + 2k + 3$ be min-homogeneous for h and homogeneous for g. Let $z_1 < z_2 < z_3$ be the last three elements of H_0, and set $H_1 = H_0 - \{z_1, z_2, z_3\}$. If g on $[H_1]^2$ were constantly $j > 0$, then we can derive a contradiction as in the previous lemma. Thus, we can assume that $h = f$ on $[H_1]^{n+1}$, and $x < y$ both in H_1 implies $x^x \leqslant y$.

Now let H_2 consist of the last $2k$ elements of H_1. Then $x < y$ both in H_2 implies $2^{(e+1)x} \leqslant x^x \leqslant y$ since $2^{e+1} \leqslant x$, so that $x \leqslant q(y)$. Thus, if H_3 consists of every other element of H_2, then H_3 constitutes the desired k diagonal indiscernibles since H_2 is min-homogeneous for f.

Now the corresponding model-theoretic result:

Theorem 4.4. *Suppose that $n > 0$ and $M \models \mathrm{PRA}$ & $[a, b] \to (c)^{n+1}_{\mathrm{reg}}$, where $c \in M - N$. Then there is an $I < M$ with $a < I < b$ such that $I \models \mathrm{I}\Sigma_n$.*

Proof. We proceed as in the proof of Theorem 2.2. First, we can get diagonal indiscernibles $\langle c_i \mid i \in N \rangle$ in the interval $[a, b]$ for all Σ_0 formulas but in at most $n + 1$ free variables, using Proposition 4.3. Let $I < M$ be determined by the c_i's, and note that I is closed under addition and multiplication. (The proof in Theorem 2.2 works only for $n \geqslant 2$, but we could cite Proposition 4.3 where we spread out the indiscernibles directly.) Finally, the verification of the Induction schema for Σ_n formulas proceeds as before, since we only need n alterations of quantifiers.

Corollary 4.5. *If $n > 0$, $(*)_{n+1}$ is not provable in $\mathrm{I}\Sigma_n + \Pi_1(N)$.*

Proof. As for Corollary 2.3.

Proceeding as with $(*)$, we can observe that

$$Y(a, b) = \max c([a, b] \to (c)^{n+1}_{\text{reg}})$$

is an indicator for models of $\mathrm{I}\Sigma_n$, and that in PRA, $(*)_{n+1}$ implies $\mathrm{Con}(\mathrm{I}\Sigma_n + T_1)$, the 1-Consistency of $\mathrm{I}\Sigma_n$.

We have the following corollaries:

Corollary 4.6. *The function $\nu_2(n) =$ the least m such that $m \to (n)^2_{\text{reg}}$ eventually dominates every primitive recursive function; its rate of growth is approximately that of the Ackermann function.*

Corollary 4.7. *If $n > 0$, the following are equivalent in $\mathrm{I}\Sigma_1$:*
(a) $(\mathrm{PH})_{n+1}$,
(b) $(*)_{n+1}$,
(c) $\mathrm{Con}(\mathrm{I}\Sigma_n + T_1)$.

Corollary 4.6 is our Theorem C; the proof of its first part is entirely analogous to that of Corollary 2.4. The proof of the second part is part of the proof that in PRA, $(*)_{n+1}$ implies $\mathrm{Con}(\mathrm{I}\Sigma_n + T_1)$. Corollary 4.7 follows from the result of Paris [12] mentioned earlier, that $(\mathrm{PH})_{n+1}$ is equivalent to $\mathrm{Con}(\mathrm{I}\Sigma_n + T_1)$. Our argument with indiscernibles may be more direct than Paris' argument from $(\mathrm{PH})_{n+1}$. However, at present we see no way to establish $\mathrm{Con}(\mathrm{I}\Sigma_n + T_1)$ implies $(*)_{n+1}$ other than to go through his argument using concepts from [4] and developing the [4] scheme for $(*)$. Very recently (March 1985), Paris has provided a clever combinatorial argument to show that $(*)_{n+1}$ implies $(\mathrm{PH})_{n+1}$ directly.

We now discuss a simple way to extend $(*)$ based on the growth rate of functions, in the spirit of [8]. Given any $F: N \to N$, $n \in N$, and $X \subseteq N$, say that a function $f: [X]^n \to N$ is *F-regressive* if $f(s) < F(\min(s))$ for all s such that $F(\min(s)) > 0$. For $n, k, m \in N$, $m \to (k)^n_{F\text{-reg}}$ means that whenever $f: [m]^n \to N$ is F-regressive, there is an $H \in [m]^k$ min-homogeneous for f. Consider now the propositions

$$\text{For any } n, k, \in N, \text{ there is an } m \in N \text{ such that } m \to (k)^n_{F\text{-reg}}. \qquad (*)_F$$

Thus, $(*)$ is just the special case when F is taken to be the identity function, and for any F, $(*)_F$ follows from the Erdös-Rado Theorem 1.1 by the same sort of argument as for $(*)$. The corresponding generalization of (PH) discussed in [8] results from replacing relatively large by $|H| \geqslant F(\min(H))$.

The following characterization makes clear how $(*)_F$ can be incorporated into the known contexts:

Proposition 4.8. *For any increasing function $F: N \to N$, $(*)_F$ is equivalent to: For any $n, k \in N$, there is an $m \in N$ such that whenever $f: [m]^n \to m$ is regressive, there* *is an $H \in [m]^k$ which is min-homogeneous for f and has the property that $x < y$ both in H implies $F(x) \leqslant y$.*

Proof. In the forward direction, define $g : [N]^2 \to N$ by

$$g(x, y) = \begin{cases} 0 & \text{if } F(x) \leqslant y. \\ y & \text{otherwise.} \end{cases}$$

g is clearly F-regressive, and whenever H finite of cardinality >2 is min-homogeneous for g, then g on $[H - \{\max(H)\}]^2$ is constantly 0. By $(*)_F$ we can find an m sufficiently large such that: given any $f : [m]^n \to m$ regressive, we can combine it with $g \mid [m]^2$ by a version of Proposition 3.1, and find a min-homogeneous set for both f and g of cardinality k.

For the converse, the argument of [13, Lemma 2.14] can be used. Define $h : N \to N$ by $h(x) =$ largest y such that $F(y) \leqslant x$ (and 0 if there is no such y). Now given $n, k \in N$, let m be as provided, and suppose that $f : [m]^n \to N$ is F-regressive. Define $\bar{f}$ on $[m]^n$ by:

$$\bar{f}(x_0, \ldots, x_{n-1}) = \begin{cases} f(h(x_0), \ldots, h(x_{n-1})), & \text{if } h(x_i) \neq h(x_j) \text{ for } 0 \leqslant i < j < n, \\ 0 & \text{otherwise.} \end{cases}$$

Since this value is $< F(h(x_0)) \leqslant x_0$, whenever $F(h(x_0) \neq 0$, $\bar{f}$ is a regressive function. So, let $\bar{H} \in [m]^k$ be min-homogeneous for $\bar{f}$, satisfying: $x < y$ both in $\bar{H}$ implies $F(x) \leqslant y$. Consider $H = \{h(x) \mid x \in \bar{H}\}$; the last condition on $\bar{H}$ guarantees that h is one-to-one on $\bar{H}$, so that $|H| = k$. H is clearly min-homogeneous for f.

We can now proceed as in [8] to formulate for each $n \in N$ a function G_n based on some complete Π_n-formula so that over PA, $(*)_{G_n}$ is equivalent to the n-Consistency of PA, i.e. Con(PA + T_n) where T_n is the set of Π_n-sentences true according to a standard complete Π_n-formula. Moreover, one can continue this process into the transfinite as in [8] with a hierarchy of faster and faster growing functions.

We conclude with some remarks on regressive partition relations and the Reverse Mathematics program of Friedman, Simpson and others. It has been observed that over the base theory RCA_0 (Recursive Comprehension Axiom), the system ACA_0 (Arithmetic Comprehension Axiom) is equivalent to the system axiomatized by the principle $N \to (N)^3_2$. It is not known whether the supercript 3 can be replaced by 2. This is the so-called '3-2 problem,' which has the following recursion-theoretic formulation: Is there a recursive map $f : [N]^2 \to 2$ such that for any infinite homogeneous H, $0' \leqslant_T H$? Peter Clote has observed that over RCA_0, the system axiomatized by $N \to (N)^2_{\text{reg}}$ is equivalent to ACA_0. Thus the exponent can be lowered if regressive partitions are used.

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