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DIAMONDS, LARGE CARDINALS, AND ULTRAFILTERS*

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That Jensen's Diamond Principle can be decided by sufficiently strong large cardinal hypotheses has been known for quite some time. Here, we provide a leisurely discussion of the results and questions concerning this general theme, with particular focus on a strong variant of Diamond and ultrafilters over a measurable cardinal.

Henceforth, we will denote by κ a regular uncountable cardinal, and by E a stationary subset of κ consisting of limit ordinals. *Diamond for* E is:

$\Diamond_\kappa(E)$: There is a sequence $\langle S_\alpha \mid \alpha \in E\rangle$, where $S_\alpha \subseteq \alpha$, such that: for any $X \subseteq \kappa$, $\{\alpha \in E \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.

$\Diamond_\kappa$ is simply this principle where we take $E = \kappa$. Jensen [Jen] established that if $V = L$, then $\Diamond_\kappa(E)$ holds for every κ and E; he first isolated these principles in his famous proof of the failure of Souslin's Hypothesis in L. These principles have turned out to be very convenient and useful for consistency results, as they encapsulate in succinct form a significant aspect of constructibility. It is quite noteworthy that sufficiently strong large cardinal axioms about some κ, axioms antithetical to $V = L$, also imply $\Diamond_\kappa$, roughly because they impose a uniform superstructure on the power set. (In what follows, we assume a familiarity with the basic facts about the well-known large cardinal axioms; see [Jec] or [KM] for details.) Let us take a look at the weakest natural such axiom, formulated by Jensen and Kunen [JK]:

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* This was requested, but not the topic of the author's talk at the conference.

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DEFINITION: κ is *subtle* iff whenever $\langle S_\alpha \mid \alpha<\kappa\rangle$ with $S_\alpha \subseteq \alpha$ and closed unbounded $C \subseteq \kappa$ are given, there are $\alpha<\beta$ both in C such that $S_\beta \cap \alpha = S_\alpha$.

Subtlety is a natural weakening of the better known concept of ineffability, which figures prominently in the study of the Generalized Kurepa's Hypothesis in L (see [Jen]). We provide the proof due to Kunen (see [JK]) of the next result for a later generalization:

PROPOSITION 1: *If* κ *is subtle, then* $\diamond_\kappa$ *holds.*

Proof: Define $\langle S_\alpha, C_\alpha\rangle$ for limit ordinals $\alpha<\kappa$ by induction as follows: Suppose that we have already provided for $\alpha<\beta$. If there is an $S \subseteq \beta$ and a closed unbounded $C \subseteq \beta$ such that $\alpha \in C$ implies $S \cap \alpha \neq S_\alpha$, let us call this the *nontrivial case* and set $\langle S_\beta, C_\beta\rangle$ = one such pair $\langle S, C\rangle$. Otherwise, set $\langle S_\beta, C_\beta\rangle = \langle \phi, \beta\rangle$.

Let us now show that $\langle S_\alpha \mid \alpha<\kappa\rangle$ verifies $\diamond_\kappa$: Assume to the contrary that there is an $X \subseteq \kappa$ and a closed unbounded $C \subseteq \kappa$ such that $\alpha \in C$ implies $X \cap \alpha \neq S_\alpha$. Let $\overline{C}$ consist of the limit points of C. Notice that for any $\beta \in \overline{C}$, $\alpha \in C \cap \beta$ implies $(X \cap \beta) \cap \alpha = X \cap \alpha \neq S_\alpha$, so that the existence of the pair $\langle X \cap \beta, C \cap \beta\rangle$ insures that the nontrivial case of the definition occurred at β. However, by subtlety, let $\alpha<\beta$ both in $\overline{C}$ satisfy $S_\beta \cap \alpha = S_\alpha$ and $C_\beta \cap \alpha = C_\alpha$. (Both of these conditions can be met, by applying subtlety to a closed unbounded subset of $\overline{C}$ consisting of ordinals closed under the Gödel pairing function.) The latter implies $\alpha \in C$ so that $S_\beta \cap \alpha \neq S_\alpha$, a contradiction. □

It is well-known that κ is measurable implies κ is ineffable, which in turn implies that κ is subtle. Thus, all these hypotheses imply $\diamond_\kappa$.

In the direction of weaker axioms, although the least subtle cardinal is not weakly compact (since the subtlety of κ has a Π^1_1 description over $\langle V_\kappa, \varepsilon\rangle$), it is known that there are many weakly compact cardinals below any subtle cardinal. The following is a prominent open question in this area:

QUESTION 1: *If* κ *is weakly compact, does* $\diamond_\kappa$ *hold?*

It is likely that the answer is no, and efforts of Woodin have shown that it is consistent to have a Mahlo cardinal κ so that $\diamond_\kappa$ fails.

In the direction of stronger versions of $\diamond_\kappa$, Baumgartner [B1] observed that the ineffability of κ denies the principle now commonly known as $\diamond^*_\kappa$: There is a sequence $\langle S_\alpha \mid \alpha<\kappa\rangle$, where $S_\alpha \subseteq P(\alpha)$ and $|S_\alpha| = |\alpha|$, such that for any $X \subseteq \kappa$, $\{\alpha \mid X \cap \alpha \in S_\alpha\}$ contains a closed unbounded set. $\diamond^*_\kappa$ implies $\diamond_\kappa$, and it is shown in [B1] that if $V = L$, then κ is ineffable *iff* $\diamond^*_\kappa$

fails. Thus, the ineffability of κ makes a fine distinction between $\Diamond_\kappa$ and $\Diamond^*_\kappa$.

It is another direction of strengthening $\Diamond_\kappa$ which is the main concern of this article: We investigate the possibilities for $\Diamond_\kappa(\mathrm{Cof}^\lambda_\kappa)$, where $\lambda<\kappa$ is regular and

$$\mathrm{Cof}^\lambda_\kappa = \{\alpha<\kappa \mid \mathrm{cf}(\alpha)=\lambda\}.$$

$\mathrm{Cof}^\lambda_\kappa$ is certainly a natural stationary set to consider, but the questions raised seem to be complicated and difficult.

We first provide some correlating characterizations. By a *ladder system* we mean an indexed set $\{t_\alpha \mid \alpha\in X\}$ for some set X of limit ordinals such that each t_α is a cofinal subset of α of ordertype $\mathrm{cf}(\alpha)$. Ladder systems figure prominently in the study of weak versions of $\Diamond_\kappa$, for example in Shelah [S]. The first part of the following is a well-known result from Ostaszewski [O].

PROPOSITION 2:

(a) $\Diamond_\kappa(E)$ *iff* $\kappa^{<\kappa}=\kappa$ *and PRINCIPLE P1: there is a ladder system* $\{t_\alpha \mid \alpha\in E\}$ *such that for any unbounded* $X\subseteq\kappa$, *there is an* $\alpha\in E$ *such that* $t_\alpha\subseteq X$.

(b) $\Diamond_\kappa(E)$ *iff* $\Diamond_\kappa$ *and PRINCIPLE P2: there is a ladder system* $\{t_\alpha \mid \alpha\in E\}$ *such that for any stationary* $X\subseteq\kappa$, *there is an* $\alpha\in E$ *such that* $t_\alpha\subseteq X$.

Proof: For (a), the forward direction is straightforward; it is well-known that $\Diamond_\kappa$ implies $\kappa^{<\kappa}=\kappa$.

For the converse, let $\{t_\alpha \mid \alpha\in E\}$ verify P1, and using $\kappa^{<\kappa}=\kappa$, let $\langle T_\xi \mid \xi\in\kappa\rangle$ enumerate the bounded subsets of κ so that each set occurs cofinally often. Setting $S_\alpha = \cup\{T_\xi \mid \xi\in t_\alpha\}$ for $\alpha\in E$, we show that $\langle S_\alpha \mid \alpha\in E\rangle$ verifies $\Diamond_\kappa(E)$:

Given any $X\subseteq\kappa$ and closed unbounded $C\subseteq\kappa$, first define ordinals ξ_η for $\eta<\kappa$ by induction as follows: With ξ_ζ for $\zeta<\eta$ already provided, let $\gamma = \sup_{\zeta<\eta}\xi_\zeta$, find a $\rho>\gamma$ such that $\rho\in C$, and finally let $\xi_\eta>\rho$ such that $T_{\xi_\eta} = X\cap\rho$. By P1, there is an $\alpha\in E$ such that $t_\alpha\subseteq\{\xi_\eta \mid \eta<\kappa\}$. By construction, $\alpha = \sup t_\alpha\in C$, and moreover $S_\alpha = X\cap\alpha$.

For (b), the forward direction is again straightforward. For the converse, let $\langle S_\xi \mid \xi<\kappa\rangle$ verify $\Diamond_\kappa$ and $\{t_\alpha \mid \alpha\in E\}$ verify P2. Setting $A_\alpha = \cup\{S_\xi \mid \xi\in t_\alpha\}$ for $\alpha\in E$, we show that $\langle A_\alpha \mid \alpha\in E\rangle$ verifies $\Diamond_\kappa(E)$: Given any $X\subseteq\kappa$ and closed unbounded $C\subseteq\kappa$, $S = \{\xi<\kappa \mid X\cap\xi = S_\xi\}$ is stationary, so $\overline{S} = S\cap C$ is also stationary. By P2, let $\alpha\in E$ such that $t_\alpha\subseteq\overline{S}$.

Then $\alpha \in C$, and moreover $A_\alpha = X \cap \alpha$. □

The following seems unresolved:

QUESTION 2: *Does* P2 *imply* P1 ?

We next formulate an appropriate strengthening of subtlety for deriving $\Diamond_\kappa(E)$:

DEFINITION: κ is E-*subtle iff* there is a ladder system $\{t_\alpha \mid \alpha \in E\}$ such that whenever $\langle S_\xi \mid \xi < \kappa \rangle$ with $S_\xi \subseteq \xi$ and closed unbounded $C \subseteq \kappa$ are given, there are $\alpha < \beta$ such that $\alpha \in E$, $t_\alpha \cup \{\beta\} \subseteq C$, and $S_\beta \cap \xi = S_\xi$ for every $\xi \in t_\alpha$.

This is not a very natural concept; rather, it should be regarded as the appropriate hypothesis for generalizing Proposition 1:

PROPOSITION 3: *If* κ *is* E-*subtle, then* $\Diamond_\kappa(E)$ *holds.*

Proof: Let $\{t_\alpha \mid \alpha \in E\}$ be as in the definition of E-subtle. We will define sets $\langle S_\alpha, C_\alpha \rangle$ for $\alpha < \kappa$ by induction so that, setting $A_\alpha = \cup \{S_\xi \mid \xi \in t_\alpha\}$ for $\alpha \in E$, $\langle A_\alpha \mid \alpha \in E \rangle$ will verify $\Diamond_\kappa(E)$. Suppose that we have already provided for $\alpha < \beta$. If there is an $S \subseteq \beta$ and a closed unbounded $C \subseteq \beta$ such that $\alpha \in C \cap E$ implies $S \cap \alpha \neq \cup \{S_\xi \mid \xi \in t_\alpha\} = A_\alpha$, let us call this the *non-trivial case*, and set $\langle S_\beta, C_\beta \rangle =$ such a pair $\langle S, C \rangle$. Otherwise, set $\langle S_\beta, C_\beta \rangle = \langle \phi, \beta \rangle$.

Let us now show that $\langle A_\alpha \mid \alpha \in E \rangle$ verifies $\Diamond_\kappa(E)$. Assume to the contrary that there is an $X \subseteq \kappa$ and closed unbounded $C \subseteq \kappa$ such that $\alpha \in C \cap E$ implies $X \cap \alpha \neq A_\alpha$. Let $\overline{C}$ consist of the limit points of C. Notice that for any $\beta \in \overline{C}$, $\alpha \in (C \cap \beta) \cap E$ implies $(X \cap \beta) \cap \alpha = X \cap \alpha \neq A_\alpha$, so that the existence of the pair $\langle X \cap \beta, C \cap \beta \rangle$ insures that the nontrivial case of the definition occurred at β. However, by E-subtlety let $\alpha < \beta$ with $\alpha \in E$, $t_\alpha \cup \{\beta\} \subseteq \overline{C}$, and $S_\beta \cap \xi = S_\xi$ and $C_\beta \cap \xi = C_\xi$ for every $\xi \in t_\alpha$. (Again, both of these conditions can be met by applying E-subtlety to a closed unbounded subset of $\overline{C}$ consisting of ordinals closed under the Gödel pairing function.) Thus, $S_\beta \cap \alpha = \cup \{S_\xi \mid \xi \in t_\alpha\} = A_\alpha$, but also the condition on the C_ξ's insures that $\alpha \in C_\beta$. This is a contradiction. □

Before proceeding further, we quickly recall the relevant notation and concepts from the theory of ultrafilters over a measurable cardinal (see Kanamori [Ka] for further details). By a κ-*ultrafilter* is meant a (nonprincipal) κ-complete ultrafilter over κ, i.e. a witness to the measurability of κ. If U is such an ultrafilter, j_U denotes the corresponding ultrapower embedding, and if f is a function with domain κ, then $[f]_U$ denotes the ultrapower equivalence class of f, and $f_*(U) = \{X \mid f^{-1}(X) \in U\}$, a κ-ultrafilter if f is not constant (mod U). $id: \kappa \to \kappa$ denotes the identity function on κ; a κ-ultrafilter U is *normal iff* $[id]_U = \kappa$, i.e. the identity map is the least nonconstant function (mod U). We henceforth denote by C_κ the closed unbounded filter over κ; it is well-known that a normal κ-ultrafilter extends the filter C_κ. Moreover, for any κ-ultrafilter U, those functions $f: \kappa \to \kappa$ such that $f_*(U)$ is a κ-ultrafilter extending C_κ are important in the structural study of the ultrapower of κ via U: they are the beginnings of the "skies". Finally, if U is any κ-ultrafilter, then in the inner model $L[U]$, every κ-ultrafilter has only a finite number of skies and the set $C_\kappa \cup \{Cof_\kappa^\omega\}$ cannot be extended to a κ-ultrafilter. The following result applies just beyond this minimal relative consistency:

PROPOSITION 4: *If* λ *is regular,* $\lambda < \kappa$, *and* $C_\kappa \cup \{Cof_\kappa^\lambda\}$ *can be extended to a* κ*-ultrafilter, then PRINCIPLE P3: There is a ladder system* $\{t_\alpha \mid \alpha < \kappa$ *and* $cf(\alpha) = \lambda\}$ *such that for any closed unbounded* $C \subseteq \kappa$, *there is an* $\alpha < \kappa$ *with* $cf(\alpha) = \lambda$ *such that* $t_\alpha \subseteq C$.

Proof: It was first shown by Ketonen [Ke] that if the hypothesis obtains, then there is a κ-ultrafilter $U \supseteq C_\kappa \cup \{Cof_\kappa^\lambda\}$ for which moreover there is an increasing sequence $\langle [f_\xi]_U \mid \xi < \lambda \rangle$ cofinal in $[id]_U$ such that $f_{\xi *}(U) \supseteq C_\kappa$. Hence, $X = \{\alpha < \kappa \mid \sup_{\xi<\lambda} f_\xi(\alpha) = \alpha\} \in U$, and if we set $t_\alpha = \{f_\xi(\alpha) \mid \xi < \lambda\}$ for $\alpha \in X$, then for any closed unbounded $C \subseteq \kappa$, $C \in f_{\xi *}(U)$ for each $\xi < \lambda$, so that $\{\alpha \in X \mid t_\alpha \subseteq C\} \in U$. Thus, P3 has been verified. □

The following seem unresolved:

QUESTION 3: *Does* P3 *imply* P2?

QUESTION 4: *Is it consistent to have a measurable cardinal* κ *at which* P3 *fails?*

Interestingly enough, P3 for $\kappa = \omega_1$ is denied by the Proper Forcing Axiom — see 3.4 of Baumgartner [B2].

The following result will show that a strengthening of the ultrafilter hypothesis used within the proof of Proposition 4 (where we take all the $f_{\xi *}(U)$'s to be the same normal ultrafilter N) will imply $\Diamond_\kappa(Cof_\kappa^\lambda)$:

THEOREM 5: (i) $\to$ (ii) $\to$ (iii) $\to$ (iv), *where:*

(i) *There is a normal* κ*-ultrafilter* N *and a ladder system* $\{t_\alpha \mid \alpha < \kappa$ and $cf(\alpha) = \lambda\}$ *such that if we set* $f_\xi(\alpha) = \xi$th *element of* t_α, *then* $\{f_\xi^{-1}(X) \mid \xi < \lambda$ and $X \in N\}$ *is a* κ*-complete filter base.*

(ii) *κ is* $\mathrm{Cof}_\kappa^\lambda$*-subtle.*

(iii) $\Diamond_\kappa(\mathrm{Cof}_\kappa^\lambda)$.

(iv) *There is a ladder system* $\{t_\alpha \mid \alpha<\kappa$ and $cf(\alpha)=\lambda\}$ *such that if we set* $f_\xi(\alpha)=\xi$ th *element of* t_α *, then: whenever* $\{F_\xi \mid \xi<\lambda\}$ *are uniform* κ*-complete filters, then* $\{f_\xi^{-1}(X) \mid \xi<\lambda$ and $X\in F_\xi\}$ *is a* κ*-complete filter base.*

Proof: (i) $\rightarrow$ (ii). Suppose that $\langle S_\xi \mid \xi<\kappa\rangle$ with $S_\xi\subseteq\xi$ and closed unbounded $C\subseteq\kappa$ are given. If we set $S = [\langle S_\xi \mid \xi<\kappa\rangle]_N$, then by normality $S \subseteq [id]_N=\kappa$, and $j_N(S)\cap\kappa = S$. Thus, $X = \{\xi<\kappa \mid S\cap\xi = S_\xi\}\in N$, and also by normality, $C\in N$. By hypothesis, $Y = \bigcap_{\xi<\lambda} f_\xi^{-1}(X\cap C)$ is nonempty, so let α be a member. If β is any member of $X\cap C$ above α, then clearly $t_\alpha\cup\{\beta\}\subseteq C$, and $S_\beta\cap\xi = S\cap\xi = S_\xi$ for every $\xi\in t_\alpha$.

(ii) $\rightarrow$ (iii) is Proposition 3.

(iii) $\rightarrow$ (iv). By using a bijection: $\lambda\times\kappa\leftrightarrow\kappa$, we can invoke $\Diamond_\kappa(\mathrm{Cof}_\kappa^\lambda)$ in the form: There is a sequence $\langle A_\alpha \mid \alpha<\kappa$ and $cf(\alpha)=\lambda\rangle$ with $A_\alpha\subseteq\lambda\times\alpha$ such that: for any $X\subseteq\lambda\times\kappa$, $\{\alpha<\kappa \mid cf(\alpha)=\lambda$ and $X\cap(\lambda\times\alpha) = A_\alpha\}$ is stationary in κ. Now, choose a ladder system $\{t_\alpha \mid \alpha<\kappa$ and $cf(\alpha)=\lambda\}$ in such a way that the ξ th of t_α is in $A_\alpha\cap\{\xi\}\times\alpha$ *if* this is possible.

To verify (iv), let $\{F_\xi \mid \xi<\lambda\}$ be as hypothesized. It suffices to establish that if $X_\xi\in F_\xi$ for each $\xi<\lambda$, then $\bigcap_{\xi<\lambda} f_\xi^{-1}(X_\xi)$ is nonempty. By $\Diamond_\kappa(\mathrm{Cof}_\kappa^\lambda)$, $S = \{\alpha<\kappa \mid cf(\alpha)=\lambda$ and $X_\xi\cap\alpha = A_\alpha\cap(\{\xi\}\times\alpha)$ for every $\xi<\lambda\}$ is stationary. Also, $C = \{\alpha<\kappa \mid cf(\alpha)=\lambda$ and there is a sequence $\langle s_\alpha(\xi) \mid \xi<\lambda\rangle$ cofinal in α with $s_\alpha(\xi)\in X_\xi\}$ is unbounded and closed under the taking of lenght λ sequences. Thus, it is not hard to see that $S\cap C\neq\phi$, and if α is in this set, then $f_\xi(\alpha)\in X_\xi$ for every $\xi<\lambda$. □

Although we have elaborated the proof, (i) $\rightarrow$ (iii) should be accredited to Kunen, and (iii) $\rightarrow$ (iv) to Solovay. Since (iv) $\rightarrow$ (i) when κ is measurable by taking the F_ξ's to be a fixed normal ultrafilter, we have:

COROLLARY 6: *If κ is measurable, then* (i) – (iv) *are equivalent.*

It is certainly consistent with the measurability of κ that $\Diamond_\kappa(E)$ holds for every stationary $E\subseteq\kappa$, since this is so in $L[U]$, where U is any κ-ultrafilter, by the usual constructibility argument. Also, it is consistent with the supercompactness of that $\Diamond_\kappa(E)$ holds for every stationary $E\subseteq\kappa$ by standard arguments involving adding a Cohen subset of κ through upward Easton forcing.

In the forcing construction of Gitik [G] starting from a measurable cardinal κ such that $o(\kappa) = \lambda^+$, it is true that whenever $\{U_\xi \mid \xi < \lambda\}$ are κ-ultrafilters, there is a κ-ultrafilter V such that for every $\xi < \lambda$, $f_{\xi *}(V) = U_\xi$ for some $f_\xi : \kappa \longrightarrow \kappa$ and moreover, $\langle [f_\xi]_V \mid \xi < \lambda \rangle$ is cofinal in $[id]_V$. Hence, 5(iv) holds in a strong sense.

Related to Question 3 is:

QUESTION 4: *Is it consistent to have a measurable cardinal* κ *such that* $\Diamond_\kappa(\mathrm{Cof}^\lambda_\kappa)$ *fails for some* $\lambda < \kappa$?

It is well-known that $\Diamond_\kappa(\mathrm{Cof}^\lambda_\kappa)$ implies that the filter generated by $C_\kappa \cup \{\mathrm{Cof}^\lambda_\kappa\}$ is not 2^κ-saturated. Gitik has established that it is consistent to have a measurable cardinal κ at which the filter generated by $C_\kappa \cup \{S\}$ is κ^+-saturated, where $S \subseteq \mathrm{Cof}^\lambda_\kappa$ for some $\lambda < \kappa$, but he could not render $S = \mathrm{Cof}^\lambda_\kappa$.

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