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REGRESSIVE PARTITIONS AND BOREL DIAGONALIZATION

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Several rather concrete propositions about Borel measurable functions of several variables on the Hilbert cube (countable sequences of reals in the unit interval) were formulated by Harvey Friedman [F1] and correlated with strong set-theoretic hypotheses. Most notably, he established that a "Borel diagonalization" proposition P is equivalent to: for any $a \subseteq \omega$ and $n \in \omega$ there is an ω-model of ZFC + $\exists\kappa(\kappa$ is n-Mahlo) containing a. In later work (see the expository Stanley [St] and Friedman [F2]), Friedman was to carry his investigations further into propositions about spaces of groups and the like, and finite propositions. He discovered and analyzed mathematical propositions which turned out to have remarkably strong consistency strength in terms of large cardinal hypotheses in set theory.

In this paper, we refine and extend Friedman's work on the Borel diagonalization proposition P. First, we provide more combinatorics about regressive partitions and n-Mahlo cardinals and extend the approach to the context of the Erdös cardinals $\kappa \to (\gamma)_2^{<\omega}$. In passing, a combinatorial proof of a well-known result of Silver about these cardinals is given. Incorporating this work and sharpening Friedman's proof, we then show that there is a level-by-level analysis of P which provides for each $n \in \omega$ a proposition almost equivalent to: for any $a \subseteq \omega$ there is an ω-model of ZFC + $\exists\kappa(\kappa$ is n-Mahlo) containing a. Finally, we use the combinatorics to bracket a natural generalization S^ω of P between two large cardinal hypotheses.

To recapitulate some notation and concepts, let I be the unit interval of reals and $Q = {}^\omega I$ (the Hilbert cube) the set of countable sequences drawn from I. If $n \in \omega$ and $y, z \in {}^nQ$, say that $y \sim z$ iff there is a permutation ρ of ω, which is the identity except at finitely many arguments, such that $y(i) \circ \rho = z(i)$ for each $i < n$. Let us say that a function F with domain ${}^nQ$ is *totally invariant* if whenever $y, z \in {}^nQ$ and $y \sim z$, then $F(y) = F(z)$. A function G with domain $Q \times {}^nQ$ is *right-invariant* if whenever $x \in Q$, $y, z \in {}^nQ$, and $y \sim z$, then $G(x, y) = G(x, z)$. Next, recall that a cardinal κ is 0-*Mahlo* iff κ is (strongly) inaccessible, and inductively for $n \in \omega$, κ is $(n + 1)$-*Mahlo* iff κ is regular and every closed unbounded subset of κ contains an n-Mahlo cardinal. κ is ω-Mahlo iff it is n-Mahlo for every $n \in \omega$.

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Friedman's proposition P is $\forall n \in \omega P_n$, where

(P_n) Suppose $F: Q \times {}^nQ \to I$ is Borel and right-invariant. Then for any $m \in \omega$ there is a sequence $\langle x_k | k < m \rangle$ of distinct elements of Q such that: whenever $s < t_1 < \cdots < t_n < m$, $F(x_s, \langle x_{t_1}, \ldots, x_{t_n} \rangle)$ is the first coordinate of x_{s+1}.

As Friedman emphasized, "Borel" can be replaced by "finitely Borel," i.e. of a finite rank in the Baire hierarchy of functions, without affecting the strength of P. We observe below that this can be further reduced to rank <3.

Friedman's arguments actually show the following in ZFC − Power Set for any $n \in \omega$: (a) If for any $a \subseteq \omega$ there is an ω-model of ZFC + $\exists\kappa$ (κ is $(n + 4)$-Mahlo) containing a, then P_{n+4}. (b) If P_{n+4}, then for any $a \subseteq \omega$ there is an ω-model of ZFC + $\exists\kappa$ (κ is n-Mahlo) containing a. He also observed that P_4 implies that there is an ω-model of ZFC, and asked whether P_3 is independent of ZFC. We are initially motivated by this question and by the slack in the overall proof, but our ramification of P does not correlate directly with the P_n's, and so in particular the independence of P_3 remains unresolved.

The following formulation was motivated by the fact that in some of Friedman's arguments with indiscernibles the top few were fixed.

($\bar{P}_n$) If $F_1: Q \times {}^{n+4}Q \to I$ and $F_2: Q \times Q \to I$ are Borel and right-invariant and $F_3: {}^3Q \to I$ is Borel and totally invariant, then for any $m \in \omega$ there is a sequence $\langle x_i | i \le m + 6 \rangle$ of distinct elements of Q such that:
(a) whenever $s < t_1 < \cdots < t_n < m$, $F_1(x_s, \langle x_{t_1}, \ldots, x_{t_n}, x_m, x_{m+2}, x_{m+4}, x_{m+6} \rangle)$ only depends on x_s;
(b) whenever $s < t \le m + 6$, $F_2(x_s, x_t)$ is the first coordinate of x_{s+1}; and
(c) $F_3(x_m, x_{m+4}, x_{m+6}) = F_3(x_{m+2}, x_{m+4}, x_{m+6})$.

Thus with (a) $\bar{P}_n$ is like P_n with side conditions about some further elements $x_m, \ldots, x_{m+6}$. These are somewhat involved, primarily because we have tried to isolate in Borel fashion the minimum augmentation of P_n necessary for our proofs. From the esthetic point of view, eliminating the minor annoyance of F_3 and (c) is desirable, and may be possible with a more subtle analysis. In any case, certainly $\bar{P}_n$ follows from P_{n+4} by a Borel fusion of the three functions involved. We shall establish:

THEOREM A. *If for any $a \subseteq \omega$ there is an ω-model containing a of ZFC + $\exists\kappa\exists\delta > \kappa$ (κ is n-Mahlo and $L_\kappa[a] \prec L_\delta[a]$), then $\bar{P}_{n+2}$ holds.*

THEOREM B. *If $\bar{P}_{n+2}$ holds (even just for Borel functions of rank <3), then for any $a \subseteq \omega$ there is an ω-model of ZFC + $\exists\kappa$ (κ is n-Mahlo) containing a.*

These results bracket the strength of $\bar{P}_{n+2}$ reasonably closely. The existence of an n-Mahlo cardinal κ and a $\delta > \kappa$ such that $L_\kappa[a] \prec L_\delta[a]$ follows easily from the existence of an $(n + 1)$-Mahlo cardinal, but implies on the other hand the existence in $L[a]$ of many n-Mahlo cardinals below κ by elementarity.

Consider next the following natural generalization of P, where γ is any ordinal and

${}^{<\omega}Q = \bigcup_n {}^nQ$ the set of all finite sequences drawn from Q:

Suppose $F: Q \times {}^{<\omega}Q \to I$ is Borel and right-invariant. Then there is a sequence $\langle x_\xi \mid \xi < \gamma \rangle$ of distinct elements of Q such that:

(S^γ) (i) whenever $n \in \omega$ and $s < t_1 < \cdots < t_n < \gamma$, $F(x_s, \langle x_{t_1}, \ldots, x_{t_n} \rangle)$ only depends on x_s; and

(ii) whenever $s < t < \gamma$, $F(x_s, x_t) \in \text{Range}(t)$.

As n varies in (i), we cannot expect that $F(x_s, \langle x_{t_1}, \ldots, x_{t_n} \rangle)$ is always the first coordinate of x_{s+1}, but (ii) is a remnant of that condition. For concreteness, we shall establish:

THEOREM C. *If for any $a \subseteq \omega$ there is an ω-model of $ZFC + \exists\kappa(\kappa \to (\omega + \omega)_2^{<\omega})$ containing a, then $S^{\omega+\omega}$ holds.*

THEOREM D. *If $S^{\omega+5}$ holds (even just for Borel functions of rank < 3), then for any $a \subseteq \omega$ there is an ω-model of $ZFC + \exists\kappa(\kappa$ is ω-Mahlo) containing a.*

There is considerable slack here, and we shall discuss refinements at the end of the paper.

§1 reviews regressive partitions, provides the necessary results about them for the n-Mahlo cardinals and the Erdös cardinals $\kappa \to (\gamma)_2^{<\omega}$, and is of independent interest. The Borel diagonalization results are established by following the main line of argument of Friedman [F1], and we cite its lemmas and mainly detail the modifications necessary. §2 is devoted to a proof of Theorem A, §3 to a proof of Theorem B, and §4 to proofs of Theorems C and D.

§1. Regressive partitions. Friedman relied on characterizations in Schmerl [Sc] of the n-Mahlo cardinals via certain partition properties. In this section we review and further develop a systematic approach which clarifies the connections.

Let X be a set of ordinals and n a natural number. If f is a function with domain $[X]^n$, we write $f(\alpha_0, \ldots, \alpha_{n-1})$ for $f(\{\alpha_0, \ldots, \alpha_{n-1}\})$, with the understanding that $\alpha_0 < \cdots < \alpha_{n-1}$. Such a function is called *regressive* iff $f(\alpha_0, \ldots, \alpha_{n-1}) < \alpha_0$ whenever $\alpha_0 < \cdots < \alpha_{n-1}$ are all from X and $\alpha_0 > 0$. There is a natural motion of homogeneity for such a function f: $Y \subseteq X$ is min-*homogeneous for* f iff whenever $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$ are all from Y, $\alpha_0 = \beta_0$ implies $f(\alpha_0, \ldots, \alpha_{n-1}) = f(\beta_0, \ldots, \beta_{n-1})$. In other words, f on an n-tuple from Y depends on the first element. We write $X \to (\gamma)^n_{\text{reg}}$ iff whenever f on $[X]^n$ is regressive, there is an $Y \in [X]^\gamma$ min-homogeneous for f.

In Kanamori and McAloon [KM] the proposition

(*) for any $n, k \in \omega$ there is an $m \in \omega$ such that $m \to (k)^n_{\text{reg}}$

is shown to be equivalent to the Paris-Harrington proposition and hence unprovable in Peano arithmetic. In fact, it is shown that (*) for fixed n is equivalent to Paris-Harrington for fixed n and hence unprovable in $I\Sigma_{n-1}$, induction restricted to Σ_{n-1} formulas.

Turning to the infinite case, the following characterization was established by Hajnal, Kanamori and Shelah [HKS]:

THEOREM 1.1. *The following are equivalent for $\kappa > \omega$ and $0 < n < \omega$:*

(a) κ *is n-Mahlo.*

(b) *For any $\gamma < \kappa$ and unbounded $X \subseteq \kappa$, $X \to (\gamma)^{n+2}_{\text{reg}}$.*

(c) *For any closed unbounded $C \subseteq \kappa$, $C \to (\omega)^{n+2}_{\text{reg}}$.*

(c) here complements a previous characterization of Schmerl [Sc], stated in the present terminology as

THEOREM 1.2. *The following are equivalent for* $\kappa > \omega$ *and* $n \in \omega$:

(a) κ *is n-Mahlo.*

(b) *For any* $m \in \omega$ *and unbounded* $X \subseteq \kappa$, $X \to (m)^{n+3}_{\text{reg}}$.

(c) *For any unbounded* $X \subseteq \kappa$, $X \to (n+5)^{n+3}_{\text{reg}}$.

What will be relevant for Theorem B is 1.2 together with a variant of it, established using the following previously known lemmata. The first is a careful generalization of [HKS, 2.7],

LEMMA 1.3. *Suppose that* $n \geq 3$ *and for some limit ordinal* η, C *and* X *are subsets of* $\eta - \omega$ *with* C *closed unbounded and* $\min(C) \leq \min(X)$. *If* $C \not\to (\gamma)^n_{\text{reg}}$ *and* $X \cap \xi \not\to (\gamma)^n_{\text{reg}}$ *for every* $\xi < \eta$, *then* $X \not\to (\gamma)^n_{\text{reg}}$.

PROOF. For each $\alpha \in X$, set $\psi(\alpha) = \sup(C \cap (\alpha + 1))$, an element of C since C is closed unbounded and $\min(C) \leq \min(X)$. We first define the *type* of a member of $[X]^n$ according to C as follows: If $\alpha_0 < \cdots < \alpha_{n-1}$ are all in X, let $\{\xi_0, \ldots, \xi_k\}$ enumerate the set $\{\psi(\alpha_i) \mid i < n\}$ in increasing order, and set $r_j = |\{i \mid \psi(\alpha_i) = \xi_j\}|$ for $j \leq k$. Then the *type* of $\{\alpha_0, \ldots, \alpha_{n-1}\}$ is $\langle r_0, \ldots, r_k\rangle$, which we can assume through sequence coding is one natural number.

Next let g attest to $C \not\to (\gamma)^n_{\text{reg}}$ and g_ξ attest to $X \cap \xi \not\to (\gamma)^n_{\text{reg}}$ for $\xi < \eta$. Since C, $X \subseteq \eta - \omega$, we can assume by renumbering that the ranges of g and the g_ξ's do not contain any number coding a type. Now define G on $[X]^n$ as follows:

$$G(\alpha_0, \ldots, \alpha_{n-1}) = \begin{cases} g(\psi(\alpha_0), \ldots, \psi(\alpha_{n-1})) & \text{if } \psi(\alpha_0) < \cdots < \psi(\alpha_{n-1}), \\ g_\xi(\alpha_0, \ldots, \alpha_{n-1}) & \text{if } \psi(\alpha_1) = \cdots = \psi(\alpha_{n-1}), \\ & \text{where } \xi \text{ is the next element of } C \\ & \text{after } \psi(\alpha_1), \\ \text{type of } \{\alpha_0, \ldots, \alpha_{n-1}\}, & \text{otherwise.} \end{cases}$$

(In the second clause, that we start with $\psi(\alpha_1)$ is not a misprint; that $n \geq 3$ is called upon here.) G is regressive, so suppose that $Y \subseteq X$ is min-homogeneous for G. We can assume that Y has at least $n + 1$ elements, and let $\beta_0 < \beta_1$ be its least two elements.

Assume first that $\psi(\beta_0) = \psi(\beta_1)$. If there were a further $\beta \in Y$ such that $\psi(\beta_1) < \psi(\beta)$, then there would be two sequences of length n from Y, both starting with β_0 and with different types—one with β_1 and one without. This is contradictory, so ψ must be constant on Y. Thus, by the second clause of G, Y cannot have ordertype γ.

Assume next that $\psi(\beta_0) < \psi(\beta_1)$. Suppose first that there were a further $\beta \in Y$ such that $\psi(\beta_1) < \psi(\beta)$. Then if ψ were not one-to-one on Y, one can again generate two appropriate sequences of length n from Y, both starting with β_0 and with different types, to derive a contradiction. Thus, ψ must be one-to-one on Y, and by the first clause of G, Y cannot have ordertype γ.

In the remaining case of $\psi(\beta_0) < \psi(\beta_1)$ with $\psi(\beta) = \psi(\beta_1)$ for every further $\beta \in Y$, we can invoke the second clause of G to again show that Y cannot have ordertype γ. This completes the proof. ■

The following is part of 2.2 of [Sc]:

LEMMA 1.4. *If* $\eta > \omega$ *and* $\eta \sim \gamma \to (4)^3_{\text{reg}}$ *for every* $\gamma < \eta$, *then* η *is a strong limit cardinal.*

PROOF. If to the contrary there were a cardinal λ such that $\lambda < \eta \leq 2^\lambda$, then we can use the Sierpiński partition: Let $\{s_\alpha \mid \alpha < \eta\}$ be distinct members of ${}^\lambda 2$, and define f on $[\eta \sim \lambda]^3$ by: $f(\alpha_0, \alpha_1, \alpha_2) =$ least ξ such that $s_{\alpha_1}(\xi) \neq s_{\alpha_2}(\xi)$. There cannot be a four-element min-homogeneous set for this partition. ■

Finally, the following is a translation of Schmerl's property $S_1(n-1, n+2, n+5)$ from [Sc]:

LEMMA 1.5. *If $0 < n < \omega$, η is inaccessible, and $C \subseteq \eta$ is a closed unbounded set consisting of strong limit cardinals which are not $(n-1)$-Mahlo, then $C \not\rightarrow (n+5)^{n+3}_{\mathrm{reg}}$.*

These preliminaries lead to our desired result:

THEOREM 1.6. *If $X \cap \omega = \varnothing$, then $X \rightarrow (n+5)^{n+3}_{\mathrm{reg}}$ iff $X \cap \kappa$ is unbounded in κ for some n-Mahlo cardinal κ.*

PROOF. 1.2 confirms one direction. For the converse, let η be least such that $X \cap \eta \rightarrow (n+5)^{n+3}_{\mathrm{reg}}$. We shall show that η is n-Mahlo:

First of all, a simple argument shows that η must be a limit ordinal. It follows from 1.3 that:

(*) For any closed unbounded $C \subseteq \eta$ with $C \cap \omega = \varnothing$ and $\min(C) \leq \min(X)$,
$$C \rightarrow (n+5)^{n+3}_{\mathrm{reg}}.$$

Considering the closed unbounded sets $\{\min(X)\} \cup (\eta \sim \gamma)$ for $\gamma < \eta$, we can then conclude from 1.4 that η must be a strong limit cardinal. If η were singular, then for any closed unbounded $D \subseteq \eta$ of ordertype $\mathrm{cf}(\eta)$ with $\min(D) \geq \mathrm{cf}(\eta)$, we can define a one-to-one regressive function on $[D]^{n+3}$ with range $\subseteq \mathrm{cf}(\eta)$. But then (*) would be contradicted with $C = \{\min(X)\} \cup D$. Hence, η is inaccessible. Finally, if η were not n-Mahlo, then $n > 0$ and 1.5 contradicts (*). ■

Let us next consider the natural generalization of our partition relation to all finite sequences: If X is a set of ordinals, write $X \rightarrow (\gamma)^{<\omega}_{\mathrm{reg}}$ iff whenever f on $[X]^{<\omega}$ is regressive, there is a $Y \in [X]^\gamma$ min-homogeneous for $f \mid [X]^n$ for every n. In this context, we shall say that Y is simply min-*homogeneous for f*. We first observe that the weakest possibility here provides another characterization. Let $X \rightarrow (<\omega)^{<\omega}_{\mathrm{reg}}$ mean that $X \rightarrow (k)^{<\omega}_{\mathrm{reg}}$ for every $k \in \omega$.

THEOREM 1.7. *If $X \cap \omega = \varnothing$, then $X \rightarrow (<\omega)^{<\omega}_{\mathrm{reg}}$ iff $X \cap \kappa$ is unbounded in κ for some ω-Mahlo cardinal κ.*

PROOF. Suppose first that $X \cap \kappa$ is unbounded in κ for an ω-Mahlo cardinal κ. If f on $[X]^{<\omega}$ is regressive and $k \in \omega$, define g on $[X]^k$ by

$$g(\xi_1, \ldots, \xi_k) = \langle f(\xi_1, \xi_2), f(\xi_1, \xi_2, \xi_3), \ldots, f(\xi_1, \ldots, \xi_k)\rangle.$$

As X consists of infinite ordinals, we can regard g as regressive through coding, and for any set $\in [X]^{2k}$ min-homogeneous for g the first k members will be min-homogeneous for f.

The converse is analogous to 1.6. Take η to be least such that $X \cap \eta \rightarrow (<\omega)^{<\omega}_{\mathrm{reg}}$, and show that η must be ω-Mahlo by establishing the analogous version of 1.3 and using 1.4 and 1.5. ■

The relation $X \rightarrow (\gamma)^{<\omega}_{\mathrm{reg}}$, unlike $X \rightarrow (\gamma)^n_{\mathrm{reg}}$, turns out to be closely related to well-known partition relations requiring actually homogeneous sets. If X is a set of ordinals, recall that $X \rightarrow (\gamma)^{<\omega}_\delta$ means that whenever $f: [X]^{<\omega} \rightarrow \delta$, there is a $Y \in [X]^\gamma$ homogeneous for f, i.e. $|f``[Y]^n| = 1$ for every n. For $\gamma \geq \omega$ the *Erdös cardinal*

$\kappa(\gamma)$ of Silver [Si1] is the least κ satisfying $\kappa \to (\gamma)_2^{<\omega}$. The following initial observation is simple:

PROPOSITION 1.8. *Suppose that* $\gamma \geq \omega$ *is a limit ordinal and* $X \to (\gamma)_{\mathrm{reg}}^{<\omega}$. *If* δ *is such that for any* $\alpha < \gamma$, $\alpha + \delta < \gamma$, *then* $X \to (\gamma)_\delta^{<\omega}$.

PROOF. If $f: [X]^{<\omega} \to \delta$, define g on $[X]^{<\omega}$ by setting $g(\xi_0, \dots, \xi_n) = 0$ unless $n = 2k > 0$, in which case:

$$g(\xi_0,\dots,\xi_n) = \begin{cases} 0 & \text{if } \xi_0 \leq 3; \text{ else} \\ 1 & \text{if } f(\xi_1,\dots,\xi_k) = f(\xi_{k+1},\dots,\xi_{2k}), \\ 2 & \text{if } f(\xi_1,\dots,\xi_k) > f(\xi_{k+1},\dots,\xi_{2k}), \\ 3 & \text{if } f(\xi_1,\dots,\xi_k) < f(\xi_{k+1},\dots,\xi_{2k}). \end{cases}$$

g is regressive, so let $Y \in [X]^\gamma$ be min-homogeneous for g. $2 \in g``[Y]^{<\omega}$ would lead to an infinite descending sequence of ordinals, and $3 \in g``[Y]^{<\omega}$ would lead to too many ordinals below δ by a simple argument using the fact that $\alpha < \gamma$ implies $\alpha + \delta < \gamma$. Hence, $g``[Y]^{<\omega} \subseteq \{0, 1\}$ and consequently $Y \sim 4$, which also has ordertype γ, is homogeneous for f: For any $k \in \omega$, given $\xi_1 < \cdots < \xi_k$ and $\zeta_1 < \cdots < \zeta_k$ all in $Y \sim 4$, let $\eta_1 < \cdots < \eta_k$ be all in Y so that $\max(\xi_k, \zeta_k) < \eta_1$. Then $f(\xi_1, \dots, \xi_k) = f(\eta_1, \dots, \eta_k) = f(\zeta_1, \dots, \zeta_k)$ by definition of g. ∎

In the process of seeking further connections, we came upon a direct combinatorial proof of a well-known result of Silver [Si1]. The proof is given here for its intrinsic interest, and because its nominal generalization will lead to the next result.

PROPOSITION 1.9 (SILVER [Si1]). *If* $\gamma \geq \omega$ *is a limit ordinal and* $\delta < \kappa(\gamma)$, *then* $\kappa(\gamma) \to (\gamma)_\delta^{<\omega}$.

PROOF. Set $\kappa = \kappa(\gamma)$. First of all, that $\kappa \to (\gamma)_4^{<\omega}$ is easy to see: If $f: [\kappa]^{<\omega} \to 4$, define $g: [\kappa]^{<\omega} \to 2$ by setting $g(\xi_1, \dots, \xi_n) = 0$ if $n = 2k$ and $f(\xi_1, \dots, \xi_k) = f(\xi_{k+1}, \dots, \xi_{2k})$, and $g(\xi_1, \dots, \xi_n) = 1$ otherwise. By a simpler version of the proof of 1.8, any set homogeneous for g is also homogeneous for f.

Suppose now that we are given $f: [\kappa]^{<\omega} \to \delta$, where $\delta < \kappa$, and let $g: [\delta]^{<\omega} \to 2$ attest to $\delta \not\to (\gamma)_2^{<\omega}$. Define $h: [\kappa]^{<\omega} \to 4$ by setting $h(\xi_1, \dots, \xi_n) = 0$ unless $n = 3^i 5^j$ for some $i, j > 0$, in which case:

$$h(\xi_1,\dots,\xi_n) = \begin{cases} 0 & \text{if } f(\xi_1,\dots,\xi_i) = f(\xi_{i+1},\dots,\xi_{2i}), \\ 1 & \text{if } f(\xi_1,\dots,\xi_i) > f(\xi_{i+1},\dots,\xi_{2i}), \\ 2 & \text{if } \langle f(\xi_{ik+1},\dots,\xi_{i(k+1)}) \mid k < j\rangle \\ & \quad \text{is an ascending enumeration of} \\ & \quad j \text{ ordinals homogeneous for } g, \\ 3 & \text{otherwise.} \end{cases}$$

By the previous paragraph, there is a $Y \in [\kappa]^\gamma$ homogeneous for h. If $h``[Y]^{<\omega} = \{0\}$, then $Y$ is homogeneous for $f$ as before. So, let us assume to the contrary that, for some $\bar{n} = 3^{\bar{i}} 5^{\bar{j}}$, $h``[Y]^{\bar{n}} \neq \{0\}$, and derive a contradiction:

Note first that we also have $h``[Y]^{\bar{n}} \neq \{1\}$, else there would be an infinite descending sequence of ordinals. If $\langle \zeta_\beta \mid \beta < \gamma \rangle$ is the ascending enumeration of Y, we can define $\eta_\beta = f(\zeta_{\bar{i}\beta+1}, \dots, \zeta_{\bar{i}(\beta+1)})$ for every $\beta < \gamma$, since γ is a limit ordinal. As $h``[Y]^{\bar{n}} \neq \{0\}, \{1\}$, $\langle \eta_\beta \mid \beta < \gamma \rangle$ must be a strictly increasing sequence. In particular, for any natural number of the form $3^{\bar{i}} 5^j$ for arbitrary $j > 0$, we must also have

$h``[Y]^n \neq \{0\}, \{1\}$. We now show that $h``[Y]^n = 2$ for such n. This would complete the proof, for then $\{\eta_\beta \mid \beta < \gamma\}$ would be homogeneous for g, since every finite subset of it is, contradicting the choice of g.

To do this for a given $n = 3^{\bar{i}}5^j$ with $j > 0$, apply Ramsey's theorem j times to get an infinite $W \subseteq \{\eta_\beta \mid \beta < \omega\}$ homogeneous for every $g \mid [\delta]^k$ with $k < j$. Let $\eta_{\beta_1} < \cdots < \eta_{\beta_j}$ be the first j elements of W. Then h on any n-tuple starting with $\zeta_{\bar{i}\beta_1+1}, \ldots, \zeta_{\bar{i}(\beta_1+1)}, \zeta_{\bar{i}\beta_2+1}, \ldots, \zeta_{\bar{i}(\beta_2+1)}, \ldots, \zeta_{\bar{i}\beta_j+1}, \ldots, \zeta_{\bar{i}(\beta_j+1)}$ has value 2. Hence, $h``[Y]^n = \{2\}$ by homogeneity.

THEOREM 1.10. *Suppose $\gamma \geq \omega$ is a limit ordinal. For any unbounded $X \subseteq \kappa(\gamma)$, $X \to (\gamma)^{<\omega}_{\mathrm{reg}}$.*

PROOF. Suppose f is regressive on $[X]^{<\omega}$ and for each $\delta < \kappa(\gamma)$ let $g_\delta: [\delta]^{<\omega} \to 2$ attest to $\delta \nrightarrow (\gamma)^{<\omega}_2$. Define $h: [X]^{<\omega} \to 4$ by setting $h(\xi_0, \ldots, \xi_n) = 0$ unless $n = 3^i5^j$ for some $i, j > 0$, in which case:

$$h(\xi_0, \ldots, \xi_n) = \begin{cases} 0 & \text{if } f(\xi_0, \xi_1, \ldots, \xi_i) = f(\xi_0, \xi_{i+1}, \ldots, \xi_{2i}), \\ 1 & \text{if } f(\xi_0, \xi_1, \ldots, \xi_i) > f(\xi_0, \xi_{i+1}, \ldots, \xi_{2i}), \\ 2 & \text{if } \langle f(\xi_0, \xi_{ik+1}, \ldots, \xi_{i(k+1)}) \mid k < j\rangle \text{ is an ascending enumeration} \\ & \quad \text{of } j \text{ ordinals homogeneous for } g_{\xi_0}, \\ 3 & \text{otherwise.} \end{cases}$$

Since $|X| = \kappa(\gamma)$, there is a $Y \in [X]^\gamma$ homogeneous for h. The rest of the proof proceeds just as in 1.9 to show that Y must be min-homogeneous for f as well. ■

If actually homogeneous sets are required in the partition relation rather than min-homogeneous sets, this theorem no longer holds, and the relation in particular fails for nonstationary X. Baumgartner [B] considered this stronger partition relation and developed his *γ-Erdös cardinals* as a generalization of $\kappa(\gamma)$. 1.10 holds with $\kappa(\gamma)$ replaced by any γ-Erdös cardinal, by a straightforward modification of the proof.

§2. Getting $\bar{P}_{n+2}$.

This section is devoted to establishing

THEOREM A. *If for any $a \subseteq \omega$ there is an ω-model containing a of ZFC + $\exists\kappa\exists\delta > \kappa$ (κ is n-Mahlo and $L_\kappa[a] \prec L_\delta[a]$), then $\bar{P}_{n+2}$ holds.*

Toward $\bar{P}_{n+2}$, let us make the natural switch from I to $\mathscr{P}(\omega)$ and suppose that F_1: ${}^\omega\mathscr{P}(\omega) \times {}^{n+6}({}^\omega\mathscr{P}(\omega)) \to \mathscr{P}(\omega)$ and $F_2: {}^\omega\mathscr{P}(\omega) \times {}^\omega\mathscr{P}(\omega) \to \mathscr{P}(\omega)$ are both Borel and right-invariant, and $F_3: {}^3({}^\omega\mathscr{P}(\omega)) \to \mathscr{P}(\omega)$ is Borel and totally invariant. Let $a \subseteq \omega$ code Borel codes for F_1, F_2, and F_3, and let M be a countable ω-model of ZFC + $(V = L[a])$ + "κ is n-Mahlo, $\delta > \kappa$, and $L_\kappa[a] \prec L_\delta[a]$." $V = L[a]$ can be included since n-Mahlo cardinals relativize; for convenience we shall henceforth argue with L in place of $L[a]$ since the proof is the same.

Working in M, consider the "Levy collapse" forcing notion consisting of finite partial functions: $\delta \times \omega \to M$ such that $f(\alpha, n) \in \alpha$. (Friedman instead used conditions f such that $f(\alpha, n) \in V_\alpha$, but with $M \models V = L$ we can replace V_α by L_α in his arguments, and hence, by coding, by α.)

As in [F1], for each generic object G over M and limit ordinal $\alpha \in M$ we can define the crucial sets $J(G, \alpha) \in {}^\omega\mathscr{P}(\omega)$ associated with the collapse of α to ω. Following the analysis of [F, 5.1.14 and 5.4.1], there are finitely many axioms of ZFC, conjoined together to form a sentence σ, such that for limit ordinals $\alpha < \alpha_1 < \cdots < \alpha_{n+6} < \delta$

with $L_{\alpha_{n+6}} \models \sigma$, the set

$$H_1(\alpha, \alpha_1, \ldots, \alpha_{n+6}) = \{(k, f) \mid k \in \omega \text{ and } f \Vdash k \in F_1(\mathring{J}(G, \alpha), \langle \mathring{J}(G, \alpha_1), \ldots, \mathring{J}(G, \alpha_{n+6})\rangle)\}$$

is of form

(1) $$= \{x \in L_\alpha \mid L_{\alpha_{n+6}} \models \phi_1(x, \alpha, \alpha_1, \ldots, \alpha_{n+5})\}$$

with constants from $L_{\alpha_{n+6}}$ allowed in the formula ϕ_1. Here, the restriction to $L_{\alpha_{n+6}}$ follows from standard forcing facts, and right-invariance is used to cut down the possibilities to $x \in L_\alpha$, since only the restrictions of the conditions to the domain $(\alpha + 1) \times \omega$ are relevant.

Similarly, for limit ordinals $\alpha < \beta < \delta$ with $L_\beta \models \sigma$ the set

$$H_2(\alpha, \beta) = \{(k, f) \mid k \in \omega \text{ and } f \Vdash k \in F_2(\mathring{J}(G, \alpha), \mathring{J}(G, \beta))\}$$

is of form

(2) $$= \{x \in L_\alpha \mid L_\beta \models \phi_2(x, \alpha)\}.$$

Finally, for limit ordinals $\alpha < \beta < \gamma < \delta$ with $L_\gamma \models \sigma$ the set

$$H_3(\alpha, \beta, \gamma) = \{k \mid k \in \omega \text{ and } f \Vdash k \in F_3(\mathring{J}(\alpha, G), \mathring{J}(\beta, G), \mathring{J}(\gamma, G))\}$$

by total invariance is of form

(3) $$= \{k \in \omega \mid L_\gamma \models \phi_3(k, \alpha, \beta)\}.$$

The next task is to get appropriately homogeneous sets for H_1, H_2 and H_3. Continuing to work in M, set

$$C = \{\alpha < \kappa \mid L_\alpha \prec L_\kappa\},$$

a closed unbounded subset of κ. Next, since $L_\kappa \prec L_\delta$ so that $L_\delta \models \text{ZFC}$, we can use the reflection principle in L_δ to find ordinals $\kappa \leq \delta_\xi < \delta_\zeta < \delta_{\omega_2} < \delta_{\omega_2+1} < \delta_{\omega_2+2} \leq \delta$ for $\xi < \zeta < \omega_2$ such that each $L_{\delta_i} \models \sigma$ and they preserve ϕ_2, i.e. if $i < j \leq \omega_2 + 2$, then $L_{\delta_i} \models \phi_2(x, y)$ iff $L_{\delta_j} \models \phi_2(x, y)$ for all parameters $x, y \in L_{\delta_i}$. Towards (b) of $\bar{P}_{n+2}$, note that automatically

(4) $$\text{whenever } \alpha < \beta < \gamma \text{ are all} \in C \cup \{\delta_i \mid i \leq \omega_2 + 2\}, \quad H_2(\alpha, \beta) = H_2(\alpha, \gamma)$$

by (2) and elementarity.

Next, since $H_3: [\delta]^3 \to \mathscr{P}(\omega)$ and $|\mathscr{P}(\omega)| = \omega_1$, there are four ordinals $\xi_0 < \xi_1 < \xi_2 < \xi_3 < \omega_2$ such that, towards (c) of $\bar{P}_{n+2}$,

(5) $$H_3(\delta_{\xi_0}, \delta_{\omega_2}, \delta_{\omega_2+2}) = H_3(\delta_{\xi_2}, \delta_{\omega_2}, \delta_{\omega_2+2}).$$

To verify $\bar{P}_{n+2}$, let $m \in \omega$ be given. Setting $\alpha_m = \delta_{\xi_0}$, $\alpha_{m+1} = \delta_{\xi_1}$, $\alpha_{m+2} = \delta_{\xi_2}$, $\alpha_{m+3} = \delta_{\xi_3}$, $\alpha_{m+4} = \delta_{\omega_2}$, $\alpha_{m+5} = \delta_{\omega_2+1}$, and $\alpha_{m+6} = \delta_{\omega_2+2}$, the $\alpha_m, \ldots, \alpha_{m+6}$ will provide corresponding $x_m, \ldots, x_{m+6}$ in $\bar{P}_{n+2}$, and we will invoke the characterization 1.2 to obtain the further homogeneity for H_1. H_1 is not regressive, but there is a simple strategem available: Let $X = \{\gamma_\alpha \mid \alpha \in C\} \subseteq \kappa$ be any set such that $\alpha < \beta$ both $\in C$ implies $\alpha^+ \leq \gamma_\alpha < \gamma_\beta$, and define H_1^+ on $[X]^{n+3}$ by

$$H_1^+(\gamma_\alpha, \gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n+2}}) = H_1(\alpha, \alpha_1, \ldots, \alpha_{n+2}, \alpha_m, \alpha_{m+2}, \alpha_{m+4}, \alpha_{m+6}).$$

Since $\mathbf{P}(\alpha) \subseteq L_{\alpha^+}$, we can consider H_1^+ to be regressive on X by (1). By 1.2, let $\{\gamma_{\alpha_i} \mid i \ltimes m\}$ in ascending enumeration be a set min-homogeneous for H_1^+. Then,

$$\text{(6)} \quad \begin{aligned} &\text{whenever } s < t_1 < \cdots < t_{n+2} < m \text{ and } s < u_1 < \cdots < u_{n+2} < m, \\ &H(\alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_{n+2}}, \alpha_m, \alpha_{m+2}, \alpha_{m+4}, \alpha_{m+6}) \\ &\quad = H(\alpha_s, \alpha_{u_1}, \ldots, \alpha_{u_{n+2}}, \alpha_m, \alpha_{m+2}, \alpha_{m+4}, \alpha_{m+6}). \end{aligned}$$

Using (4), we can now complete the argument as in [F1, 5.1.16] by getting slightly different generics G_i for $i \leq m + 6$ so that $s < t \leq m + 6$ implies

$$F_2(J(G_s, \alpha_s), J(G_t, \alpha_t)) = J(G_{s+1}, \alpha_{s+1})(0).$$

Hence, with (1)–(6) the set $\{x_i \mid i \leq m + 6\}$, where $x_i = J(G_i, \alpha_i)$, satisfies $\bar{P}_{n+2}$. ■

Proposition C in [F1, §3] is a consequence of P_1. In [F1], Friedman shows that C is provable in ZF + $\mathrm{AC}_\omega \sim$ Power Set + "$\mathbf{P}(\omega)$ exists," but not provable in ZF + $(V = L) \sim$ Power Set. A simple version of the foregoing proof shows that P_1 is also provable in the first theory: In an ω-model $M \models \tau + (V = L)$, where τ is a conjunction of sufficiently many ZFC axioms excluding Power Set, consider the Levy collapse of ω_1 to ω and just use a sequence $\langle \delta_i \mid i < m \rangle$ such that $L_{\delta_i} \prec L_{\delta_j}$ for $i < j < m$ together with (2) and (4).

§3. Getting *n*-Mahlo cardinals. This section is devoted to establishing

THEOREM B. *If $\bar{P}_{n+2}$ holds (even just for Borel functions of rank < 3), then for any $a \subseteq \omega$ there is an ω-model of $ZFC + \exists\kappa(\kappa$ is n-Mahlo) containing a.*

Let $\mathscr{L}$ be the language of second-order arithmetic augmented by "class" variables for *subsets* of $\mathbf{P}(\omega)$. A formula ϕ of $\mathscr{L}$ is Σ_k^1 if it has $k - 1$ alternations of second-order quantifiers beginning with an existential quantifier, followed by only bounded numerical quantifiers. For each $x \subseteq \omega$, let $|x| = \{\{m \mid 2^n 3^m \in x\} \mid n \in \omega\} \subseteq \mathbf{P}(\omega)$; the class variables range over $|x|$'s. Modifying Friedman's notion of (n, k)-critical sequence, if $d \in \omega$ we say that $\langle x_i \mid i \leq d + 6 \rangle$ is an *n-crucial sequence* iff each $x_i \subseteq \omega$ and:

(i) for all $s < t \leq d + 6$ and Σ_2^1 formulas ϕ, we have $x_s \in |x_t|$ and $\{j \in \omega \mid |x_t| \models \phi(j, x_s)\} \in |x_{s+1}|$;

(ii) for all $s < t < u \leq d + 6$ and Σ_2^1 formulas ϕ, we have $|x_t| \models \phi(x_s)$ iff $|x_u| \models \phi(x_s)$;

(iii) for any Σ_2^1 formula ϕ, $|x_{d+6}| \models \phi(|x_d|, |x_{d+4}|) \leftrightarrow \phi(|x_{d+2}|, |x_{d+4}|)$; and

(iv) for all $s < t_1 < \cdots < t_n < d$ and $s < u_1 < \cdots < u_n < d$ and Σ_2^1 formulas ϕ, we have

$$\begin{aligned} |x_{d+6}| \models \phi(&x_s, |x_{t_1}|, \ldots, |x_{t_n}|, |x_d|, |x_{d+2}|, |x_{d+4}|) \\ &\leftrightarrow \phi(x_s, |x_{u_1}|, \ldots, |x_{u_n}|, |x_d|, |x_{d+2}|, |x_{d+4}|). \end{aligned}$$

Friedman's further parameter k was for Σ_k^1 formulas, but in our approach we only require Σ_2^1. Thus, $\bar{P}_n$ restricted to Borel functions of Baire rank < 3 will suffice to establish the following analogue of [F1, 5.1.40]:

LEMMA 3.1. *If $n > 0$ and $\bar{P}_n$ holds, then for any $d \in \omega$ there is an n-crucial sequence $\langle x_i \mid i \leq d + 6 \rangle$.*

PROOF. Let d be given. For $x \in {}^\omega\mathbf{P}(\omega)$ let $\bar{x} = \{2^n 3^m \mid m \in x(n)\}$, and let $\mathrm{Rng}(x)$ be the range of x. For any formula ϕ of $\mathscr{L}$, let $\# \phi$ denote is Gödel number in some fixed arithmetization. Now define $F_1: {}^{\omega}\mathbf{P}(\omega) \times {}^{n+4}({}^{\omega}\mathbf{P}(\omega)) \to \mathbf{P}(\omega)$ by:

$$F_1(x, \langle x_1, \ldots, x_{n+4}\rangle) = \{\#\phi \mid \phi \text{ is } \Sigma^1_2 \text{ and } \mathrm{Rng}(x_{n+4}) \models \phi(\bar{x}, \mathrm{Rng}(x_1), \ldots, \mathrm{Rng}(x_{n+3}))\}.$$

(Implicit here is that $\bar{x} \in \mathrm{Rng}(x_{n+4})$ and $\mathrm{Rng}(x_i) \subseteq \mathrm{Rng}(x_{n+4})$ for $i < n+4$, else $F_1(x, \langle x_1, \ldots, x_{n-4}\rangle) = \varnothing$; analogous remarks apply below.)

Next, define $F_2: {}^{\omega}\mathbf{P}(\omega) \times {}^{\omega}\mathbf{P}(\omega) \to \mathbf{P}(\omega)$ as follows for $x, y \in {}^{\omega}\mathbf{P}(\omega)$:

Case 1. There is a Σ^1_2 formula ϕ such that $\{j \in \omega \mid \mathrm{Rng}(y) \models \phi(j, \bar{x})\} \notin \mathrm{Rng}(y)$. Then let $\bar{\phi}$ be such a formula with $\#\phi$ the least possible, and set $F_2(x, y) = \{j \in \omega \mid \mathrm{Rng}(y) \models \bar{\phi}(j, \bar{x})\}$.

Case 2. There is no such ϕ. Then set $F_2(x, y) = \{\#\phi \mid \phi \text{ is } \Sigma^1_2 \text{ and } \mathrm{Rng}(y) \models \phi(\bar{x})\}$.

Finally, define $F_3: {}^3({}^{\omega}\mathbf{P}(\omega)) \to \mathbf{P}(\omega)$ by

$$F_3(x, y, z) = \{\#\phi \mid \phi \text{ is } \Sigma^1_2 \text{ and } \mathrm{Rng}(z) \models \phi(\mathrm{Rng}(x), \mathrm{Rng}(y))\}.$$

F_1 and F_2 are Borel, and moreover are right-invariant since only the $\mathrm{Rng}(x_i)$'s and $\mathrm{Rng}(y)$ matter. Similarly, F_3 is totally invariant. Letting $\langle x_i \mid i \leq d+6\rangle$ be as in the conclusion of $\bar{P}_n$ with $m = d$ and formulated with $\mathbf{P}(\omega)$ replacing I, we can now show that $\langle \bar{x}_i \mid i \leq d+6\rangle$ is n-crucial:

First of all, for any $s < d+6$ Case 1 of the definition does not apply to $F_2(x_s, x_{s+1})$ since it is the first coordinate of x_{s+1}. Thus, by using some simple ϕ's we can see that $\mathrm{Rng}(x_s) \subseteq \mathrm{Rng}(x_{s+1})$. It follows generally that if $s < t \leq d+6$, then $\mathrm{Rng}(x_s) \subseteq \mathrm{Rng}(x_t)$. But then, $F_2(x_s, x_t) \in \mathrm{Rng}(x_{s+1}) \subseteq \mathrm{Rng}(x_t)$, and so Case 1 does not apply. Hence, (ii) in the definition of n-crucial sequence holds for our $\bar{x}_i$'s, and so also does (i) since for $s < t \leq d+6$

$$\{a \in \omega \mid \mathrm{Rng}(x_t) \models \phi(a, \bar{x}_s)\} = \{a \in \omega \mid \mathrm{Rng}(x_{s+1}) \models \phi(a, \bar{x}_s)\} \in \mathrm{Rng}(x_{s+1}).$$

Finally, (iii) and (iv) follow from the definitions of F_1 and F_3. ■

Continuing with the overall proof, we next switch to a set-theoretic context and produce sequences of ordinals satisfying certain indiscernibility requirements. Friedman works with ω-models of a set theory T consisting of the axioms: (i) extensionality, (ii) pairing, (iii) union, (iv) transitive closures, (v) Δ_0-separation, (vi) "there is no largest ordinal," (vii) "for every ordinal α, L_α exists," (viii) $\forall x \exists \alpha (x \in L_\alpha)$, and (ix) transfinite recursion on $\in$ for all formulas. Modifying Friedman's notion of (n, k)-special sequence and [F1, 5.1.31], we establish the following, where Δ_k and Σ_k refer to the usual Levy hierarchy of formulas in set theory:

LEMMA 3.2. *Let $m \in \omega$, and set $d = m + 1$. If there is an n-crucial sequence $\langle x_i \mid i \leq d+6\rangle$, then there is an ω-model $\mathscr{A}$ of T and "ordinals" $\{\alpha_i \mid i \leq m+1\}$ in the sense of $\mathscr{A}$ such that*:

(a) $\mathscr{A} \models L_{\alpha_m} \equiv L_{\alpha_{m+1}}$, *and*

(b) *whenever $s < t_1 < \cdots < t_n < m$ and $s < u_1 < \cdots < u_n < m$, $\mathscr{A} \models \beta \leq \alpha_s$, and ϕ is Σ_1, then*

$$\mathscr{A} \models \phi(\beta, \alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_n}, \alpha_m, \alpha_{m+1}) \leftrightarrow \phi(\beta, \alpha_s, \alpha_{u_1}, \ldots, \alpha_{u_n}, \alpha_m, \alpha_{m+1}).$$

PROOF. The proof amounts to checking that the hypotheses are enough to push Friedman's argument through:

Clause (i) of n-crucial shows for $p < d+6$ that every set hyperarithmetic in x_p is in $|x_{p+1}|$ (cf. [F1, 5.1.32]). Let K_p for $p < d+6$ be the set of all $\mathscr{B} \models T$ coded in $|x_p|$

such that $\mathscr{B}$ is well-founded in the sense of $|x_{p+1}|$. Clause (ii) of n-crucial is enough to verify analogues of [F1, 5.1.33 and 5.1.34].

As in [F1, 5.1.35], for $p < M + 5$ we have

(I) $|x_{d+6}| \models$ "there is a proper initial segment of an element of K_{p+2} which is longer than all the elements of K_p."

An application of clause (iv) of n-crucial shows that

(II) K_{p+2} can be replaced by K_{p+1} in (I) for $p < d - 2 = m - 1$.

(Because of our parsimonious formulation of n-crucial, it must be checked that Σ^1_2 formulas suffice in all the foregoing; an important point is that isomorphisms between initial segments of models of T, being analogous of L_α's, are unique.)

Finally, as in [F1] we can turn K_{d+4} into a structure $\mathscr{A}$ coded in $|x_{d+5}|$ and seen to be an ω-model of T in $|x_{d+6}|$. In $\mathscr{A}$, we can define ordinals α_i for $i < m$ as the supremum of heights of models in K_i, α_m as the supremum of heights of models in K_d, and α_{m+1} as the supremum of heights of models in K_{d+2}. Then $\langle \alpha_i \mid i \leq m+1 \rangle$ is an increasing sequence by the previous paragraph, since α_i for $i < m - 1$ can be defined using structures in K_{i+1} by (II), α_{m-1} in K_d, α_m using structures in K_{d+2}, and α_{m+1} using structures in K_{d+4} by (I).

We shall show that $\mathscr{A}$ together with $\langle \alpha_i \mid i \leq m+1 \rangle$ satisfy the conclusion of the lemma. In what follows, recall first that in the Levy hierarchy bounded quantifiers $\forall x \in y$ and $\exists x \in y$ are allowed in a formula without contributing to its complexity, and then note that in models of T, by using universal formulas and least witnesses and (ix) of T, any Σ_k formula can be shown equivalent to a Σ_{k+1} formula without bounded quantifiers.

Towards (a) of the lemma, it is well known that in general the satisfaction relation for any L_β is in $L_{\beta+2}$, so $\mathscr{A} \models L_{\alpha_m} \equiv L_{\alpha_{m+1}}$ is properly affirmable. (a) follows from (iii) of n-crucial, since for any sentence τ, $\mathscr{A} \models$ "$L_{\alpha_m} = \tau$" is now seen as a Σ^1_2 (even Σ^1_1) assertion in $|x_{d+6}|$ about $|x_d|$ and $|x_{d+4}|$ by the remark about the Levy hierarchy, and $\mathscr{A} \models$ "$L_{\alpha_{m+1}} \models \tau$" is the analogous assertion in $|x_{d+6}|$ about $|x_{d+2}|$ and $|x_{d+4}|$. (b) of the lemma follows from a similar indiscernibility argument using (iv) of n-crucial, and so the proof is complete. ∎

We can now complete the proof of Theorem B. Assume its hypothesis $\bar{P}_{n+2}$, so that for any $m \in \omega$ there are $\mathscr{A}, \{\alpha_i \mid i \leq m+1\}$ as in 3.2 with n replaced by $n+2$. Fix $m > n + 5$ and a corresponding pair $\mathscr{A}, \{\alpha_i \mid i \leq m+1\}$ with $\langle \alpha_m, \ldots, \alpha_0 \rangle$ lexigraphically least in $\mathscr{A}$, and work from now on inside $\mathscr{A}$.

By the arguments of [F1, 5.1.19–5.1.23], using 3.2(b) together with the Σ_1-definability of satisfaction for L_{α_m}, we have $L_{\alpha_m} \models$ ZFC. Hence by 3.2(a) we have $L_{\alpha_{m+1}} \models$ ZFC. To conclude the proof it suffices by 1.6 to verify that

$$L_{\alpha_{m+1}} \models \alpha_m \sim \omega \to (n+5)^{n+3}_{\text{reg}}$$

(since, stepping outside of $\mathscr{A}$ for a moment, $(L_{\alpha_{m+1}})^{\mathscr{A}}$ will then be the required ω-model—note that the satisfaction relation for $L_{\alpha_{m+1}}$ inside $\mathscr{A}$ and outside are the same since $\mathscr{A}$ is an ω-model, and so the coding of formulas is the same). If this were to fail, there would be an $L_{\alpha_{m+1}}$-least counterexample map f Σ_1-definable in $\mathscr{A}$ from α_m and α_{m+1}. Now for any $0 < s < t_1 < \cdots < t_{n+2} < m$ and $s < u_1 < \cdots < u_{n+2} < m$, $L_{\alpha_{m+1}} \models \beta = f(\alpha_s, \alpha_t, \ldots, \alpha_{t_{n+2}})$ is a Σ_1 statement in $\mathscr{A}$ about $\beta, \alpha_s, \alpha_{t_1}, \ldots,$

$\alpha_{t_{n+2}}$, α_m, α_{m+1}, and $L_{\alpha_{m+1}} \models \beta = f(\alpha_s, \alpha_{u_1}, \ldots, \alpha_{u_{n+2}})$ is a corresponding statement about $\beta, \alpha_s, \alpha_{u_1}, \ldots, \alpha_{u_{n+2}}, \alpha_m, \alpha_{m+1}$. Thus, by 3.2(b), $\{\alpha_i \mid 0 < i < m\}$ is min-homogeneous for f, which is a contradiction.

For the precise statement of Theorem B, given any $a \subseteq \omega$ it can be used as a parameter in the Σ^1_2 formulas in the definition of n-crucial, so that it will be a member of $|x_1|$ and hence of the final ω-model $(L_{\alpha_{m+1}})^{\mathscr{A}}$. ■

§4. On S^γ. In this last section, we indicate how the arguments of §§2 and 3 can be used to establish corresponding results about S^γ. We first establish

THEOREM C. *If for any $a \subseteq \omega$ there is an ω-model of $ZFC + \exists\kappa(\kappa \to (\omega + \omega)^{<\omega}_2)$ containing a, then $S^{\omega+\omega}$ holds.*

The argument is in fact a simpler version of §2 and closer to the original proof of [F1], so we merely outline the main steps.

Given $F: {}^\omega\mathscr{P}(\omega) \times {}^{<\omega}({}^\omega\mathscr{P}(\omega)) \to \mathscr{P}(\omega)$ Borel and right-invariant, let $a \subseteq \omega$ be a Borel code for F and M a countable ω-model of ZFC $+ (V = L[a]) +$ "$\kappa \to (\omega + \omega)^{<\omega}_2$." From Silver [Si2] the property $\lambda \to (\gamma)^{<\omega}_2$ relativizes to $L[a]$ for any $\gamma < \omega_1^{L[a]}$. We will take κ to be $\kappa(\omega + \omega)$ in the sense of M, and, for convenience, henceforth argue with L in place of $L[a]$ since the proof is the same.

Working in M, consider the Levy collapse of κ to ω_1, and for each generic object G over M and limit ordinal $\alpha \in M$ define sets $J(G, \alpha) \in {}^\omega\mathscr{P}(\omega)$ associated with the collapse of α to ω as in [F1]. As before, for any $n \in \omega$ and limit ordinals $\alpha < \alpha_1 < \cdots < \alpha_n < \kappa$ with $L_{\alpha_n} \models$ ZFC, the set

$$H(\alpha, \alpha_1, \ldots, \alpha_n) = \{(k, f) \mid k \in \omega \text{ and } f \Vdash k \in F(\mathring{J}(G, \alpha), \langle \mathring{J}(G, \alpha_1), \ldots, \mathring{J}(G, \alpha_n)\rangle)\}$$

is of form

$$= \{x \in L_\alpha \mid L_{\alpha_n} \models \phi(x, \alpha, \alpha_1, \ldots, \alpha_{n-1})\} \tag{1}$$

with constants from L_{α_n} allowed in the formula ϕ.

It is well known that $\kappa = \kappa(\omega + \omega)$ is inaccessible, so $C = \{\alpha < \kappa \mid L_\alpha \models \text{ZFC}\}$ has cardinality κ. Using the H^+ strategem of §2 and 1.10, we can therefore extract a $Y \in [C]^{\omega+\omega}$ min-homogeneous for H. The sequence $\langle J(G, \alpha) \mid \alpha \in Y\rangle$ (here there are no adjustments of G) then satisfies $S^{\omega+\omega}$: condition (ii) is automatic from the definition of the $J(G, \alpha)$'s and (1). ■

Now the analogue of Theorem B:

THEOREM D. *If $S^{\omega+5}$ holds (even just for Borel functions of rank <3) then for any $a \subseteq \omega$ there is an ω-model of $ZFC + \exists\kappa(\kappa$ is ω-Mahlo$)$ containing a.*

Following §3, we say that $\langle x_i \mid i \leq \omega + 4\rangle$ is a *crucial sequence* iff each $x_i \subseteq \omega$ and

(i) for all $s < t \leq \omega + 4$ and Σ^1_2 formulas ϕ we have $\{j \in \omega \mid |x_t| \models \phi(j, x_s)\} \in |x_t|$, and

(ii) for all $n \in \omega$, $s < t_1 < \cdots < t_n < \omega + 4$ and $s < u_1 < \cdots < u_n < \omega + 4$, and all Σ^1_2 formulas ϕ, we have

$$|x_{\omega+4}| \models \phi(x_s, |x_{t_1}|, \ldots, |x_{t_n}|) \leftrightarrow \phi(x_s, |x_{u_1}|, \ldots, |x_{u_n}|).$$

Arguing as in 3.1, $S^{\omega+5}$ establishes that there is such a sequence.

Proceeding as in 3.2, we next verify that there is an ω-model $\mathscr{A}$ of T and "ordinals" $\{\alpha_i \mid i \leq \omega + 1\}$ in the sense of $\mathscr{A}$ such that:

(a) $\mathscr{A} \models L_{\alpha_\omega} \equiv L_{\alpha_{\omega+1}}$, and

(b) whenever $n \in \omega$, $s < t_1 < \cdots < t_n \leq \omega + 1$ and $s < u_1 < \cdots < u_n \leq \omega + 1$, $\mathscr{A} \models \beta \leq \alpha_s$, and ϕ is Σ_1, then

$$\mathscr{A} \models \phi(\beta, \alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_n}) \leftrightarrow \phi(\beta, \alpha_s, \alpha_{u_1}, \ldots, \alpha_{u_n}).$$

The details are simpler than in 3.2 because of the full indiscernibility in (b): Defining K_i as before, $\mathscr{A}$ is the structure resulting from $K_{\omega+2}$, and $\langle \alpha_i \mid i \leq \omega + 1 \rangle$ is taken with α_i the supremum of the heights of models in K_i. (a) and (b) now follow directly; for $L_{\alpha_\omega} \equiv L_{\alpha_{\omega+1}}$, given a sentence τ, that $\mathscr{A} \models$ "$L_{\alpha_\omega} \models \tau$" is an assertion about x_i, $|x_\omega|$, $|x_{\omega+2}|$, $|x_{\omega+4}|$ for any fixed x_i with $i < \omega$ used as a dummy variable, and that $\mathscr{A} \models$ "$L_{\alpha_{\omega+1}} \models \tau$" is the analogous assertion about x_i, $|x_{\omega+1}|$, $|x_{\omega+2}|$, $|x_{\omega+4}|$.

To complete the argument, taking $\langle \alpha_\omega, \alpha_2, \alpha_1, \alpha_0 \rangle$ the lexigraphically least possible in $\mathscr{A}$ is enough to insure that $L_{\alpha_\omega} \models$ ZFC by Friedman's arguments, and the rest goes through as before to show that $L_{\alpha_{\omega+1}} \models \alpha_\omega \sim \omega \rightarrow (<\omega)^{<\omega}_{\text{reg}}$ since the initial segments of $\langle \alpha_i \mid i < \omega \rangle$ will be min-homogeneous for the $L_{\alpha_{\omega+1}}$-least counterexample. The proof is then complete because of 1.7. ■

We make some concluding remarks on Theorems C and D. First of all, in Theorem C we took $\kappa(\omega + \omega)$ for convenience because of 1.10, but we could have started with the hypothesis "for any unbounded $X \subseteq \kappa$, $X \rightarrow (\omega + 5)^{<\omega}_{\text{reg}}$ to get $S^{\omega+5}$ in anticipation of Theorem D. Note that Theorem C holds in general with $\omega + \omega$ replaced by any limit ordinal which is standard in every ω-model of ZFC, say ordinals $< \omega_1^{CK}$. In any case, by going to a more involved principle $\bar{S}^\omega$, we could have established a result analogous to Theorem A from an ω-model of ZFC + $\exists \delta > \kappa(\omega)(L_{\kappa(\omega)}[a] \prec L_\delta[a])$. Above all, it is of course desirable to strengthen the conclusion of Theorem D from $\exists \kappa(\kappa \rightarrow (<\omega)^{<\omega}_{\text{reg}})$ to $\exists \kappa(\kappa \rightarrow (\omega)^{<\omega}_{\text{reg}})$ to achieve a near equivalence, but we saw no way of getting infinite homogeneous sets inside the ω-models using Borel methods.

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