Kunen the Expositor

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Kenneth Kunen (1943-2020) was a remarkable and influential mathematician whose work in set theory and set-theoretic topology was of crucial significance in advancing these fields to their modern sophistication. In *Kunen and set theory* [5], I chronicled his singular research progress over a broad swath of set theory, fully airing his penetrating work of dramatic impact in the formative years 1968-1972. In succeeding years, while he broadened his research activity into set-theoretic topology, he provided incisive expository texts in set theory. On that, it is exceptional for a research mathematician of such high caliber to come out with a graduate-level text like his 1980 *Set Theory* [7], which had a significant impact on the field. Here, in commemoration and to complement my previous account of his research, I describe Kunen's expository work, bringing out both his way of assimilating and thinking about set theory and how it had a meaningful hand in its promulgation into the next generations.

With the dexterity and synthetic sense that had brought him to the frontiers of set-theoretic research, Kunen was also strongly disposed to present assimilated mathematics in a coherent, logical way, and this he first did in contributions to handbooks, as described in Section 1. That initiative led to the full-blown 1980 graduate text *Set Theory* as described in Section 2, in which he brought together ways of proceeding through recent results and advances in a dynamic way. Section 3 describes how, much later, Kunen brought out two texts, the 2009 *The Foundations of Mathematics* and the 2011 *Set Theory*, these reflecting through details, shades and aspects his seasoned engagement.

1 Handbook Chapters

With his prodigious work in set theory in place by the early 1970s, Kunen in the next decade provided handbook chapters, one broadly synthetic and the other penetratingly sophisticated, that boosted research in the expanding field. For the 1977 *Handbook of Mathematical Logic* [1], edited by his colleague Jon Barwise and the mother of all handbooks in logic, Kunen coordinated the contributed chapters on set theory and himself contributed the chapter *Combinatorics* [6]. Going through the stuff of stationary sets, enumeration principles, trees, almost disjoint sets, partition calculus, and large cardinals, the chapter presented a remarkably integrative view of classical initiatives and modern developments. For example, discussing the Continuum Hypothesis as an enumeration principle, the 1914 Luzin set is pivotally brought in; its intrinsic properties are cast as on a linear ordering and so related to the classical Suslin's Hypothesis; and then Jensen's $\diamondsuit$ is smoothly incorporated to sharpen the context. From today's vantage point, trees, partition calculus, and large cardinals cohere together and are the bulwark of the broad field of combinatorial set theory, and one has to harken back to that earlier time of Kunen's chapter to appreciate how spurring it would have been.

Kunen edited, together with Jerry Vaughan, the gargantuan 1984 *Handbook of Set-Theoretic Topology* [11], and he himself contributed the incisive chapter *Random and Cohen Reals* [8]. Measure and category are classical notions for reckoning sets of reals, and the advent of forcing brought into prominence the meagre ideal, in connection with Cohen reals, and the null ideal, in connection with Solovay's random reals. Lucidly setting out Solovay's understandings, Kunen in his chapter exhibited the structural similarities as well as the differences in measure and category of the corresponding generic extensions. For example, he (p.905) pointed to a subtle conflation, one which can be seen to turn on the difference between the measure-theoretic product of two measure algebras and the Boolean algebraic completion of the product order, which is not a measure algebra.

2 Set Theory (1980)

Of his expository work, Kunen's 1980 *Set Theory* [7] was evidently the most pivotal, being a timely graduate-level text that spurred interest and research in the subject. Baumgartner in an extended review [2] of the book provided a dynamic account of its debut and impact, and rhetorically we dovetail in and out of it in framing a telling narrative.

Baumgartner [2] began, somewhat romantically:

> Once upon a time, not so very long ago, logicians hardly ever wrote anything down. Wonderful results were being obtained almost weekly, and no one wanted to miss out on the next theorem by spending the time to write up the last one. Fortunately there was a Center where these results were collected and organized, but even for the graduate students at the Center life was hard. They had no textbooks for elementary courses, and for advanced courses they were forced to rely on handwritten proof outlines, which were usually illegible and incomplete; handwritten seminar notes which were usually wrong; and Ph.D. dissertations, which were usually out of date. Nevertheless they prospered.
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> Now the Center I have in mind was Berkeley and the time was the early and middle 1960's, but similar situations have surely occurred many times before. In this case, to the good fortune both of the graduate students and of the logicians not lucky enough to be in California, all the wonderful results were eventually written down. But it took a surprisingly long time.

Kunen was one of those "lucky enough" to be in California, at Stanford, and his 1968 Ph.D. dissertation was a notable landmark that had "wonderful results" in set theory.

Baumgartner proceeded to mention a couple of texts emerging in the early 1970's that were formative. He then wrote:

> In 1978 the clouds parted as Jech produced his encyclopedic text *Set theory* [4] and immediately supplanted all the earlier competitors. Graduate students went without food in order to put a copy of this fat green book on their shelves. But even though it was indispensable as a reference, Jech's book still left a gap at the first-year graduate level. In order to pack into it as much material as possible, the author was forced to shorten many proofs, sometimes to the point of only supplying hints. While this presents no difficulty to the reader already sophisticated in set theory it makes the book rather less accessible to the novice.
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> Nowadays, on those graduate student bookshelves one finds not only a fat green book but also a slender yellow one, Kunen's contribution to the education of young set theorists and the book here under review.

While Jech's 621-page book was capacious, wide-ranging, and developed a broad structural framework, Kunen's 313-page book was directed, combinatorial, and focused on procedures and proofs, particularly for establishing relative consistency.

Affirming in the preface that his book is indeed "intended to be used as a text in beginning graduate courses", Kunen began the introduction with a remarkable pronouncement:

> Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set-theoretic principles. From such axioms, all known mathematics may be derived. However, there are some questions which the axioms fail to settle, and that failure is the subject of this book.

With this clarion call, Kunen proceeded forthwith to set out the ZF axioms, and then, in 46 pages, quite informally established "the foundations of set theory". Summarily in the style of a review, he set out the basic constructs, and ordinals, cardinals, and transfinite recursion. Setting up a regime of ending chapters with challenging exercises, he left to exercises with hints proofs of the Schroder-Bernstein Theorem, equivalents of the Axiom of Choice, and the Milner-Rado Paradox.

Kunen then discussed "infinitary combinatorics", bringing in topics broached in his *Combinatorics* chapter [6], but now with a pervading treatment and integration of Martin's Axiom, already discussed in his 1968 thesis, and at the end an account of $\diamondsuit^+$, which he and Jensen had worked out in 1969. This early inclusion of the forcing-inspired Martin's Axiom, which seemed novel at the time, served Kunen's scheme to set the stage with consequential combinatorics that can be accessed before forcing is presented, and moreover, exemplified his emerging way of rounding out and making connections across topics. The chapter exercises here have been stimulating and inspiring for generations of graduate students, and this has been an significant factor in the text's pedagogical success.

Proceeding, Kunen discussed "easy consistency proofs" based on relativization and absoluteness, attending to the class WF of well-founded sets, these ramified into ranks $R(\alpha)$, and the $H(\kappa)$ for infinite cardinals κ. Setting out "defining definability" combinatorially in terms of set operations, he then established the Gödel relative consistency results with L of AC and GCH, and moreover, of $\diamondsuit^+$. Kunen was in good company with his combinatorial approach to definability, for Gödel himself took this approach in his monograph [3]; however, as the obscuring proof for $\diamondsuit^+$ brings out, a full syntactic embrace of definability would have served better, admittedly, for conceptualizing relative consistencies via L.

Past all this as prologue, Kunen presented the central technique for establishing relative consistency, Cohen's forcing method. For this he decisively took the approach of Shoenfield [12], another fine expositor, and this choice is worth some discussion. In the 1960's the development of the theory of forcing had proceeded through Solovay's general articulation in terms of partial orders and Levy's concept of generic filter, and was thence elaborated by Scott and Solovay in the sophisticated algebraic terms of Boolean-valued models. Kunen himself in his 1968 thesis had proceeded in terms of Boolean-valued models for establishing the forcing results, and Jech's 1978 book [4] incorporated Boolean-valued models early on in the treatment of forcing. But already in his 1967 account [12] Shoenfield had shown that the gist of the Boolean-valued approach can be captured by working with partial orders and generic filters through countable transitive models. It can be fairly said that by adopting the Shoenfield entré, Kunen had a significant hand in making forcing accessible through guiding heuristics about conditions and truth.

Squarely into the game, Kunen worked through the formalism and metamathematics of forcing, sketching with a sure hand what is at the heart, the recursive definition of the forcing relation for the atomic formulas. Afterwards, he did incorporate Boolean-valued models and make careful historical remarks about the development of forcing. He then provided a rich range of exercises that further rounded out the discussion.

In the final chapter, on iterated forcing, Kunen realized explanatory possibilities of his approach by setting out a range of modes of iteration and establishing relative consistency results. In particular, he gave the consistency proof for Martin's Axiom with finite-support iteration, thematically completing his early combinatorial introduction of the principle. Furthermore, he speculated on possible generalizations of the principle to be secured with countable-support iteration in a very general setting for iteration. At the end, he provided a remarkable range of sophisticated exercises both to establish further relative consistencies as well as to articulate his general setting. While many of these seemed to be beyond the reach of the typical graduate student at the time, they at least indicated the potency of forcing and its richness of possibility.

As was brought out above, Kunen made several deliberate choices as to the expository path to take, choices which could be seen to reflect his thinking as to how to achieve a first assimilation to be rounded out by discussion. With its dynamic and stimulating way of presenting combinatorics and forcing, Kunen's text had a significant impact, and Baumgartner (p.463) alluded to this anecdotally:

> ... Kunen's success has been truly world-wide. I had some concrete experience with that fact recently when I visited Shanghai to deliver two weeks of lectures on combinatorial set theory. I arrived full of missionary zeal, and with a briefcase full of notes which, while not exactly lifted from Kunen, nonetheless had a certain affinity with his approach. Imagine my shock when I discovered not only that all students had copies of Kunen but also that they were very impatient with arguments that did not measure up to the Kunen standard. I spent the whole two weeks tiptoeing around Kunen, and I am sure the same experience awaits others who may be touring parts of the world previously thought to be outside the orbit of set theory.

3 Late Texts

In the fullness of time, Kunen, after retiring from his university in 2008, published in 2009 *The Foundations of Mathematics* [9], a text on mathematical logic mostly written by 2005, and in 2011 *Set Theory* [10], a "total rewrite", in his words, of his 1980 text. The interest here is now on how Kunen in his maturity reflectively approached the subjects, still with incentives at play to promote them in the next generations.

The 252-page *The Foundations of Mathematics* [9] is a text that (p.3) "describes the basics of set theory, model theory, proof theory, and recursion theory". There is a range of mathematical logic texts that cover model theory and recursion theory, and a few that cover set theory as well, and Kunen's is the only one starting with set theory. True to his pronouncement in the introduction to his 1980 *Set Theory*, he took set theory to be the foundation of mathematics, and proceeded to develop in about eighty pages the basic theory, this time with many examples of mathematical objects and structures subsumed as sets. With many exercises interwoven and proofs included, ordinal and cardinal arithmetic and the cumulative hierarchy get full treatment. Significantly, the set HF of hereditarily finite sets is emphasized as the domain for finite mathematics, and Kunen brought in computer programming expressions, e.g. defining the rational numbers as the sets of form $\langle i, \langle m, n\rangle\rangle$ with $i = 0$ or 1 and m and n natural numbers satisfying $\gcd(m, n) = 1$.

Kunen then in the next hundred pages covered elementary model theory together with the attendant formal proof theory. With set theory in place in foundational aspect, many motivating examples of mathematical structures are presented; set-theoretic constructs and arguments are deployed for formalization; and transfinite cardinals are freely deployed as parameters. Kunen meticulously formalized the syntax of logic in Polish notation; defined structures and satisfaction; and formulated deducibility according to a Hilbert-style system. He then established the Completeness Theorem in the usual Henkin way, drawing the Compactness and Löwenheim-Skolem conclusions. Rather unexpectedly, there is throughout a remarkable attention to detail in the formalization and proofs. After discussing equational and Horn theories and elementary submodels, Kunen concluded with a rigorous account of models of various set theories, providing the details of the bi-interpretability of Peano Arithmetic with $\langle \mathrm{HF}, \in\rangle$.

The last chapter, although entitled "Recursion Theory", is a brief fifty pages really on undecidability in logic. Here, Kunen interestingly took up HF, "the domain for finite mathematics", as the universe for abstract computation, and Δ_1 definability thereon as the criterion for decidability. The traditional Turing machines were thus bypassed, as in any case he was not going to be delving into Turing degrees and reducibility. Working on HF, rather than the set of natural numbers, Kunen carried forth with the easier coding of formulas and their semantics in closer analogy with compiling with computers. Then, again interestingly, Kunen first established the 1936 Church Undecidability and then incorporated the 1931 Gödel First Incompleteness, and afterwards, with the leverage of HF, established Tarski's Undefinability of Truth and Gödel's Second Incompleteness.

Yes, *The Foundations of Mathematics* is a text on mathematical logic, presumably honed over the years in beginning graduate courses. On the other hand, as brought out above, it is quite distinctive in that it pursues an original line of logical development, this reminiscent of the 1980 *Set Theory*, and in that it exudes the author's conviction that set theory is the foundation of mathematics, this evident in the deployment of set-theoretic constructs in the rigorization of model theory and a telling presentation of undecidability.

The 402-page, 2011 *Set Theory* [10], is Kunen's mature presentation of the subject. A third again as long as his 1980 text, it may best be regarded as a rich source for an advanced graduate course. Generally speaking, there is considerable discursive rounding out of topics and updating with respect to recent developments. More specifically, there is a new attention to the syntax and semantics of model theory and its detailed presentation. At the interstices, the exercises are now integrated into the text according to topic, instead of being a formidable challenge grouped together at the end of chapters. In what follows, we provide a description of the new text, in comparative relation to the old.

Kunen does not begin this time with any pronouncement, but does declare (p.2) "by now it is clear that modern mathematics is the study of *models* of ZFC, which is the subject of this book". While he had stated in his 1980 text that (p.7) "Pedagogically, it is much easier to develop ZFC from a platonistic point of view, and we shall do so throughout this book", he now states (p.6) "In this book, our *official* philosophy is *Formalism* — mainly because it helps to clarify the logical structure of independence proofs." With these modulating adaptations, Kunen proceeded to set out axiomatic set theory more systematically, in the style of his *The Foundation of Mathematics*, to which he refers. Toward his study of models of set theory, he carefully specified how the existence of set-theoretic constructs follow from which axioms, e.g. giving different proofs for the Cartesian product, one from Replacement and the other from Power Set. Then over fifty pages Kunen provided a rigorous, rounded account of the ordinals, transfinite recursion on well-founded relations, rank for the well-founded sets, cardinals and their arithmetic, and the Axiom of Choice.[1] Then in what actually would be the most consequential move for his new text, Kunen worked in a substantial part of the syntax and semantics of model theory from his *Foundations of Mathematics* and focused it on an account of models of set theory, absoluteness, and relative interpretability.

With this preparation, in "Easy Consistency Proofs", now one extended chapter, Kunen with remarkable rigor displayed his heralded "formalistic" approach to models of set theory. Not only did he attend to WF, $R(\alpha)$ and $H(\kappa)$ as before, but he smoothly worked in L with full syntactic embrace of definability, as well as the class of relatively constructible sets $L[A]$.

In "Infinitary Combinatorics", now an extended chapter of nearly one hundred pages, Kunen adroitly wove in the recent structural elaborations and results of combinatorial set theory. This he did in substantial part to establish fertile ground for the exercise of forcing but also simply to expositorily enrich a subject which he had consistently fostered. He began by motivating and working out relations among the prominent cardinal characteristics (invariants) of the continuum. He then brought in Martin's Axiom as before as a combinatorial principle, but this time considerably aired it with various posets, like the amoeba, and various poset properties, establishing in context Bell's Theorem.[2] He continued at length on trees in a general setting, working out Suslin line and other topological interactions. Finally, he worked the proof of $\diamondsuit^+$ in L through its full definability presentation, and pursuing elementary substructures, provided a motivated proof of Arhangelskii's Theorem.[3]

In the central chapter on forcing, Kunen covered much the same ground as before, but this time, with his earlier elaboration of model theory, he in

[1] A telling point is that Kunen this time carefully described how the proof of transfinite recursion requires Replacement. In his 1980 text, he (pp.26f) misleadingly mentioned the need for the axiom not in the proof, but later in an application.

[2] The least cardinal for which Martin's Axiom fails for some σ-centered poset equals the pseudo-intersection number.

[3] Every first-countable Hausdorff space has cardinality at most that of the continuum.

his "formalist" spirit was more rigorous and explicit, e.g. about the recursive definition of the forcing relation for the atomic formulas. Attuned to current practice, he also rigorously recast the countable transitive model approach to the preferred $V[G]$ approach for the presentation of forcing results. The many challenging exercises at the end were cast as topics, some given full proofs.

The final chapter, on iterated forcing, Kunen enriched and expanded over the 1980 chapter in two directions. First, he throughout brought in, as planned, combinatorics of the cardinal characteristics and so forth in the study of the forcing models. Second, where he had formerly ended with speculations about possible generalizations of Martin's Axiom to be secured with countable-support iterations, he now sketched a consistency proof for one such generalization, Baumgartner's Axiom.

The final pages of the chapter set it autonomously at a new, higher plane. Kunen first discussed in detail the independence from Martin's Axiom afforded by the Semi-Open Coloring Axiom, bringing in, to good effect, his own concept of weakly Luzin set. He then breezily discussed the Proper Forcing Axiom, reaching up to the current heights of forcing axioms and consequences of large cardinal hypotheses.

All in all, Kunen's two late texts exude considerable expertise and care in formal presentation. But beyond that, unlike most texts in mathematical logic and set theory, they exhibit through discursive rounding out, cross-referencing, and range of exercises a dynamic, progressive approach to the topics as regulated by proofs and procedures. Kunen's texts all feature his following a distinctive expository path, one that individuates the presentation to the author. This individuation is moreover accentuated by his conviction that set theory serves as the foundation of mathematics and his sometime incorporation of his own researched connections. The result is a highly successful conveyance of mathematical spirit and practice, and as such Kunen's expository work is a compelling and influential part of his mathematical achievement.

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