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Laver and set theory

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Abstract In this commemorative article, the work of Richard Laver is surveyed in its full range and extent.

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Richard Joseph Laver (20 October 1942–19 September 2012) was a set theorist of remarkable breadth and depth, and his tragic death from Parkinson's disease a month shy of his 70th birthday occasions a commemorative and celebratory account of his mathematical work, work of an individual stamp having considerable significance, worth, and impact. Laver established substantial results over a broad range in set theory from those having the gravitas of resolving classical conjectures through those about an algebra of elementary embeddings that opened up a new subject. There would be crisp observations as well, like the one, toward the end of his life, that the ground model is actually definable in any generic extension. Not only have many of his results as facts become central and pivotal for set theory, but they have often featured penetrating methods or conceptualizations with potentialities that were quickly recognized and exploited in the development of the subject as a field of mathematics.

In what follows, we discuss Laver's work in chronological order, bringing out the historical contexts, the mathematical significance, and the impact on set theory. Because of his breadth, this account can also be construed as a mountain hike across

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heights of set theory in the period of his professional life. There is depth as well, as we detail with references the earlier, concurrent, and succeeding work.

Laver became a graduate student at the University of California at Berkeley in the mid-1960s, just when Cohen's forcing was becoming known, elaborated and applied. This was an expansive period for set theory with a new generation of mathematicians entering the field, and Berkeley particularly was a hotbed of activity. Laver and fellow graduate students James Baumgartner and William Mitchell, in their salad days, energetically assimilated a great deal of forcing and its possibilities for engaging problems new and old, all later to become prominent mathematicians. Particularly influential was Fred Galvin, who as a post-doctoral fellow there brought in issues about order types and combinatorics. In this milieu, the young Laver in his 1969 thesis, written under the supervision of Ralph McKenzie, exhibited a deep historical and mathematical understanding when he affirmed a longstanding combinatorial conjecture with penetrating argumentation. Section 1 discusses Laver's work on Fraïssé's Conjecture and subsequent developments, both in his and others' hands.

For the two academic years 1969–1971, Laver was a post-doctoral fellow at the University of Bristol, and there he quickly developed further interests, e.g. on consistency results about partition relations from the then *au courant* Martin's Axiom. Section 3 at the beginning discusses this, as well as his pursuit in the next several years of saturated ideals and their partition relation consequences.

For the two academic years 1971–1973, Laver was an acting assistant professor at the University of California at Los Angeles; for Fall 1973 he was a research associate there; and then for Spring 1973 he was a research associate back at Berkeley. During this time, fully engaged with forcing, Laver established the consistency of another classical conjecture, again revitalizing a subject but also stimulating a considerable development of forcing as method. Section 2 discusses Laver's work on the Borel Conjecture as well as the new methods and results in its wake.

By 1974, Laver was comfortably ensconced at the University of Colorado at Boulder, there to pursue set theory, as well as his passion for mountain climbing, across a broad range. He was Assistant Professor 1974–1977, Associate Professor 1977–1980, and Professor from 1980 on; and there was prominent faculty in mathematical logic, consisting of Jerome Malitz, Donald Monk, Jan Mycielski, William Reinhardt, and Walter Taylor. Laver not only developed his theory of saturated ideals as set out in Sect. 3, but into the 1980s established a series of pivotal or consolidating results in diverse directions. Section 4 describes this work: indestructibility of supercompact cardinals; functions $\omega \to \omega$ under eventual dominance; the $\aleph_2$-Suslin Hypothesis; nonregular ultrafilters; and products of infinitely many trees.

In the mid-1980s, Laver initiated a distinctive investigation of elementary embeddings as given by very strong large cardinal hypotheses. Remarkably, this led to the freeness of an algebra of embeddings and the solvability of its word problem, and stimulated a veritable cottage industry at this intersection of set theory and algebra. Moving on, Laver clarified the situation with even stronger embedding hypotheses, eventually coming full circle to something basic to be seen anew, that the ground model is definable in any generic extension. This is described in the last, Sect. 5.

In the preparation of this account, several chapters of Kanamori et al. [45], especially Jean Larson's, proved to be helpful, as well as her compiled presentation of Laver's work at Luminy, September 2012. Just to appropriately fix some recurring terminology, a *tree* is a partially ordered set with a minimum element such that the predecessors of any element are well-ordered; the *αth level* of a tree is the set of elements whose predecessors have order type α; and the *height* of a tree is the least α such that the αth level is empty. A forcing poset has the *κ-c.c. (κ-chain condition)* when every antichain (subset consisting of pairwise incompatible elements) has size less than κ, and a forcing poset has the *c.c.c. (countable chain condition)* if it has the $\aleph_1$-c.c.

1 Fraïssé's conjecture

Laver [54,55] in his doctoral work famously established Fraïssé's Conjecture, a basic-sounding statement about countable linear orderings that turned out to require a substantial proof. We here first reach back to recover the historical roots, then describe how the proof put its methods at center stage, and finally, recount how the proof itself became a focus for analysis and for further application.

Cantor at the beginnings of set theory had developed the ordinal numbers [*Anzalen*], later taking them as order types of well-orderings, and in his mature *Beiträge* presentation [11] also broached the order types of linear orderings. He (§§9–11) characterized the order types θ of the real numbers and η of the rational numbers, the latter as the type of the countable dense linear ordering without endpoints. With this as a beginning, while the transfinite numbers became incorporated into set theory as the (von Neumann) ordinals, there remained an *indifference to identification* for linear order types as primordial constructs about order, as one moved variously from canonical representatives to equivalence classes according to order isomorphism or just taking them as *une façon de parler* about orderings.

The first to elaborate the transfinite after Cantor was Hausdorff, and in a series of articles he enveloped Cantor's ordinal and cardinal numbers in a rich structure festooned with linear orderings. Well-known today are the "Hausdorff gaps", but also salient is that he had characterized the scattered [*zerstreut*] linear order types, those that do not have embedded in them the dense order type η. Hausdorff [40, §§10–11] showed that for regular $\aleph_\alpha$, the scattered types of cardinality $< \aleph_\alpha$ are generated by starting with 2 and regular ω_ξ and their converse order types ω_ξ^* for $\xi < \alpha$, and closing off under the taking of sums $\Sigma_{i\in\varphi}\varphi_i$, the order type resulting from replacing each i in its place in φ by φ_i. With this understanding, scattered order types can be ranked into a *hierarchy*.

The study of linear order types under order-preserving embeddings would seem a basic and accessible undertaking, but there was little scrutiny until the 1940s. Ostensibly unaware of Hausdorff's work, Dushnik and Miller [25] and Wacław Sierpiński [113,114], in new groundbreaking work, exploited order completeness to develop uncountable types embedded in the real numbers that exhibit various structural properties.

Then in 1947 Roland Fraïssé, now best known for the Ehrenfeucht-Fraïssé games and Fraïssé limits, pointed to basic issues for *countable* order types in four conjectures. For types φ and ψ, write $\varphi \leq \psi$ *iff* there is an (injective) order-preserving embedding of φ into ψ and $\varphi < \psi$ *iff* $\varphi \leq \psi$ yet $\psi \not\leq \varphi$. Fraïssé's [31] first conjecture, at first surprising, was that there is no infinite $<$-descending sequence of countable types. Laver would affirm this, but in a strong sense as brought out by the emerging theory and the eventual method of proof.

A general notion applicable to classes ordered by embeddability, Q with a $\leq_Q$ understood is *quasi-ordered* if $\leq_Q$ is a reflexive, transitive relation on Q. Reducing with the equivalence relation $q \equiv r$ *iff* $q \leq_Q r$ and $r \leq_Q q$, one would get a corresponding relation on the equivalence classes which is anti-symmetric and hence a partial ordering; the preference however is to develop a theory doing without this, so as to be able to work directly with members of Q.[1] Q is *well-quasi-ordered* (wqo) if for any $f: \omega \to Q$, there are $i < j < \omega$ such that $f(i) \leq_Q f(j)$. Higman [41] came to the concept of wqo via a finite basis property and made the observation, simple with Ramsey's Theorem, that Q is wqo *iff* (a) Q is *well-founded*, i.e. there are no infinite $<_Q$-descending sequences (where $q <_Q r$ *iff* $q \leq_Q r$ yet $r \not\leq_Q q$), and (b) there are no infinite *antichains*, i.e. sets of pairwise $\leq_Q$-incomparable elements. For a Q quasi-ordered by $\leq_Q$, the subsets of Q can be correspondingly quasi-ordered by: $X \leq Y$ *iff* $\forall x \in X \exists y \in Y (x \leq_Q y)$. With this, Higman established that if Q is wqo, then so are the *finite* subsets of Q. In his 1954 dissertation Kruskal [51] also came to well-quasi-ordering, coining the term, and settled a conjecture about trees: For trees T_1 and T_2, $T_1 \leq T_2$ *iff* T_1 is homeomorphically embeddable into T_2, i.e. there is an injective $f: T_1 \to T_2$ satisfying $f(x \wedge y) = f(x) \wedge f(y)$, where $\wedge$ indicates the greatest lower bound. Kruskal established that the *finite* trees are wqo.

Pondering the delimitations to the finite, particularly that there had emerged a simple example of a wqo Q whose full power set $\mathcal{P}(Q)$ is not wqo, Nash-Williams [102] came up with what soon became a pivotal notion. Identifying subsets of ω with their increasing enumerations, say that a set B of non-empty finite subsets of ω is a *block* if every infinite subset of ω has an initial segment in B. For non-empty finite subsets s, t of ω, write $s \lhd t$ *iff* there is a $k < \min(t)$ such that s is a proper initial segment of $\{k\} \cup t$. Finally, Q with $\leq_Q$ is *better-quasi-ordered* (bqo) if for any block B and function $f: B \to Q$, there are $s \lhd t$ both in B such that $f(s) \leq_Q f(t)$. bqo implies wqo, since $\{\{i\} \mid i \in \omega\}$ is a block and $\{i\} \lhd \{j\}$ *iff* $i < j$, and this already points to how bqo might be a useful strengthening in structured situations. Nash-Williams observed that if Q is bqo then so is $\mathcal{P}(Q)$, and established that the infinite trees of height at most ω are bqo.

With this past as prologue, Laver [54,55] in 1968 dramatically established Fraïssé's Conjecture in the strong form: *the countable linear order types are bqo.* Of course, it suffices to consider only the scattered countable types, since any countable type is embeddable into the dense type η. In a remarkably synthetic proof, Laver worked up a hierarchical analysis building on the Hausdorff characterization of scattered types; develop a labeled tree version of Nash-Williams's tree theorem; and established a main preservation theorem, Q bqo $\longrightarrow$ $Q^{\mathcal{M}}$ bqo, the latter consisting of Q-labeled ordered types in a class $\mathcal{M}$. Actually, Laver established his result for the large class $\mathcal{M}$ of *σ-scattered* order types, countable unions of scattered types, working up a specific hierarchy for these devised by Fred Galvin.

Laver's result, both in affirming that the countable linear order types have the basic wqo connecting property and being affirmed with a structurally synthetic and penetrating proof, would stand as a monument, not the least because of a clear and

[1] A quasi-order is also termed a *pre-order*, and in iterated forcing, to the theory of which Laver would make an important contribution (cf. Sect. 2), one also prefers to work with pre-orders of conditions rather than equivalence classes of conditions.

mature presentation in [55]. wqo and bqo were brought to the foreground; the result was applied and analyzed; and aspects and adaptations of both statement and proof would be investigated. Laver himself [56,59,64] developed the theory in several directions.

In [56], Laver proceeded to a decomposition theorem for order types. As with ordinals, an order type φ is *additively indecomposable* (AI) *iff* whenever it is construed as a sum $\psi + \theta$, then $\varphi \leq \psi$ or $\varphi \leq \theta$. Work in [55] had shown that any scattered order type is a finite sum of AI types and that the AI scattered types can be generated via "regular unbounded sums". Generalizing homeomorphic embedding to a many-one version, Laver established a tree representation for AI scattered types as a decomposition theorem, and then drew the striking conclusion that *for σ-scattered φ, there is an $n \in \omega$ such that for any finite partition of φ, φ is embeddable into a union of at most n parts*. In [59], Laver furthered the wqo theory of finite trees; work there was later applied by Kozen [50] to establish a notable finite model property. Finally in [64], early in submission but late in appearance, Laver made his ultimate statement on bqo. He first provided a lucid, self-contained account of bqo theory through to Nash-Williams's subtle "forerunning" technique. A tree is *scattered* if the complete binary tree is not embeddable into it, and it is *σ-scattered* if it is a countable union of scattered, downward-closed subtrees. As a consequence of a general preservation result about labeled trees, Laver established: *the σ-scattered trees are bqo*. Evidently stimulated by this work, Shelah [110] investigated a bqo theory for uncountable cardinals based on whenever $f : \kappa \to Q$ there are $i < j < \kappa$ such that $f(i) \leq_Q f(j)$, discovering new parametrized concepts and a large cardinal connection.

"Fraïssé's Conjecture", taken to be the (proven) proposition that countable linear orders are wqo, would newly become a focus in the 1990s with respect to the reverse mathematics of provability in subsystems of second-order arithmetic.[2] Shore [112] established that the countable *well*-orderings being wqo already entails the system ATR_0. Since the latter implies that any two countable well-orderings are comparable, there is thus an equivalence. Montalbán [99] proved that every hyperarithmetic linear order is mutually embeddable with a recursive one and [100] showed that Fraïssé's Conjecture is equivalent (over the weak theory RCA_0) to various propositions about linear orders under embeddability, making it a "robust" theory. However, whether Fraïssé's Conjecture is actually equivalent to ATR_0 is a longstanding problem of reverse mathematics, with e.g. Marcone and Montalbán [91] providing a partial result. The proposition, basic and under new scrutiny, still has the one proof that has proved resilient, the proof of Laver [55] going through the hierarchy of scattered countable order types and actually establishing bqo through a preservation theorem for labeled order types.

Into the 21st Century, there would finally be progress about possibilities for extending Laver's result into non-σ-scattered order types and trees. Laver [64] had mentioned that Aronszajn trees (cf. Sect. 4.3) are not wqo assuming Ronald Jensen's principle $\diamondsuit$ and raised the possibility of a relative consistency result. This speculation would stand for decades until in 2000 Todorcevic [120] showed that no, there are $2^{\aleph_1}$ Aronszajn trees pairwise incomparable under (just) injective order-preserving embeddability. Recently, on the other hand, Martinez-Ranero [94] established that under the Proper

[2] See [92] for a survey of the reverse mathematics of wqo and bqo theory.

Forcing Axiom (PFA), Aronszajn lines *are* bqo. Aronszajn lines are just the linearizations of Aronszajn trees, so this is a contradistinctive result. Under PFA, Moore [101] showed that there is a universal Aronszajn line, a line into which every Aronszajn line is embeddable, and starting with this analogue of the dense type η, Martinez-Canero proceeded to adapt the Laver proof. Generally speaking, a range of recent results have shown PFA to provide an appropriately rich context for the investigation of general, uncountable linear order types; Ishiu and Moore [42] even discussed the possibility that the Laver result about σ-scattered order types, newly apprehended as prescient as to how far one can go, is sharp in the sense that it cannot be reasonably extended to a larger class of order types.

2 Borel conjecture

Following on his Fraïssé's Conjecture success, Laver [57,60] by 1973, while at the University of California at Los Angeles, had established another pivotal result with an even earlier classical provenance and more methodological significance, the consistency of "the Borel Conjecture". A subset X of the unit interval of reals has *strong measure zero* (Laver's term) *iff* for any sequence $\langle \epsilon_n \mid n \in \omega \rangle$ of positive reals there is a sequence $\langle I_n \mid n \in \omega \rangle$ of intervals with the length of I_n at most ϵ_n such that $X \subseteq \bigcup_n I_n$. Laver established with iterated forcing the relative consistency of $2^{\aleph_0} = \aleph_2$ + "Every strong measure zero set is countable". We again reach back to recover the historical roots and describe how the proof put its methods at center stage, and then how both result and method stimulated further developments.

At the turn of the 20th Century, Borel axiomatically developed his notion of measure, getting at those sets obtainable by starting with the intervals and closing off under complementation and countable union and assigning corresponding measures. Lebesgue then developed his extension of Borel measure, which in retrospect can be formulated in simple set-theoretic terms: A set of reals if *null iff* it is a subset of a Borel set of measure zero, and a set is *Lebesgue measurable iff* it has a null symmetric difference with some Borel set, in which case its Borel measure is assigned. With null sets having an amorphous feel, Borel [10] studied them constructively in terms of rates of convergence of decreasing measures of open covers, getting to the strong measure zero sets. Actually, he only mentioned them elliptically, writing that they would have to be countable but that he did not possess an "entirely satisfactory proof".[3] Borel would have seen that no uncountable closed set of reals can have strong measure zero, and so, that no uncountable Borel set can have strong measure zero. More broadly, a perfect set (a non-empty closed set with no isolated points), though it can be null,[4] is seen not to have strong measure zero. So, it could have been deduced by then that no uncountable analytic set, having a perfect subset, can have strong measure zero. While all this might have lent an air of plausibility to strong measure zero sets having to be countable, it was also known by then that the Continuum Hypothesis (CH) implies

[3] Borel [10, p. 123]: "Un ensemble énumérable a une mesure asymptotique inférieure à toute série donnée a l'avance; la réciproque me paraît exacte, mai je n'en possède pas de démonstration entièrement satisfaisante."

[4] The Cantor ternary set, defined by Cantor in 1883, is of course an example.

the existence of a *Luzin set*, an uncountable set having countable intersection with any meager set. A Luzin set can be straightforwardly seen to have strong measure zero, and so Borel presumably could not have possessed a "satisfactory proof".

In the 1930s strong measure zero sets, termed Wacław Sierpiński's "sets with Property C", were newly considered among various special sets of reals formulated topologically.[5] Abram Besicovitch came to strong measure zero sets in a characterization result, and he provided, in terms of his "concentrated sets", a further articulated version of CH implying the existence of an uncountable strong measure zero set. Then Sierpiński and Fritz Rothberger, both in 1939 papers, articulated the first of the now many cardinal invariants of the continuum, the bounding number. (A family F of functions: $\omega \to \omega$ is *unbounded* if for any $g\colon \omega \to \omega$ there is an f in F such that $\{n \mid g(n) \leq f(n)\}$ is infinite, and the *bounding number* $\mathfrak{b}$ is the least cardinality of such a family.) Their results about special sets established that (without CH but just) $\mathfrak{b} = \aleph_1$ implies the existence of an uncountable strong measure zero set. Strong measure zero sets having emerged as a focal notion, there was however little further progress, with Rothberger [107] retrospectively declaring "the principal problem" to be whether there are uncountable such sets.[6]

Whatever the historical imperatives, two decades later Laver [57,60] duly established the relative consistency of "the Borel Conjecture", that all strong measure zero sets can be countable. Cohen, of course, had transformed set theory in 1963 by introducing forcing, and in the succeeding decades there were broad advances made through the new method involving the development both of different forcings and of forcing techniques. Laver's result featured both a new forcing, for adding a *Laver real*, and a new technique, adding reals at each stage in a *countable support iteration*.

For adding a Laver real, a condition is a tree of natural numbers with a finite trunk and all subsequent nodes having infinitely many immediate successors. A condition is stronger than another if the former is a subtree, and the longer and longer trunks union to a new, generic real: $\omega \to \omega$. Thus a Laver condition is a structured version of the basic Cohen condition, which corresponds to just having the trunk, and that structuring revises the Sacks condition, in which one requires that every node has an eventual successor with two immediate successors. Already, a Laver real is seen to be a dominating real, i.e. for any given ground model $g\colon \omega \to \omega$ a Laver condition beyond the trunk can be pruned to always take on values larger than those given by g. Thus, the necessity of making the bounding number $\mathfrak{b}$ large is addressed. More subtly, Laver conditions exert enough infinitary control to assure that for any uncountable set X of reals in the ground model and with f being the Laver real, there is no sequence $\langle I_n \mid n \in \omega\rangle$ of intervals in the extension with length $I_n < \frac{1}{f(n)}$ such that $X \subseteq \bigcup_n I_n$.

Laver proceeded from a model of CH and adjoined Laver reals iteratively in an iteration of length ω_2. The iteration was with countable support, i.e. a condition at the αth stage is a vector of condition names at earlier stages, with at most countably many of them being non-trivial. This allowed for a tree "fusion" argument across the iteration that determined more and more of the names as actual conditions and so showed that

[5] cf. Steprāns [116, pp. 92–102] for a historical account.

[6] Rothberger [107, p. 111] "…the principal problem, viz., to prove with the axiom of choice only (without any other hypothesis) the existence of a non-denumerable set of property C, this problem remains open."

e.g. for any countable subset of the ground model in the extension, there is a countable set in the ground model that covers it. Consequently, ω_1 is preserved in the iteration and so also the $\aleph_2$-c.c., so that all cardinals are preserved and $2^{\aleph_0} = \aleph_2$ in the extension. Specifically for the adjoining of Laver reals, Laver crowned the argument as follows:

Suppose that X is an $\aleph_1$ size set of reals in the extension. Then it had already occurred at an earlier stage by the chain condition, and so at that stage the next Laver real provides a counterexample to X having strong measure zero. But then, there is enough control through the subsequent iteration with the "fusion" apparatus to ensure that X still will not have strong measure zero.

Laver's result and paper [60] proved to be a turning point for iterated forcing as method. Initially, the concrete presentation of iteration as a quasi-order of conditions that are vectors of forcing names for local conditions was itself revelatory. Previous multiple forcing results like the consistency of Martin's Axiom had been cast in the formidable setting of Boolean algebras. Henceforth, there would be a grateful return to Cohen's original heuristic of conditions approximating a generic object, with the particular advantage in iterated forcing of seeing the dynamic interaction with forcing names, specifically names for later conditions. More centrally, Laver's structural results about countable support iteration established a scaffolding for proceeding that would become standard fare. While the consistency of Martin's Axiom had been established with the finite support iteration of c.c.c. forcings, the new regimen admitted other forcings and yet preserved much of the underlying structure of the ground model.

Several years later Baumgartner and Laver [67] elaborated the countable support iteration of Sacks forcing, and with it established consistency results about selective ultrafilters as well as about higher Aronszajn trees (cf. Sect. 4.3). They established: *If κ is weakly compact and κ Sacks reals are adjoined iteratively with countable support, then in the resulting forcing extension $\kappa = \omega_2$ and there are no $\aleph_2$-Aronszajn trees.* Groundbreaking for higher Aronszajn trees, that they could be no $\aleph_2$-Aronszajn trees had first been pointed out by Jack Silver as a consequence of forcing developed by Mitchell (cf. Mitchell [98, p. 41]) and significantly, that forcing was the initial instance of a countable support iteration. However, it worked in a more involved way with forcing names, and the Baumgartner–Laver approach with the Laver scaffolding made the result more accessible.

By 1978 Baumgartner had axiomatically generalized the iterative addition of reals with countable support with his "Axiom A" forcing, and in an influential account [7] set out iterated forcing and Axiom A in an incisive manner. Moreover, he specifically worked through the consistency of the Borel Conjecture by iteratively adjoining Mathias reals with countable support, a possible alternate approach to the result pointed out by Laver [60, p. 168]. All this would retrospectively have a precursory air, as Shelah in 1978 established a general, subsuming framework with his *proper forcing*. With its schematic approach based on countable elementary substructures, proper forcing realized the potentialities of Laver's initial work and brought forcing to a new plateau. Notably, a combinatorial property of Laver forcing, "the Laver property", was shown to be of importance and preserved through the iteration of proper forcings.[7]

[7] cf. Bartoszyński and Judah [4, 6.3.E].

As for Laver reals and Laver's specific [60] model, Arnold Miller [96] showed that in that model there are no q-point ultrafilters, answering a question of the author. Later, in the emerging investigation of cardinal invariants, Laver forcing would become *the* forcing "associated" with the bounding number $\mathfrak{b}$,[8] in that it is the forcing that increases $\mathfrak{b}$ while fixing the cardinal invariants not immediately dependent on it. Judah and Shelah [43] exhibited this with the Laver [60] model.

And as for the Borel Conjecture itself, the young Hugh Woodin showed in 1981 that adjoining any number of random reals to Laver's model preserves the Borel Conjecture, thereby establishing the consistency of the conjecture with the continuum being arbitrarily large. The sort of consistency result that Laver had achieved has become seen to have a limitative aspect in that countable support iteration precludes values for the continuum being larger than $\aleph_2$, and at least for the Borel Conjecture a way was found to further increase the size of the continuum. Judah et al. [44] provided systematic iterated forcing ways for establishing the Borel Conjecture with the continuum arbitrarily large.

3 Partition relations and saturated ideals

Before he established the consistency of Borel's conjecture, Laver, while at the University of Bristol (1969–1971), had established [58] relative consistency results about partition relations low in the cumulative hierarchy. Through the decade to follow, he enriched the theory of saturated ideals in substantial part to get at further partition properties. This work is of considerable significance, in that large cardinal hypotheses and infinite combinatorics were first brought together in a sustained manner.

In the well-known Erdős–Rado partition calculus, the simplest case of the ordinal partition relation is $\alpha \longrightarrow (\beta)^2_2$, the proposition that for any partition $[\alpha]^2 = P_0 \cup P_1$ of the 2-element subsets of α into two cells P_0 and P_1, there is a subset of α of order type β all of whose 2-element subsets are in the same cell. The unbalanced relation $\alpha \longrightarrow (\beta, \gamma)^2$ is the proposition that for any partition $[\alpha]^2 = P_0 \cup P_1$, either there is a subset of α order type β all of whose 2-element subsets are in P_0 or there is a subset of α of order type γ all of whose 2-elements subsets are in P_1. Ramsey's seminal 1930 theorem amounts to $\omega \longrightarrow (\omega)^2_2$, and sufficiently strong large cardinal properties for a cardinal κ imply $\kappa \longrightarrow (\kappa)^2_2$, which characterizes the weak compactness of κ. Laver early on focused on the possibilities of getting the just weaker $\kappa \longrightarrow (\kappa, \alpha)^2$ for small, accessible κ and a range of $\alpha < \kappa$.

In groundbreaking work, Laver [58] showed that Martin's Axiom (MA) has consequences for partition relations of this sort for $\kappa \leq 2^{\aleph_0}$. Laver was the first to establish relative consistency results, rather than outright theorems of ZFC, about partition relations for accessible cardinals. Granted, Prikry's [105] work was important in this direction in establishing a negation of a partition relation consistent, particularly as he did this by forcing a significant combinatorial principle that would subsequently be shown to hold in L. Notably, Erdős bemoaned how the partition calculus would now have to acknowledge relative consistency results. Laver's work, in first applying MA,

[8] cf. Bartoszyński and Judah [4, 7.3.D].

was also pioneering in adumbration of arguments for the central theorem of Baumgartner and András Hajnal [5], that $\omega_1 \longrightarrow (\alpha)^2_2$ for every $\alpha < \omega_1$, a ZFC theorem whose proof involved appeal to MA and absoluteness. As for the stronger, unbalanced relation, the young Todorcevic [118] by 1981 established the consistency of MA + $2^{\aleph_0} = \aleph_2$ together with $\omega_1 \longrightarrow (\omega_1, \alpha)^2$ for every $\alpha < \omega_1$.

By 1976, Laver saw how saturated ideals in a strong form can drive the argumentation to establish unbalanced partition relations for cardinals. Briefly, I is a *κ-ideal iff* it is an ideal over κ (a family of subsets of κ closed under the taking of subsets and unions) which is non-trivial (it contains $\{\alpha\}$ for every $\alpha < \kappa$ but not κ) and κ-complete (it is closed under the taking of unions of fewer than κ of its members). Members of a κ-ideal are "small" in the sense given by I, and mindful of this, such an ideal is *λ-saturated iff* for any λ subsets of κ not in I there are two whose intersection is still not in I. Following the founding work of Robert Solovay on saturated ideals in the 1960s, they have become central to the theory of large cardinals primarily because they can carry strong consistency strength yet appear low in the cumulative hierarchy. κ is a measurable cardinal, as usually formulated, just in case there is a 2-saturated κ-ideal, and e.g. if κ Cohen reals are adjoined, then in the resulting forcing extension: $\kappa = 2^{\aleph_0}$ and there is an $\aleph_1$-saturated κ-ideal. Conversely, if there is a κ^+-saturated κ-ideal for some κ, then in the inner model relatively constructed from such an ideal, κ is a measurable cardinal.

In a first, parametric elaboration of saturation, Laver formulated the following property: A κ-ideal I is *(λ, μ, ν)-saturated iff* every family of λ subsets of κ not in I has a subfamily of size μ such that any ν of its members has still has intersection not in I. In particular, a κ-ideal is λ-saturated *iff* it is $(\lambda, 2, 2)$-saturated. In the abstract [65], for a 1976 meeting, Laver announced results subsequently detailed in [62] and [69].

In [62] Laver established that if $\gamma < \kappa$ and there is a (κ, κ, γ)-saturated κ-ideal (which entails that κ must be a regular limit cardinal) and $\beta^\gamma < \kappa$ for every $\beta < \kappa$, then $\kappa \longrightarrow (\kappa, \alpha)^2$ holds for every $\alpha < \gamma^+$. He then showed, starting with a measurable cardinal κ, how to cleverly augment the forcing for adding many Cohen subsets of a $\gamma < \kappa$ to retain such κ-ideals with κ newly accessible, a paradigmatic instance being a $(2^{\aleph_1}, 2^{\aleph_1}, \aleph_1)$-saturated $2^{\aleph_1}$-ideal with $\beta < 2^{\aleph_1}$ implying $\beta^{\aleph_0} < 2^{\aleph_1}$. From this one has the consistency of $2^{\aleph_1} \longrightarrow (2^{\aleph_1}, \alpha)^2$ for every $\alpha < \omega_2$, and this is sharp in two senses, indicative of what Laver was getting at: A classical Sierpiński observation is that $2^{\aleph_1} \longrightarrow (\omega_2)^2$ fails, and the well-known Erdős–Rado Theorem implies that $(2^{\aleph_1})^+ \longrightarrow ((2^{\aleph_1})^+, \omega_2)^2$ holds. Years later, Todorcevic [119] established the consistency, relative only to the existence of a weakly compact cardinal, of $2^{\aleph_0} \longrightarrow (2^{\aleph_0}, \alpha)^2$ for every $\alpha < \omega_1$, as well as of $2^{\aleph_1} \longrightarrow (2^{\aleph_1}, \alpha)^2$ for every $\alpha < \omega_2$.

In [69] Laver established the consistency of a substantial version of his saturation property holding for a κ-ideal with κ a *successor* cardinal, thereby establishing the consistency of a partition property for such κ. In the late 1960s, while having a κ^+-saturated κ-ideal for some κ had been seen to be equi-consistent to having a measurable cardinal, Kunen had shown that the consistency strength, were κ posited to be a *successor* cardinal, was far stronger. In a *tour de force*, Kunen [52] in 1972 established: *If κ is a huge cardinal, then in a forcing extension $\kappa = \omega_1$ and there is an $\aleph_2$-saturated ω_1-ideal.* In the large cardinal hierarchy huge cardinals are consistency-wise much stronger than the better known supercompact cardinals, and Kunen had unabashedly appealed to the strongest embedding hypothesis to date for carrying out a forcing construction. From the latter 1970s on, Kunen's argument, as variously elaborated and amended, would become and remain a prominent tool for producing strong phenomena at successor cardinals, though dramatic developments in the 1980s would show how weaker large cardinal hypotheses suffice to get $\aleph_2$-saturated ω_1-ideals themselves. Laver in 1976 was first to amend Kunen's argument, getting [69]: *If* κ *is a huge cardinal, then in a forcing extension* $\kappa = \omega_1$ *and there is an* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated* ω_1*-ideal.* Not only had Laver mastered Kunen's sophisticated argument with elementary embedding, but he had managed to augment it, introducing "Easton supports".

Laver [69] (see also [48]) established that the newly parametrized saturation property has a partition consequence: *If* $\kappa^{<\kappa} = \kappa$ *and there is a* $(\kappa^+, \kappa^+, <\kappa)$*-saturated (with the expected meaning)* κ*-ideal, then* $\kappa^+ \longrightarrow (\kappa + \kappa + 1, \alpha)$ *for every* $\alpha < \kappa^+$. This partition relation is thus satisfied at measurable cardinals κ, and with CH holding in Laver's [69] model it satisfies

$$\omega_2 \longrightarrow (\omega_1 + \omega_1 + 1, \alpha) \text{ for every } \alpha < \omega_2.$$

This result stood for decades as best possible for successor cardinals larger than ω_1. Then Matthew Foreman and Hajnal in [27] extended the ideas to a stronger conclusion, albeit from a stronger ideal hypothesis. A κ-ideal I is λ*-dense iff* there is a family D of λ subsets of κ not in I such that for any subset X of κ not in I, there is a $Y \in D$ almost contained in X, i.e. $Y - X$ is in I. This is a natural notion of density for the Boolean algebra $\mathcal{P}(\kappa)/I$, and evidently a κ-dense κ-ideal is $(\kappa^+, \kappa^+, <\kappa)$-saturated. Foreman and Hajnal managed to prove that if $\kappa^{<\kappa} = \kappa$ and there is a κ-dense κ-ideal, then $\kappa^+ \longrightarrow (\kappa^2 + 1, \alpha)$ for every $\alpha < \kappa^+$. Central work by Woodin in the late 1980s had established the existence of an ω_1-dense ω_1-ideal relative to large cardinals, and so one had the corresponding improvement, $\omega_2 \longrightarrow (\omega_1^2 + 1, \alpha)$ for every $\alpha < \omega_2$, of the Laver [69] result and the best possible to date for ω_2.

4 Consolidations

In the later 1970s and early 1980s Laver, by then established at the University of Colorado at Boulder, went from strength to strength in exhibiting capability and willingness to engage with *au courant* concepts and questions over a broad range. In addition to the saturated ideals work, Laver established pivotal, consolidating results, each in a single incisive paper, and in what follows we deal with these and frame their significance.

4.1 Indestructibility

In a move that exhibited an exceptional insight into what might be proved about supercompact cardinals, Laver in 1976 established their possible "indestructibility" under certain forcings. This seminal result, presented in a short 4-page paper [63], would not only become part and parcel of method about supercompact cardinals but would become a concept to be investigated in its own right for large cardinals in general.

In 1968, Robert Solovay and William Reinhardt (cf. [115]) formulated the large cardinal concept of supercompactness as a generalization of the classical concept of measurability once its elementary embedding characterization was attained. A cardinal κ is *supercompact iff* for every $\lambda \geq \kappa$, κ is λ-supercompact, where in turn κ is λ-*supercompact iff* there is an elementary embedding $j : V \to M$ such that the least ordinal moved by j is κ and moreover M is closed under arbitrary sequences of length λ. That there is such a j is equivalent to having a *normal ultrafilter* over $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$; from such a j such a normal ultrafilter can be defined, and conversely, from such a normal ultrafilter U a corresponding elementary embedding j_U can be defined having the requsite properties. κ is κ-supercompact exactly when κ is measurable, as quickly seen from the embedding formulation of the latter.

In 1971, Silver established the relative consistency of having a measurable cardinal κ satisfying $\kappa^+ < 2^\kappa$. That this would require strong hypotheses had been known, and for Silver's argument having an elementary embedding j as given by the κ^{++}-supercompactness of κ suffices. Silver introduced two motifs that would become central to establishing consistency results from strong hypotheses. First, he forced the necessary structure of the model below κ, but *iteratively*, proceeding upward to κ. Second, in considering the j-image of the process he developed a *master condition* so that forcing through it would lead to an extension of j in the forcing extension, thereby preserving the measurability (in fact the κ^{++}-supercompactness) of κ.

Upon seeing Silver's argument as given e.g. in Menas [95] and implementing the partial order approach from the Borel Conjecture work, Laver saw through to a generalizing synthesis, first establishing a means of universal anticipation below a supercompact cardinal and then applying it to render the supercompactness robust under further forcing. The first result exemplifies what reflection is possible at a supercompact cardinal: *Suppose that κ is supercompact. Then there is one function $f : \kappa \to V_\kappa$ such that for all $\lambda \geq \kappa$ and all sets x hereditarily of cardinality at most λ, there is a normal ultrafilter U over $\mathcal{P}_\kappa\lambda$ such that $j_U(f)(\kappa) = x$.* Such a function has been called a "Laver function" or "Laver diamond"; indeed, the proof is an elegant variant of the proof of the diamond principle $\diamondsuit$ in L which exploits elementary embeddings and definability of least counterexamples.

With this, Laver [63] established his "indestructibility" result. A notion of forcing P is κ-*directed closed iff* whenever $D \subseteq P$ has size less that κ and is directed (i.e. any two members of D have a lower bound in D), D has a lower bound. Then: *Suppose that κ is supercompact. Then in a forcing extension κ is supercompact and remains so in any further extension via a κ-directed closed notion of forcing.* The forcing done is an iteration of forcings along a Laver function. To show that any further κ-directed closed forcing preserves supercompactness, master conditions are exploited to extend elementary embeddings.

For relative consistency results involving supercompact cardinals, Laver indestructibility leads to technical strengthenings as well as simplifications of proofs, increasing their perspicuity. At the outset as pointed out by Laver himself, while [95] had shown that for κ supercompact and $\lambda \geq \kappa$ there is a forcing extension in which κ remains supercompact and $2^\kappa \geq \lambda$, once a supercompact cardinal is "Laverized", from that single model 2^κ can be made arbitrarily large while preserving supercompactness. Much more substantially and particularly in arguments involving several large cardinals, Laver indestructibility was seen to set the stage after which one can proceed with iterations that preserve supercompactness without bothering with specific preparatory forcings. Laver indestructibility was thus applied in the immediately subsequent, central papers for large cardinal theory, Magidor [89], Foreman et al. [28,29].

The Laver function itself soon played a crucial role in a central relative consistency result. Taking on Shelah's proper forcing, the Proper Forcing Axiom (already mentioned at the end of Sect. 1) asserts that for any proper notion of forcing P and sequence $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ of dense subsets of P, there is a filter F over P that meets every D_α. Early in 1979 Baumgartner (cf. [22]) established: *Suppose that κ is supercompact. Then in a forcing extension $\kappa = \omega_2 = 2^{\aleph_0}$ and* PFA *holds.* Unlike for Martin's Axiom, to establish the consistency of PFA requires handling a proper class of forcings, and it sufficed to iterate proper forcings given along a Laver function, these anticipating all proper forcings through elementary embeddings. PFA is known to have strong consistency strength, and to this day Baumgartner's result, with its crucial use of a Laver function, stands as the bulwark for consistency.

Laver functions have continued to be specifically used in consistency proofs (e.g. Cummings and Foreman [15, 2.6]) and have themselves become the subject of investigation for a range of large cardinal hypotheses (e.g. Corazza [14]). As for the indestructibility of large cardinals, the concept has become part of the mainstream of large cardinals not only through application but through concerted investigation. Gitik and Shelah [33] established a form of indestructibility for strong cardinals to answer a question about cardinal powers. Apter and Hamkins [1] showed how to achieve universal indestructibility, indestructibility simultaneously for the broad range of large cardinals from weakly compact to supercompact cardinals. Hamkins [37] developed a general kind of Laver function for any large cardinal and, with it, a general kind of Laver preparation forcing to achieve a broad range of new indestructibilities. Starting with [2], Arthur Apter has pursued the indestructibility particularly of partially supercompact and strongly compact cardinals through over 20 articles. Recently, Bagaria and Hamkins [3] showed that very large cardinals, superstrong and above, are never Laver indestructible, so that there is a ceiling to indestructibility.

In retrospect, it is quite striking that Laver's modest 4-page paper should have had such an impact.

4.2 Eventual dominance

Hugh Woodin in 1976, while still an undergraduate, made a remarkable reduction of a proposition ("Kaplansky's Conjecture") of functional analysis about the continuity of homomorphisms of Banach algebras to a set-theoretic asserton about embeddability into $\langle {}^\omega\omega, <^* \rangle$, the family of functions: $\omega \to \omega$ ordered by eventual dominance (i.e. $f <^* g$ *iff* $f(n) < g(n)$ for sufficiently large n). Solovay, the seasoned veteran, soon established the consistency of the set-theoretic assertion, and thereby, the relative consistency of the proposition.[9] In the process, Solovay raised a question, which Laver [66] by 1978 answered affirmatively by establishing the relative consistency of: $\aleph_1 < 2^{\aleph_0}$ and every linear ordering of size $\leq 2^{\aleph_0}$ is embeddable into $\langle {}^\omega\omega, <^* \rangle$.

Linear orderings of size $\leq \aleph_1$ are in any case embeddable into $\langle {}^\omega\omega, <^* \rangle$, yet if to a model of CH one e.g. adjoins many Cohen reals then ω_2 is still not embeddable into $\langle {}^\omega\omega, <^* \rangle$. Martin's Axiom (MA) implies that every well-ordering of size $< 2^{\aleph_0}$ is embeddable into $\langle {}^\omega\omega, <^* \rangle$, yet Kunen in incisive 1975 work had shown that MA is consistent with the existence of a linear ordering of size $2^{\aleph_0}$ *not* being embeddable into $\langle {}^\omega\omega, <^* \rangle$. Schematically proceeding as for the consistency of MA itself, Laver [66] in fact operatively showed: *For any cardinal κ satisfying $\kappa^{<\kappa} = \kappa$, there is a c.c.c. forcing extension in which $2^{\aleph_0} = \kappa$ and the saturated linear order of size $2^{\aleph_0}$ (i.e. the extant size $2^{\aleph_0}$ linear order into which every other linear order of size $\leq 2^{\aleph_0}$ embeds) is embeddable into $\langle {}^\omega\omega, <^* \rangle$.*

Laver's construction would have to do with the classical work of Hausdorff on order types and gaps at the beginnings of set theory. For a linear order $\langle L, < \rangle$ and $A, B \subseteq L$, $\langle A, B \rangle$ is a *gap iff* every element of A is $<$ any element of B yet there is no member of L $<$-between A and B. Such a gap is a (κ, λ^*)-*gap iff* A has $<$-increasing order type κ and B has $<$-decreasing order type λ. Hausdorff famously constructed what is now well-known as a "Hausdorff gap", an (ω_1, ω_1^*)-gap in $\langle {}^\omega\omega, <^* \rangle$ which is not fillable in any forcing extension preserving $\aleph_1$.

To establish his theorem Laver proceeded, in an iterative way with finite support, to adjoin $f_\alpha \in {}^\omega\omega$ so that $\langle \{f_\alpha \mid \alpha < \kappa\}, <^* \rangle$ will be the requisite saturated linear order. At stage β, if there is a gap $\langle A, B \rangle$ with $A \cup B = \{f_\alpha \mid \alpha < \beta\}$, Laver adjoined a generic f_β to fill the gap. As Laver astutely pointed out, his construction would have to avoid prematurely creating a Hausdorff gap, and it does so by iteratively creating a saturated linear order *generically* with finite support. Although Laver does not mention it, his construction affirmatively answered, consistency-wise, the first question of Hausdorff [39, §6]: Is there a pantachie with no (ω_1, ω_1^*) gaps? (For Hausdorff a *panachie* is a maximal linear sub-ordering of $\langle {}^\omega\mathbb{R}, <^* \rangle$, i.e. with the functions being real-valued, but Laver's construction can be adapted.) Historically, Hausdorff's question was the first in ongoing mathematics whose positive answer entailed $2^{\aleph_0} = 2^{\aleph_1}$ and hence the failure of the Continuum Hypothesis.

On topic, Woodin soon augmented Laver's construction to incorporate MA as well. This sharpened the situation, since as mentioned earlier Kunen had shown the consistency of MA and the proposition that there is a linear ordering of size $2^{\aleph_0}$ not embeddable into $\langle {}^\omega\omega, <^* \rangle$. Baumgartner [8, 4.5] later pointed out that the Proper Forcing Axiom directly implies this proposition.

A decade later Laver [80] pursued the study the space of functions $: \omega_1 \to \omega_1$ under eventual dominance modulo finite sets.

[9] See Dales and Woodin [17] for an account of Kaplansky's Conjecture, Woodin's reduction, and Woodin's own version of the relative consistency incorporating Martin's Axiom.

4.3 κ-Suslin trees

Laver and Shelah [68] showed: *If κ is weakly compact, then in a forcing extension $\kappa = \omega_2$,* CH*, and the $\aleph_2$-Suslin Hypothesis holds.* The proof establishes an analogous result for the successor of any regular cardinal less than κ. Laver had first established the result with "weakly compact" replaced by "measurable", and then Shelah refined the argument. This was the first result appropriately affirming a higher Suslin hypothesis, and as such would play an important, demarcating role in the investigation of generalized Martin's axioms.

A *κ-Aronszajn tree* is a tree with height κ all of whose levels have size less that κ yet there no chain (linearly ordered subset) of size κ; a *κ-Suslin tree* is an κ-Aronszajn tree with no antichain (subset of pairwise incomparable elements) of size κ as well; and the *κ-Suslin Hypothesis* asserts that there are *no* κ-Suslin trees. Without the "κ-" it is to be understood that $\kappa = \aleph_1$.

A classical 1920 question of Mikhail Suslin was shown to be equivalent to the Suslin Hypothesis (SH), and Nathan Aronszajn observed in the early 1930s that in any case there are Aronszajn trees. In the post-Cohen era the investigation of SH led to formative developments in set theory: Stanley Tennenbaum showed how to force ¬SH, i.e. to adjoin a Suslin tree; he and Solovay showed how to force ¬CH + SH with an inaugural multiple forcing argument, one that straightforwardly modified gives the stronger ¬CH + MA; Jensen showed that $V = L$ implies that there is a Suslin tree, the argument leading to the isolation of the diamond principle $\diamond$; and Jensen established the consistency of CH + SH, the argument motivating Shelah's eventual formulation of proper forcing.

With this esteemed, central work at $\kappa = \aleph_1$, Laver one level up faced the $\aleph_2$-Suslin Hypothesis. A contextualizing counterpoint was Silver's deduction through forcing developed by Mitchell (cf. Mitchell [98, p. 41]) that if κ is weakly compact, then in a forcing extension $\kappa = \omega_2$ and there are no $\aleph_2$-Aronszajn trees at all. But here CH fails, and indeed CH implies that there is an $\aleph_2$-Aronszajn tree. So, the indicated approach would be to start with CH, do forcing that adjoins no new reals and yet destroys all $\aleph_2$-Suslin trees, perhaps using a large cardinal.

In the Solovay-Tennenbaum approach, Suslin trees were destroyed one at a time by forcing through long chains; the conditions for a forcing were just the members of a Suslin tree under the tree ordering, and so one has the c.c.c., which can be iterated with finite support. One level up, one would have to have countably closed forcing (for preseving CH) that, iterated with countable support, would maintain the $\aleph_2$-c.c. (for preserving e.g. the necessary cardinal structure). However, Laver [68, p. 412] saw that there could be countably closed $\aleph_2$-Suslin trees whose product may not have the $\aleph_2$-c.c.

Laver then turned to the clever idea of destroying an $\aleph_2$-Suslin tree not by injecting a long chain but a large *antichain*, simply forcing with antichains under inclusion. But for this approach too, Laver astutely saw a problem. For a tree T, with its αth level denoted T_α, a *κ-ascent path* is a sequence of functions $\langle f_\alpha \mid \alpha \in A\rangle$ where A is an unbounded subset of $\{\alpha \mid T_\alpha \neq \emptyset\}$, each $f_\alpha : \kappa \to T_\alpha$, and: if $\alpha < \beta$ are both in A, then for sufficiently large $\xi < \kappa$, $f_\alpha(\xi)$ precedes $f_\beta(\xi)$ on the tree. Laver [68, p. 412] noted that if an $\aleph_2$-Suslin tree has an ω-ascent path, then the forcing for adjoining a large antichain does not satisfy the $\aleph_2$-c.c., and showed that it is relatively consistent to have an $\aleph_2$-Suslin tree with an ω-ascent path. In the subsequent elaboration of higher Suslin trees, the properties and constructions of trees with ascent paths became a significant topic in itself; cf. Cummings [16] from which the terminology is drawn.

Laver saw how, then, to proceed. With conceptually resonating precedents like [98], Laver first (Levy) collapsed a large cardinal κ to render it ω_2 and then carried out the iterative injection of large antichains to destroy $\aleph_2$-Suslin trees. The whole procedure is countably closed so that $2^{\aleph_0} = \aleph_1$ is preserved, and the initial collapse incorporates the κ-c.c. throughout to preserve κ as a cardinal.

Especially with this result in hand, the question arises, analogous to MA implying SH, whether there is a version of MA adapted to $\aleph_2$ that implies the $\aleph_2$-Suslin Hypothesis. Laver in 1973 was actually the first to propose a generalized Martin's axiom; Baumgartner in 1975 proposed another; and then Shelah [109] did also (cf. Tall [117, p. 216]). These various axioms are consistent (relative to ZFC) and can be incorporated into the Laver-Shelah construction. However, none of them can *imply* the $\aleph_2$-Suslin Hypothesis, since [108] soon showed, as part of extensive work on forcing principles and morasses, that 4CH + $\aleph_2$-Suslin Hypothesis implies that the (real) $\aleph_2$ is inaccessible in L. In particular, *some* large cardinal hypothesis is necessary to implement Laver-Shelah.

The Laver idea of injecting large antichains rather than long chains stands resilient; while generalized Martin's axioms do not apply to such forcings, the $\aleph_2$-Suslin Hypothesis can be secured. It is still open whether, analogous to Jensen's consistency of CH + SH, it is consistent to have CH + $2^{\aleph_1} = \aleph_2$ + $\aleph_2$-Suslin Hypothesis.

With respect to ($\aleph_1$-)Suslin trees, Shelah [111] in the early 1980s showed that forcing to add a single Cohen real actually adjoins a Suslin tree. This was a surprising result about the fragility of SH that naturally raised the question about other generic reals. After working off and on for several years, Laver finally clarified the situation with respect to Sacks and random reals.

As set out in Carlson–Laver [75], Laver showed that if CH, then adding a Sacks real forces $\diamond$, and hence that a Suslin tree exists, i.e. $\neg$SH. Tim Carlson specified a strengthening of MA, which can be shown consistent, and then showed that if it holds, then adding a Sacks real forces $\mathrm{MA}_{\aleph_1}$, Martin's Axiom for meeting $\aleph_1$ dense sets, and hence SH. Finally, Laver [73] showed that if $\mathrm{MA}_{\aleph_1}$ holds, then adding any number of random reals does *not* adjoin a Suslin tree, i.e. SH is maintained.

4.4 Nonregular ultrafilters

With his experience with saturated ideals and continuing interest in strong properties holding low in the cumulative hierarchy, Laver [70] in 1982 established substantial results about the existence of nonregular ultrafilters over ω_1. This work became a pivot point for possibility, as we emphasize by first describing the wake of emerging results, including Laver [61] on constructibility, and then the related subsequent work, tucking in a reference to the joint Foreman–Laver [74] on downwards transfer.

For present purposes, an ultrafilter U over κ which is uniform (i.e. every element of U has size κ) is *regular iff* there are κ sets in U any infinitely many of which have empty intersection. Regular ultrafilters were considered at the beginnings of the study of ultraproduct models in the early 1960s, in substantial part as they ensure large ultrapowers, e.g. if U over κ is regular, then its ultrapower of ω must have size 2^κ. With the expansion of set theory through the 1960s, the regularity of ultrafilters became topical, and [104] astutely established by isolating a combinatorial principle that holds in L that if $V = L$, then every ultrafilter over ω_1 is regular.

Can there be, consistently, a uniform nonregular ultrafilter over ω_1? Given the experience of saturated ideals and large cardinals, perhaps one can similarly collapse a large cardinal e.g. to ω_1 while retaining the ultrafilter property and the weak completeness property of nonregularity. This was initially stimulated by a result [46] of the author, that if there were such a nonregular ultrafilter over ω_1, then there would be one with the large cardinal-like property of being *weakly normal*: If $\{\alpha < \omega_1 \mid f(\alpha) < \alpha\} \in U$, then there is a $\beta < \omega_1$ such that $\{\alpha < \omega_1 \mid f(\alpha) < \beta\} \in U$. Using this, Ketonen [49] showed in fact that if there were such an ultrafilter, then $0^\#$ exists. Magidor [90] was first to establish the existence of a nonregular ultrafilter, showing that if there is a huge cardinal, then e.g. in a forcing extension there is a uniform ultrafilter U over ω_2 such that its ultrapower of ω has size only $\aleph_2$ and hence is nonregular.

Entering the fray, Laver first provided incisive commentary in a two-page paper [61] on Prikry's result [104] about regular ultrafilters in L. Jensen's principle $\diamondsuit^*$ is a strengthening of $\diamondsuit$ that he showed holds in L. Laver established: *Assume* $\diamondsuit^*$. *Then for every* $\alpha < \omega_1$, *there is a partition* $\{\xi \mid \alpha < \xi < \omega_1\} = X_{\alpha 0} \cup X_{\alpha 1}$ *such that for any function* $h \colon \omega_1 \to 2$ *there is an* $\aleph_1$ *size subset of* $\{X_{\alpha h(\alpha)} \mid \alpha < \omega_1\}$ *such that any countably many of these has empty intersection.* Thus, while Prikry had come up with a new combinatorial principle holding in L and used it to establish that every uniform ultrafilter over ω_1 is regular there, Laver showed that the combinatorial means had already been isolated in L, one that led to a short, elegant proof! Laver's proof, as does Prikry's, generalizes to get analogous results at all successor cardinals in L.

Laver [70] subsequently precluded the possibility that saturated ideals themselves could account for nonregular ultrafilters. First he characterized those κ-c.c. forcings that preserve κ^+-saturated κ-ideals, a result rediscovered and exploited by Baumgartner and Taylor [6]. Laver then applied this to show that if there is an $\aleph_2$-saturated ω_1-ideal, then in a forcing extension there is such an ideal and moreover every uniform ultrafilter over ω_1 is regular.

Laver [70] then answered the pivotal question by showing that, consistently, there can be a uniform nonregular ultrafilter over ω_1. Woodin had recently shown that starting from strong determinacy hypotheses a ZFC model can be constructed which satisfies: $\diamondsuit$ + "There is an ω_1-dense ω_1-ideal".[10] From this, Laver [70] established that it follows that there is a uniform nonregular ultrafilter over ω_1. In fact, he applied $\diamondsuit$ to show that the filter dual to such an ideal can be extended to an ultrafilter by just adding $\aleph_1$ sets and closing off. Such an ultrafilter must be nonregular, and in fact the size of its ultrapower of ω is only $\aleph_1$.

With this achievement establishing nonregular ultrafilters on the landscape, they later figured in central work that reduced the strong hypotheses needed to get strong

10 In the 1990s Woodin would reduce the hypothesis to (just) the Axiom of Determinacy.

properties to hold low in the cumulative hierarchy. Reorienting large cardinal theory, Foreman et al. [28] reduced the sufficient hypothesis for getting the consistency of an $\aleph_2$-saturated ω_1-ideal from Kunen's initial huge cardinal to just having a supercompact cardinal. Moreover, Foreman et al. [29] established that if there is a supercompact cardinal, then in a forcing extension there is a nonregular ultrafilter over ω_1, and that analogous results hold for successors of regular cardinals. It was in noted [47] by the author that both this and the Laver result could be refined to get ultrafilters with "finest partitions", which made evident that the size of their ultrapowers is small.

In extending work done by the summer of 1992, Foreman [30] showed that if there is a huge cardinal, then in a forcing extension there is an $\aleph_1$-dense ideal over ω_2 in a strong sense, from which it follows that there is a uniform ultrafilter over ω_2 such that its ultrapower of ω has size only $\aleph_1$.

Earlier, Foreman and Laver [74] by 1988 had incisively refined Kunen's original argument for getting an $\aleph_2$-saturated ω_1-ideal from a huge cardinal by incorporating Foreman's thematic κ-centeredness into the forcing to further get strong downwards transfer properties. A prominent such property was that every graph of size and chromatic number $\aleph_2$ has a subgraph of size and chromatic number $\aleph_1$. Foreman [30] showed that having a nonregular ultrafilter over ω_2 directly implies this graph downwards transfer property. This work still stands in terms of consistency strength in need not just of supercompactness but hugeness to get strong propositions low in the cumulative hierarchy.

4.5 Products of infinitely many trees

Laver [71] by 1983 established a striking partition theorem for infinite products of trees which, separate from being of considerable combinatorial interest, answered a specific question about possibilities for product forcing. The theorem is the infinite generalization of the Halpern–Laüchli Theorem [34], a result to which Laver in 1969 had arrived at independently in a reformulation, in presumably his first substantive result in set theory. He worked off and on for many years on the infinite possibility, and so finally establishing it must have been a particularly satisfying achievement.

For present purposes, a *perfect tree* is a tree of height ω such that every element has incomparable successors, and $T(n)$ denotes the n-level of a tree T. For $A \subseteq \omega$ and a sequence of trees $\langle T_i \mid i < d\rangle$, let $\bigotimes^A \langle T_i \mid i < d\rangle = \bigcup_{n\in A} \Pi_{i<d} T_i(n)$, the set of d-tuples across the trees at the levels indexed by A. Finally, for $d \leq \omega$ let HL_d be the proposition:

> If $\langle T_i \mid i < d\rangle$ is a sequence of perfect trees and $\bigotimes^\omega \langle T_i \mid i < d\rangle = G_0 \cup G_1$, then there are $j < 2$, infinite $A \subseteq \omega$, and downwards closed perfect subtrees T_i' of T_i for $i < d$ such that $\bigotimes^A \langle T_i' \mid i < d\rangle \subseteq G_j$.

That HL_d holds for $d < \omega$ is essentially the Halpern–Läuchli Theorem [34], which was established and applied to get a model for the Boolean Prime Ideal Theorem together with the failure of the Axiom of Choice.[11] Laver in 1969 from different

[11] cf. Halpern and Levy [35].

motivations (see below) and also David Pincus by 1974[12] arrived at a incisive "dense set" formulation from which HD_d readily follows. For a sequence of trees $\langle T_i \mid i < d\rangle$, $\langle X_i \mid i < d\rangle$ is *n-dense iff* for some $m \geq n$, $X_i \subseteq T_i(m)$ for $i < d$, and moreover, for $i < d$ every member of $T_i(n)$ is below some member of X_i. For $\overrightarrow{x} = \langle x_i \mid i < d\rangle \in \bigotimes^{\omega}\langle T_i \mid i < d\rangle$, $\langle X_i \mid i < d\rangle$ is $\overrightarrow{x}$*-n-dense iff* it is n-dense in $\langle (T_i)_{x_i} \mid i < d\rangle$, where $(T_i)_{x_i}$ is the subtree of T_i consisting of the elements comparable with x_i. Let LP_d be the proposition:

> If $\langle T_i \mid i < d\rangle$ is a sequence of perfect trees and $\bigotimes^{\omega}\langle T_i \mid i < d\rangle = G_0 \cup G_1$, then either (a) for all $n < \omega$ there is an n-dense $\langle X_i \mid i < d\rangle$ with $\bigotimes^{\omega}\langle X_i \mid i < d\rangle \subseteq G_0$, or (b) for some $\overrightarrow{x} = \langle x_i \mid i < d\rangle$ and all $n < \omega$ there is an $\overrightarrow{x}$-n-dense $\langle X_i \mid i < d\rangle$ with $\bigotimes^{\omega}\langle X_i \mid i < d\rangle \subseteq G_1$.

LP_d and HD_d for finite d are seen to be mutually derivable, but unaware of HD_d Laver in 1969 had astutely formulated and proved LP_d for finite d in order to establish a conjecture of Galvin. In the late 1960s at Berkeley, Galvin had proved that if the rationals are partitioned into finitely many cells, then there is a subset of the same order type η whose members are in at most two of the cells. Galvin then conjectured that if the r-element sequences of rationals are partitioned into finitely many cells, then there are sets of rationals $X_0, X_1, \ldots, X_{r-1}$ each of order type η such that the members of $\Pi_{i<r} X_i$ are in at most $r!$ of the cells. Laver while a graduate student affirmed this, soon after he had established Fraïssé's Conjecture.[13] Notably, Keith Milliken [97], in his UCLA thesis with Laver on the committee, applied LP_d to derive a "pigeonhole principle", actually a partition theorem in terms of "strongly embedded trees" rather than perfect subtrees.

Finally to Laver's [71] result after all this set up, he after years of returning to it finally established HL_ω, the infinite generalization of Halpern–Läuchli. With its topicality, Tim Carlson (cf. [12]) also established HL_ω in a large context of "dual Ramsey theorems". HL_ω is seen as an infinitary Ramsey theorem, but in any case, Laver had an explicit motivation from forcing, for Baumgartner had raised the issue of HL_ω in the late 1970s. Extending the combinatorics for Sacks reals and HL_d for finite d, Baumgartner had observed that HL_ω implies that when adding κ Sacks reals "side-by-side", i.e. with product forcing, any subset of ω contains or is disjoint from an infinite subset of ω in the ground model. This now became an impressive fact about the stability of product Sacks forcing.

In retrospect, what slowed Laver's progress to HL_ω was his inability to establish the ostensibly stronger LP_ω despite numerous attempts. He finally saw that by patching together a technical weakening of LP_ω, he could get to HL_ω. As he moved on to further triumphs, the one problem he would bequeath to set theory is the infinitary generalization of his earliest result in the combinatorics of the infinite: Does LP_ω hold?

[12] cf. Pincus and Halpern [103].

[13] All this is noted in Erdős and Hajnal [26, p. 275]. Laver latterly published his proof of Galvin's conjecture from LP_d as Theorem 2 in [71].

5 Embeddings of rank into rank

Some time in the mid-1980s, Laver [72] began tinkering with elementary embeddings $j: V_\delta \to V_\delta$, combining them and looking at how they move the ordinals. On the one hand, that there are such embeddings at all amounts to asserting a consistency-wise very strong hypothesis, and on the other hand, there was an algebraic simplicity in the play of endomorphisms and ordinals. Laver persisted through a proliferation of embeddings and ordinals moved to get at patterns and issues about algebras of embeddings. He [76] then made enormous strides in discerning a normal form, and, with it, getting at the freeness of the algebras, as well as the solvability of their word problems. Subsequently, Laver [78] was able to elaborate the structure of iterated embeddings and formulate new finite algebras of intrinsic interest. Laver not only brought in distinctively algebraic incentives into the study of strong hypotheses in set theory, but opened up separate algebraic vistas that stimulated a new cottage industry at this intersection of higher set theory and basic algebra. Moving on however, Laver [81,83] considerably clarified the situation with respect to even stronger embedding hypotheses, and eventually he [86] returned, remarkably, to something basic about forcing, that the ground model is definable in any generic extension.

In what follows, we delve forthwith into strong elementary embeddings and successively describe Laver's work. There is less in the way of historical background and less that can be said about the algebraic details, so the comparative brevity of this section belies to some extent the significance and depth of this work. It is assumed and to be implicit in the notation that elementary embeddings j are not the identity, and so, as their domains satisfy enough set-theoretic axioms, they have a *critical point* $\mathrm{cr}(j)$, a least ordinal α such that $\alpha < j(\alpha)$.

5.1 Algebra of embeddings

Kunen in 1970 had famously delimited the large cardinal hypotheses by establishing an outright inconsistency in ZFC, that there is no elementary embedding $j: V \to V$ of the universe into itself. The existence of large cardinals as strong axioms of infinity had turned on their being critical points of elementary embeddings $j: V \to M$ with M being larger and larger inner models, and Kunen showed that there is a limit to such formulations with M being V itself. Of course, it is all in the proof, and with $\kappa = \mathrm{cr}(j)$ and λ the supremum of $\kappa < j(\kappa) < j^2(\kappa) < \cdots$, Kunen had actually showed that having a certain combinatorial object in $V_{\lambda+2}$ leads to a contradiction. Several hypotheses just skirting Kunen's inconsistency were considered, the simplest being that $\mathcal{E}_\lambda \neq \emptyset$ for some limit λ, where $\mathcal{E}_\lambda = \{j \mid j: V_\lambda \to V_\lambda \text{ is elementary}\}$. The λ here is taken anew, but from Kunen's argument it is understood that if $j \in \mathcal{E}_\lambda$ and $\kappa = \mathrm{cr}(j)$, the supremum of $\kappa < j(\kappa) < j^2(\kappa) < \ldots$ would have to be λ.

Laver [72] in 1985 explored $\mathcal{E}_\lambda$, initially addressing definability issues, under two binary operations. Significantly, in this he worked a conceptual shift from critical points as large cardinals to the embeddings themselves and their interactions. One operation was *composition*: if $j, k \in \mathcal{E}_\lambda$, then $j \circ k \in \mathcal{E}_\lambda$. The other, possible as embeddings are sets of ordered pairs, was *application*: if $j, k \in \mathcal{E}_\lambda$, then $j \cdot k = \bigcup_{\alpha<\lambda} j(k \cap V_\alpha) \in \mathcal{E}_\lambda$

with $\mathrm{cr}(j \cdot k) = j(\mathrm{cr}(k))$. Application was first exploited by Martin [93], with these laws easily checked:

$$(\Sigma) \qquad i \circ (j \circ k) = (i \circ j) \circ k; \quad (i \circ j) \cdot k = i \cdot (j \cdot k);$$
$$i \cdot (j \circ k) = (i \cdot j) \circ (i \cdot k); \text{ and } i \circ j = (i \cdot j) \circ i.$$

From these follows the *left distributive law* for application: $i \cdot (j \cdot k) = (i \cdot j) \cdot (i \cdot k)$. A basic question soon emerged as to whether these are the only laws, and Laver a few years later in 1989 showed this. For $j \in \mathcal{E}_\lambda$, Let $\mathcal{A}_j$ be the closure of $\{j\}$ in $\langle \mathcal{E}_\lambda, \cdot \rangle$, and let $\mathcal{P}_j$ be the closure of $\{j\}$ in $\langle \mathcal{E}_\lambda, \cdot, \circ \rangle$. Laver [76] established: *$\mathcal{A}_j$ is the free algebra $\mathcal{A}$ with one generator satisfying the left distributive law, and $\mathcal{P}_j$ is the free algebra $\mathcal{P}$ with one generator satisfying Σ.*

Freeness here has the standard meaning. For $\mathcal{P}_j$ and $\mathcal{P}$, let W be the set of terms in one constant a in the language of $\cdot$ and $\circ$. Define an equivalence relation $\equiv$ on W by stipulating that $u \equiv v$ *iff* there is a sequence $u = u_0, u_1, \ldots, u_n = v$ with each u_{i+1} obtained from u_i by replacing a subterm of u_i by a term equivalent to it according to one of the laws of Σ. Then $\cdot$ and $\circ$ are well-defined for equivalence classes, and Laver's result asserts that the resulting structure on $W/\equiv$ and $\mathcal{P}_j$ are isomorphic via the map induced by sending the equivalence class of a to j.

For $u, v \in W$, define $u <_L v$ *iff* u is an iterated left divisor of v, in the sense that for some $w_1, \ldots, w_{n+1} \in W$,

$$v \equiv ((\cdots (u \cdot w_1) \cdot w_2) \cdot \cdots \cdot w_n) \cdot w_{n+1} \text{ , or}$$
$$v \equiv ((\cdots (u \cdot w_1) \cdot w_2) \cdot \cdots \cdot w_n) \circ w_{n+1} \text{ .}$$

When in the mid-1980s Laver was trying to understand the proliferation of critical points of members of $\mathcal{A}_j$, he had worked with equivalence relations on embeddings based on partial agreement and had shown that *if $\mathcal{E}_\lambda \neq \emptyset$ for some λ, then $<_L$ is irreflexive*, i.e. $u <_L u$ always fails. Assuming, then, the irreflexivity of $<_L$, Laver [76] showed that every equivalence class in $W/\equiv$ has a unique member in a certain normal form; that the lexicographic ordering of these normal forms is a linear ordering; that this lexicographic ordering then agrees with $<_L$; and hence that $<_L$ on W is a linear ordering. This structuring of the freeness leads to *the solvability of the word problem for $W/\equiv$*, i.e. there is an effective procedure for deciding whether or not $u \equiv v$ for arbitrary $u, v \in W$. For $\mathcal{A}$, with just one operation, there is no normal form, but Laver showed that $\mathcal{P}$ is conservative over $\mathcal{A}$ in that two terms in the language of $\cdot$ are equivalent as per the laws Σ exactly when they are equivalent as per just the left distributive law. Considerable interest was generated by a hypothesis bordering on the limits of consistency entailing solvability in finitary mathematics, particularly because of the peculiar and enticing possibility that some strong hypothesis may be necessary. By 1990 Laver [77] had extended his normal form result systematically to get, for any $p <_L q \in \mathcal{P}$, a unique, recursively defined "p-division form" equivalent to q, so that there is a $<_L$-largest p_0 with $p \cdot p_0 \leq_L q$, and one can conceptualize the process as a division algorithm.

Patrick Dehornoy, having been pursuing similar initiatives, made important contributions.[14] He first provided in some 1989 work an alternative proof [18] of Laver's [76] freeness and solvability results, one that does with less than the irreflexivity of $<_L$ at the cost of foregoing normal forms. Then in 1991, he [19,20] established that irreflexivity outright in ZFC by following algebraic incentives and bringing out a realization of $\mathcal{A}$ within the Artin braid group B_∞ with infinitely many strands. Consequently, the various structure results obtained about $W/\equiv$ by Laver [76] became theorems of ZFC.

The braid group connection was soon seen explicitly. The braid group B_n, with $2 \leq n \leq \infty$, is generated by elements $\{\sigma_i \mid 0 < i < n\}$ satisfying $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Define the "Dehornoy bracket" on B_∞ by: $g[h] = g\,\mathrm{sh}(h)\,\sigma_1\,(\mathrm{sh}(g))^{-1}$, where sh is the shift homomorphism given by $\mathrm{sh}(\sigma_i) = \sigma_{i+1}$. The Dehornoy bracket is left distributive, and one can assign to each $u \in A$ a $\overline{u} \in B_\infty$ by assigning the generator of A to σ_1, and recursively, $\overline{uv} = \overline{u}[\overline{v}]$. For the irreflexivity of $<_L$, assume that $u <_L u$. Then the corresponding assertion about $\overline{u}$ leads to a "σ_1-positive" element, an element with an occurrence of σ_1 but none of σ_1^{-1}, which represents the identity of B_∞. But one argues that this cannot happen, that B_∞ is torsion-free. Laver's student Larue [53] provided a straightforward argument of this last, and so a shorter, more direct proof of the irreflexivity of $<_L$.

The investigation of elementary embeddings continued full throttle, and this led strikingly to new connections and problems in finitary mathematics. Dougherty [24], coming forthwith to the scene, persisted with a detailed investigation of the proliferation of critical points. Fixing a $j \in \mathcal{E}_\lambda$ with $\kappa = \mathrm{cr}(j)$, define a corresponding function f on ω by:

$$f(n) = |\{\mathrm{cr}(k) \mid k \in \mathcal{A}_j \ \wedge \ j^n(\kappa) < \mathrm{cr}(k) < j^{n+1}(\kappa)\}| \,.$$

Laver had seen that $f(0) = 0$, $f(1) = 0$, and $f(2) = 1$, but that $f(3)$ is suddenly large because of application. Dougherty [24] established a large lower bound for $f(3)$ and showed moreover that f eventually dominates the Ackermann function, and hence cannot be primitive recursive.

In tandem, with the $f(n)$'s not even evidently finite, Laver [78] duly established that $f(n)$ is finite for all n. Let $j_{[1]} = j$, and $j_{[n+1]} = j_{[n]} \cdot j$. Then Laver showed that the sequence $\langle \mathrm{cr}(j_{[n]}) \mid n \in \omega \rangle$ enumerates, increasingly with repetitions, the first ω critical points of embeddings in $\mathcal{A}_j$. A general result of John Steel implies that the supremum of this sequence is $\lambda = \sup_n j^n(\kappa)$, so that the sequence must have all the critical points. Hence the Laver-Steel conclusion is that indeed the $f(n)$'s are finite.

Toward his result about the critical points of the $j_{[n]}$'s, Laver [78] worked with finite left distributive algebras, algebras which can be presented without reference to elementary embeddings. For $k \in \omega$, let $A_k = \{1, 2, \ldots, 2^k\}$ with the operation $*_k$ given by the cycling $a *_k 1 = a + 1$ for $1 \leq a < 2^k$ and $2^k *_k 1 = 1$, and the left distributive $a *_k (b *_k 1) = (a *_k b) *_k (a *_k 1)$. Then the $\langle A_k, *_k \rangle$ turn out to be *the* finite algebras satisfying the left distributive law as well as the cycling laws $a *_k 1 = a + 1$ for $1 \leq a < 2^k$ and $2^k *_k 1 = 1$. The multiplication table or "Laver table" for $k = 3$ is as follows:

[14] See Dehornoy [21] for an expository account of the eventually developed theory from his perspective, one from which material in what follows is drawn.

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Each element in A_k is periodic with period a power of 2. Laver in his asymptotic analysis of $\mathcal{A}_j$ showed that *for each $a \in \omega$, the period of a in A_k tends to infinity with* k, and e.g. *for every k the period of* 2 *in A_k is at least that of* 1.

Remarkably, such observations about these finite algebras are not known to hold just in ZFC. Appealing to the former's earlier work, Dougherty and Jech [23] did show that the number of computations needed to guarantee that the period of 1 in A_k grows faster in k than the Ackermann function. While Dehornoy's work had established the irreflexivity of $<_L$ and so the solvability the word problem in ZFC, it is a striking circumstance that Laver's arguments using a strong large cardinal hypothesis still stand for establishing some seemingly basic properties of the finite algebras A_k.

In [79] Laver impressively followed the trail into braid group actions. Dehornoy [20] had shown that the ordering $<_L$ naturally induces linear orderings of the braid groups. With the positive braids being those not having any inverses of the strands appearing, Laver applied his structural analysis of $<_L$ to show that for the braid groups B_n with n finite the Dehornoy ordering actually well-orders the positive braids. This would stand as a remarkable fact that would frame the emerging order theory of braid groups, and with a context set, e.g. Carlucci et al. [13] investigated unprovability results according to the order type of long descending sequences in the Dehornoy order.

Laver's last papers [85,87,88] were to be on left-distributivity. With John Moody, Laver [85] stated conjectures about the free left distributive algebras $\mathcal{A}^{(k)}$ extending $\mathcal{A}$ by having k generators. These conjectures, about how a comparison process must terminate, would establish the following, still open for $\mathcal{A} = \mathcal{A}^{(1)}$: If $w \in \mathcal{A}^{(k)}$, the set $\{u \in A^{(k)} \mid \exists v(u \cdot v = w)\}$ of (direct) left divisors of w is well-ordered by $<_L$. With his student Sheila Miller, Laver [87] applied his division algorithm [77] to get at the possibility of comparisons and well-orderings, establishing that in $\mathcal{A}$, if $ab = cd$ and a and b have no common left divisors and c and d have no common left divisors, the $a = c$ and $b = d$. Laver and Miller [88] further simplify the division algorithm and provide a mature account of the theory of left distributive algebras.

5.2 Implications between very large cardinals

In the later 1990s Laver [81,83,86], moving on to higher pastures, developed the definability theory of elementary embedding hypotheses even stronger than $\mathcal{E}_\lambda \neq \emptyset$, getting into the upper reaches near Kunen's inconsistency, reaches first substantially broached by Woodin for consistency strength in the 1980s.

Kunen's inconsistency argument showed in a sharp form that in ZFC there is no elementary embedding $j: V_{\lambda+2} \to V_{\lambda+2}$. Early on, the following "strongest hypotheses" approaching the known inconsistency were considered, the last setting the stage for Laver's investigation of the corresponding algebra of embeddings.

$E_\omega(\lambda)$: There is an elementary $j: V_{\lambda+1} \to V_{\lambda+1}$.
$E_1(\lambda)$: There is an elementary $j: V \to M$ with $\mathrm{cr}(j) < \lambda = j(\lambda)$
and $V_\lambda \subseteq M$.
$E_0(\lambda)$: There is an elementary $j: V_\lambda \to V_\lambda$. ($\mathcal{E}_\lambda \neq \emptyset$.)

In all of these it is understood that, with $\mathrm{cr}(j) = \kappa$ the corresponding large cardinal, λ must be the supremum of $\kappa < j(\kappa) < j^2(\kappa) < \cdots$ by Kunen's argument. Thrown up *ad hoc* for stepping back from inconsistency, these strong hypotheses were not much investigated except in connection with a substantial application by Martin [93] to determinacy.

In work dating back to his first abstract [72] on embeddings, Laver [81] established a hierarchy up through $E_\omega(\lambda)$ with definability. In the language for second-order logic with $\in$, a formula is Σ^1_0 if it contains no second-order quantifiers and is Σ^1_n if it is of the form $\exists X_1 \forall X_2 \cdots Q_n X_n \Phi$ with second-order variables X_i and Φ being Σ^1_0. A $j: V_\lambda \to V_\lambda$ is Σ^1_n *elementary* if for any Σ^1_n Φ in one free second-order variable and $A \subseteq V_\lambda$, $V_\lambda \models \Phi(A) \leftrightarrow \Phi(j(A))$.

It turns out that $E_1(\lambda)$ above is equivalent to having an elementary $j: V_\lambda \to V_\lambda$ which is Σ^1_1. Also, if there is an elementary $j: V_\lambda \to V_\lambda$ which is Σ^1_n for every n, then j witnesses $E_\omega(\lambda)$. Incorporating these notational anticipations, define:

$$E_n(\lambda) : \text{There is a}\,\Sigma^1_n\text{elementary}\, j: V_\lambda \to V_\lambda.$$

Next, say that for parametrized large cardinal hypotheses, $\Psi_1(\lambda)$ *strongly implies* $\Psi_2(\lambda)$ if for every λ, $\Psi_1(\lambda)$ implies $\Psi_2(\lambda)$ and moreover there is a $\lambda' < \lambda$ such that $\Psi_2(\lambda')$.

Taking compositions and inverse limits of embeddings, Laver [72,81] established that in the sequence,

$$E_0(\lambda),\ E_3(\lambda),\ E_5(\lambda),\ \ldots,\ E_\omega(\lambda)$$

each hypothesis strongly implies the previous ones, each $E_{n+2}(\lambda)$ in fact providing for many $\lambda' < \lambda$ such that $E_n(\lambda')$ as happens in the hierarchy of large cardinals. Martin had essentially shown that any Σ^1_{2n+1} elementary $j: V_\lambda \to V_\lambda$ is Σ^1_{2n+2}, so Laver's results complete the hierarchical analysis for second-order definability.

In [81] Laver reached a bit higher in his analysis, and in [83] he went up to a very strong hypothesis formulated by Woodin:

$W(\lambda)$: There is an elementary $j: L(V_{\lambda+1}) \to L(V_{\lambda+1})$

with $\mathrm{cr}(j) < \lambda$.

That $W(\lambda)$ holds for some λ, just at the edge of the Kunen inconsistency, was formulated by Woodin in 1984, and, in the first result securing a mooring for the Axiom of Determinacy (AD) in the large cardinals, shown by him to imply that the axiom holds in the inner model $L(\mathbb{R})$, $\mathbb{R}$ the reals. Just to detail, $L(\mathbb{R})$ and $L(V_{\lambda+1})$ are constructible closures, where the *constructible closure* of a set A is the class $L(A)$ given by $L_0(A) = A$; $L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(A))$, the first-order definable subsets of $L_\alpha(A)$; and $L(A) = \bigcup_\alpha L_\alpha(A)$. Woodin, in original work, developed and pursued an analogy between $L(\mathbb{R})$ and $L(V_{\lambda+1})$ taking V_λ to be the analogue of ω and $V_{\lambda+1}$ to be analogue of $\mathbb{R}$ and established AD-like consequences for $L(V_{\lambda+1})$ from $W(\lambda)$.[15]

Consider an elaboration of $W(\lambda)$ according to the constructible hierarchy $L(V_{\lambda+1}) = \bigcup L_\alpha(V_{\lambda+1})$:

$W_\alpha(\lambda)$: There is an elementary $j\colon L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ with $\mathrm{cr}(j) < \lambda$.

Laver in his [81] topped his hierarchical analysis there by showing that $W_1(\lambda)$ strongly implies $E_\omega(\lambda)$, again in a strong sense. In [83] Laver impressively engaged with some of Woodin's work with $W(\lambda)$ to extend hierarchical analysis into the transfinite, showing that $W_{\lambda^+ + \omega+1}(\lambda)$ strongly implies $W_{\lambda^+}(\lambda)$ and analogous results with the "λ^+" replaced e.g. by the supremum of all second-order definable prewellorderings of $V_{\lambda+1}$.

In the summarizing [84], Laver from his perspective set out the landmarks of the work on elementary embeddings as well as provided an outline of Woodin's work on the AD-like consequences of $W(\lambda)$, and stated open problems for channeling the further work. Notably, Laver's speculations reached quite high, beyond $W(\lambda)$; Woodin [123, p. 117] mentioned "Laver's Axiom", an axiom providing for an elementary embedding to provide an analogy with the strong determinacy axiom $\mathrm{AD}_{\mathbb{R}}$.

In what turned out to be his last paper in these directions, Laver [86] in 2004 established results about the large cardinal propositions $E_n(\lambda)$ for $n \le \omega$ and forcing. For each $n \le \omega$, let $E_n(\kappa, \lambda)$ be $E_n(\lambda)$ further parametrized by specifying the large cardinal $\kappa = \mathrm{cr}(j)$. Laver established: *If $V[G]$ is a forcing extension of V via a forcing poset of size less than κ and $n \le \omega$, then $V[G] \models \exists\lambda E_n(\kappa, \lambda)$ implies $V \models \exists\lambda E_n(\kappa, \lambda)$.* The converse direction follows by well-known arguments about small forcings preserving large cardinals. The Laver direction is not surprising, on general grounds that consistency strength should not be created by forcing. However, as Laver notes by counterexamples, a λ satisfying $E_n(\kappa, \lambda)$ in $V[G]$ need not satisfy $E_n(\kappa, \lambda)$ in V, and a j witnessing $E_n(\kappa, \lambda)$ in $V[G]$ need not satisfy $j \restriction V_\lambda \in V$.

Laver established his result by induction on n deploying work from [81], and what he needed at the basis and first proved is the lemma: *If $V[G]$ is a forcing extension of V via a forcing poset of size less than κ and j witnesses $E_0(\kappa, \lambda)$, then $j \restriction V_\alpha \in V$ for every $\alpha < \lambda$.* Laver came up with a proof of this using a result $(**)$ he proved about models of ZFC that does not involve large cardinals, and this led to a singular development.

[15] This work would remain unpublished by Woodin. On the other hand, in his latest work [123] on suitable extenders Woodin considerably developed and expanded the $L(V_{\lambda+1})$ theory with $W(\lambda)$ in his quest for an ultimate inner model.

As Laver [81] described it, Joel Hamkins pointed out how the methods of his [38], also on extensions not creating large cardinals, can establish (∗∗) in a generalized form, and Laver wrote this out as a preferred approach. Motivated by Hamkins [38], Laver, and Woodin independently and in his scheme of things, established *ground model definability*: *Suppose that V is a model of* ZFC*, $P \in V$, and $V[G]$ is a generic extension of V via P. Then in $V[G]$, V is definable from a parameter.* (With care, the parameter could be made $\mathcal{P}^{V[G]}(P)$ through Hamkins' work.)

The ground model is definable in any generic extension! Although a parameter is necessary, this is an illuminating result about forcing as method. Was this issue raised decades earlier at the inception of forcing? In truth, for a particular forcing a term $\dot{V}$ can be introduced into the forcing language for assertions about the ground model, so there may not have been an earlier incentive. The argumentation for ground model definability provided *one* formula that defines the ground model in any generic extension in terms of a corresponding parameter. Like Laver indestructibility, this available uniformity stimulated renewed investigations and conceptualizations involving forcing.

Motivated by ground model definability, Hamkins and Jonas Reitz formulated the Ground Axiom: The universe of sets is not the forcing extension of any inner model W by a (nontrivial) forcing $P \in W$. Reitz [106] investigated this axiom, and [36] established its consistency with $V \neq$ HOD. [32] then extended the investigations into "set-theoretic geology", digging into the remains of a model of set theory once the layers created by forcing are removed. On his side, Woodin [122, §8] used ground model definability to formalize a conception of the "generic-multiverse"; the analysis here dates back to 2004. The definability is a basic ingredient in his latest work [121] toward an ultimate inner model.

Ground model definability serves as an apt and worthy capstone to a remarkable career. It encapsulates the several features of Laver's major results that made them particularly compelling and potent: it has a succinct basic-sounding statement, it nonetheless requires a proof of substance, and it gets to a new plateau of possibilities. With it, Laver circled back to his salad days.

6 Envoi

Let me indulge in a few personal reminiscences, especially to bring out more about Rich Laver.

A long, long time ago, I was an aspiring teenage chess master in the local San Francisco chess scene. It was a time fraught with excitement and inventiveness, as well as encounters with eclectic, quirky personalities. In one tournament, I had a gangly opponent who came to the table with shirt untucked and opened 1.g4, yet I still managed to lose. He let on that he was a graduate student in mathematics, which mystified me at the time (what's new in subtraction?).

During my Caltech years, I got wind that Rich Laver was on the UC Berkeley team that won the national collegiate chess championship that year. I eventually saw a 1968 game he lost to grandmaster Pal Benko when the latter was trotting out his gambit, a game later anthologized in [9]. A mutual chess buddy mentioned that Laver had told him that his thesis result could be explained to a horse—only years later did I take in that he had solved Fraïssé's Conjecture.[16]

In 1971 when Rich was a post-doc at Bristol and heard that I was up at King's College, Cambridge, he started sending me postcards. In one he suggested meeting up at the big Islington chess tourney (too complicated) and in others he mentioned his results and problems about partition relation consequences of MA. I was just getting up to speed, and still could not take it in.

A few years later, I was finally up and running, and when I sent him my least function result [46] for nonregular ultrafilters, he was very complementary and I understood then that we were on par. When soon later I was writing up the Solovay-Reinhardt work on large cardinals, Laver pointedly counseled me against the use of the awkward "n-hypercompact" for "n-huge", and I forthwith used the Kunen term.

By then at Boulder, Rich would gently suggest going mountain climbing, but I would hint at a constitutional reluctance. He did mention how he was a member of a party that took Paul Erdős up a Flatiron (mountain) near Boulder and how Erdős came in his usual light beige clothes and sandals. In truth, our paths rarely crossed as I remained on the East Coast. Through the 1980s Rich would occasionally send me preprints, sometimes with pencil scribblings. One time, he sent me his early thinking about embeddings of rank into rank. Regrettably, I did not follow up.

The decades went by with our correspondence turning more and more to chess, especially fanciful problems and extraordinary grandmaster games. In a final email to me, which I can now time as well after the onset of Parkinson's, Rich posed the following chess problem: Start with the initial position and play a sequence of legal moves until Black plays 5 . . . N × R mate. I eventually figured out that the White king would have to be at f2, and so sent him: 1. f3, Nf6 2. Kf2, Nh5 3. d3, Ng3 4. Be3, a6 5. Qe1, N × R mate. But then, Rich wrote back, now do it with an intervening check! This new problem kicked around in my mental attic for over a year, and one bright day I saw: 1. f3, Nf6 2. e4, N × e4 3. Qe2, Ng3 4. Q × e7ch, Q × Qch 5. Kf2, N × R mate. But by then it was too late to write him.

7 Doctoral students of Richard Laver

Stephen Grantham, *An analysis of Galvin's tree game*, 1982.
Carl Darby, *Countable Ramsey games and partition relations*, 1990.
Janet Barnett, *Cohen reals, random reals and variants of Martin's Axiom*, 1990.
Emanuel Knill, *Generalized degrees and densities for families of sets*, 1991.
David Larue, *Left-distributive algebras and left-distributive idempotent algebras*, 1994.
Rene Schipperus, *Countable partition ordinals*, 1999.
Sheila Miller, *Free left distributive algebras*, 2007.

[16] According to artsandsciences.colorado.edu/magazine/2012/12/by-several-calculations-a-life-well-lived/ the crucial point came to Laver in an epiphanous moment while he, mountain climbing, was stranded for a night "on a ledge in darkness" at Yosemite.

In addition to having these doctoral students at Boulder, Laver was on the thesis committees of, among many: Keith Devlin (Bristol), Maurice Pouzet (Lyon), Keith Milliken (UCLA), Joseph Rebholz (UCLA), Carl Morgenstern (Boulder), Stewart Baldwin (Boulder), Steven Leth (Boulder), Kai Hauser (Caltech), Mohammed Bekkali (Boulder), and Serge Burckel (Caen).

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