THE COMPLEAT $0^+$

Akihiro Kanamori

Soon after Solovay [So] and Silver [Si] established the basic transcendence properties over $L$ of the existence of $0^+$, Solovay formulated an analogous set of integers $0^+$ ("zero dagger") for inner models of measurability. Its definition became known through a timely survey of set theory by Mathias which appeared in typescript in 1968 and later as Mathias [M]. Results about $0^+$ appeared in Kunen[K], and it was further described in Dodd[Do]. Here, we finally provide a detailed presentation of the theory of $0^+$, establishing intimate connections between the various generating classes of indiscernibles for inner models of measurability.

As for the ambient context, we assume familiarity with the theory of $0^+$. Kanamori-Magidor[KM], Jech[J], and Devlin[De] all provide the necessary details; we shall follow the development of the first. We also assume familiarity with the basic inner model structure theory from Kunen[K]. In order to establish some terminology we review the major results: An inner model of measurability is an inner model of ZFC of form $L[U]$, where for some ordinal $\kappa$, $L[U] \models U$ is a normal ultrapower over $\kappa$. Incorporating $U$ as a predicate, we call $(L[U], c, U)$ the $\kappa$-model, since it is known that the only dependence is on $\kappa$: If $L[U]$ and $L[U']$ are both inner models of measurability for the same $\kappa$, then $U = U'$. For convenience, we say that $(L[U], c, \kappa, \kappa, i_{\alpha\beta})_{\alpha < \beta < \omega_1}$ is the iteration of $(L[U], c, U)$ meaning that the $\alpha$th iterated ultrapower of $(L[U], c, U)$ is the $\kappa_{\alpha}$-model $(L[U_{\alpha}], c, U_{\alpha})$ with $i_{\alpha\beta}$ the corresponding elementary embedding of the $\kappa_{\alpha}$-model into the $\kappa_{\beta}$-model. The $\kappa_{\alpha}$s comprise a closed unbounded class of ordinals, and every $\rho$-model for $\rho > \kappa$ appears in the iteration.

As for the organization of this paper, in §1 we formulate the necessary Ehrenfeucht-Mostowski theory, show how a sufficiently strong hypothesis generates inner models of measurability with indiscernibles, and formulate $0^+$. In §2 we establish connections between the classes of indiscernibles for the $\kappa$-models for various $\kappa$. Finally, in §3 we review various characterizations of the existence of $0^+$.

§1 Indiscernibles for the $\kappa$-models

If $(L[U], c, U)$ is the $\kappa$-model for some ordinal $\kappa$, there exists under sufficient assumptions a set $U^\# \subseteq \kappa$ analogous to $0^+$ that generates a closed unbounded class of indiscernibles for the structure $(L[U], c, U, \xi)_{\xi \leq \kappa}$. However, as the $\kappa$-models for various $\kappa$ are merely iterated ultrapowers of each other, one might expect a unifying transcendence principle. This is successfully realized by the existence of the set of integers $0^+$. The basic idea behind $0^+$ is to develop a canonical theory for structures of form $(L[U], c, U) \models "U is a normal ultrapower over \kappa"$ with two sets of indiscernibles, one below $\kappa$ and one above, that together generate the structure. Proceeding to the development, we follow the main steps of Kanamori-Magidor[KM]§7.

If $A$ is a structure and $X$ and $Y$ are subsets of the domain of $A$ so that $X \cup Y$ is linearly ordered by a relation $<$, then $(X, Y, <)$ (or in context, just $(X, Y)$) is a double set of
indiscernibles for \( A \) iff for every formula \( \psi(v_1, \ldots, v_{n+s}) \) in the language of \( A \); \( x_1 < \ldots < x_n \) and \( \bar{x}_1 < \ldots < \bar{x}_n \) all \( \in X \); and \( y_1 < \ldots < y_s \) and \( \bar{y}_1 < \ldots < \bar{y}_s \) all \( \in Y \),

\[ A \models \psi[x_1, \ldots, x_n, y_1, \ldots, y_s] \iff A \models \psi[\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_s]. \]

We next recall that there is a formula \( \phi(v_0, v_1) \) that for any set \( A \) defines in \( L[A] \) a well-ordering \( <_{L[A]} \) of \( L[A] \) such that: for any limit \( \delta > \omega \) and \( x, y \in L_\delta[A] \), \( x <_{L[A]} y \) iff \( \langle L_\delta[A], \in, A \cap L_\delta[A] \rangle \models \phi(x, y) \). In such \( \langle L_\delta[A], \in, A \cap L_\delta[A] \rangle \) we can define Skolem functions for every formula \( \psi \) by taking \( <_{L[A]} \)-least witnesses; it is crucial to observe that using \( \phi \), the definition of the Skolem function for \( \psi \) can be taken to be the same for all such structures. Consequently, if a structure \( \langle M, E, R \rangle \) is elementarily equivalent to one of form \( \langle L_\delta[A], \in, A \rangle \) for some limit ordinal \( \delta > \omega \), then for any \( X \subseteq M \) we can consider the Skolem hull of \( X \) in \( \langle M, E, R \rangle \) to be well-defined, and given by Skolem terms closed under composition.

Let \( \overline{L} \) be the language of set theory augmented by one unary predicate symbol \( \dot{U} \), and let \( \overline{L}^* \) be \( \overline{L} \) further augmented by new constants \( \{ c_k \mid k \in \omega \} \cup \{ d_k \mid k \in \omega \} \). By an EM blueprint in this context we mean the theory in \( \overline{L}^* \) of some structure

\[ \langle L_\zeta[U], \in, U, x_k, y_k \rangle_{k \in \omega} \]

where \( \zeta \) a limit ordinal \( > \omega \); for some ordinal \( \kappa \), \( \langle L_\zeta[U], \in, U \rangle \models U \) is a normal ultrafilter over \( \kappa \); and \( \langle x_k \mid k \in \omega \rangle \) and \( \langle y_k \mid k \in \omega \rangle \) are ascending sequences of ordinals such that \( \langle \langle x_k \mid k \in \omega \rangle, \langle y_k \mid k \in \omega \rangle \rangle \) is a double set of indiscernibles for \( \langle L_\zeta[U], \in, U \rangle \) satisfying

\[ x_k < \kappa < y_k \text{ for every } k. \]

A basic observation to keep in mind is that any structure \( \langle L_\zeta[U], \in, U \rangle \) where \( \zeta \) a limit ordinal \( > \omega \) and \( \langle L_\zeta[U], \in, U \rangle \models U \) is a normal ultrafilter over \( \kappa \), a double set \( \langle X, Y \rangle \) of ordinal indiscernibles satisfying \( X \subseteq \kappa \) and \( Y \cap (\kappa + 1) = \emptyset \) uniquely determines an EM blueprint, so long as \( X \) and \( Y \) are both infinite.

The next two lemmata are as for \( 0^\# \). For any theory \( T \) in \( \overline{L}^* \), let \( T^- \) denote its restriction to \( \overline{L} \). Note for here and later that if \( \dot{U} \) is interpreted by a normal ultrafilter in a structure, then \( \cup \dot{U} \) is a way of denoting the corresponding measurable cardinal in the structure.

1.1 Lemma: Suppose that \( T \) is an EM blueprint. Then for any \( \alpha \) and \( \gamma \), there is a model \( M = M(T, \alpha, \gamma) \) of \( T^- \) unique up to isomorphism such that:

(a) There is a double set \( \langle X, Y \rangle \) of indiscernibles for \( M \) with \( X \cup Y \subseteq O_n^M \), \( X \) of ordertype \( \alpha \) and \( Y \) of ordertype \( \gamma \) under \( <^M \), and \( x <^M \cup \dot{U}^M <^M y \) for every \( x \in X \) and \( y \in Y \). Moreover, for any formula \( \psi(v_1, \ldots, v_{n+s}) \) in \( \overline{L} \), \( x_1 <^M \ldots <^M x_n \) all \( \in X \), and \( y_1 <^M \ldots <^M y_s \) all \( \in Y \),

\[ M \models \psi[x_1, \ldots, x_n, y_1, \ldots, y_s] \iff \psi(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in T. \]
(b) The Skolem hull of $X \cup Y$ in $\mathcal{M}$ is again $\mathcal{M}$. 

If $\mathcal{M}(T, \alpha, \gamma)$ is well-founded, then its transitive collapse is of form $(L_\delta[U], \in, U)$ for some limit ordinal $\delta > \omega$. In this case,

we identify $\mathcal{M}(T, \alpha, \gamma)$ with $(L_\delta[U], \in, U)$.

1.2 Lemma: Suppose that $T$ is an EM blueprint. Then $\mathcal{M}(T, \alpha, \gamma)$ is well-founded for every $\alpha, \gamma$ iff

(I) $\mathcal{M}(T, \alpha, \gamma)$ is well-founded for every $\alpha, \gamma < \omega_1$.

We next describe a sufficient hypothesis that leads quickly to EM blueprints with potent properties.

1.3 Lemma: Suppose that there is a $\kappa$-model for some ordinal $\kappa$ and a Ramsey cardinal $\lambda > \kappa$. Then there is an EM blueprint satisfying (I) of 1.2.

Proof: Let $(L[U], \in, U)$ be the $\kappa$-model with iteration $(L[U_\alpha], U_\alpha, \kappa_\alpha, \iota_{\alpha\beta})_{\alpha \leq \beta \in \text{On}}$, and $\nu$ a Ramsey cardinal $\lambda > \kappa$. Let $\lambda$ be a cardinal such that $\kappa_\lambda = \lambda < \nu$. (This is for later extractions from this proof; $\lambda = \omega_1$ suffices for present purposes.) In what follows, we rely on Kunen[70]3.9 concerning how the $\iota_{\alpha\beta}$’s move ordinals. The set

$$Z = \{ \theta < \nu \mid \theta > \lambda \land i_{\iota_\lambda}(\theta) = \theta \}$$

has cardinality $\nu$, so let $Y \in [Z]^\nu$ be a set of indiscernibles for the structure

$$(L_\nu[U_\lambda], \in, U_\lambda, \kappa_\nu)_{\nu \in \omega}.$$ 

Set $X = \{ \kappa_\alpha \mid \alpha < \lambda \}$. $i_{\iota_\lambda}$ fixes every member of $Z \cup \{ \nu \}$, so Kunen[70]3.3 implies that $X$ is a set of indiscernibles for $(L_\nu[U_\lambda], \in, U_\lambda)$ allowing parameters from $Z$. Consequently, a simple argument shows that $(X, Y)$ is a double set of indiscernibles for $(L_\nu[U_\lambda], \in, U_\lambda)$. Hence, $(X, Y)$ determines an EM blueprint, and since $X$ and $Y$ are uncountable, this EM blueprint satisfies (I) by an argument as for $0^\#$. 

Assuming the hypothesis of 1.3, we can deduce the existence of an EM blueprint fully analogous to $0^\#$. (Actually, a weaker partition property than Ramsey will do, but this involves distracting technical details.) On the basis of the proof of 1.3, specify that

(i) $\lambda < \nu$ are uncountable cardinals (in $V$) and $(L_\nu[U], \in, U) \models U$ is a normal ultrafilter over $\lambda$,

(ii) $(X, Y)$ is a double set of indiscernibles with $X \in [\lambda]^\lambda$ and $Y \in [\nu \sim (\lambda + 1)]^\nu$, and
(iii) $X$ and $Y$ have the least possible $\omega$th elements. Finally,
(iv) $T_0$ is the corresponding EM blueprint.

1.4 Lemma: The following conditions hold for $T = T_0$:

(IIa) For any $n + s$-ary Skolem term $t$, $T$ contains the sentence:

$$t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in \bigcup \hat{U} \rightarrow t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) < c_n.$$  

(IIb) For any $n + s$-ary Skolem term $t$, $T$ contains the sentence:

$$t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in \Omega \rightarrow t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) < d_s.$$  

(IIIa) For any $m + n + s + 1$-ary Skolem term $t$, $T$ contains the sentence:

$$t(c_0, \ldots, c_{m+n}, d_0, \ldots, d_{s-1}) < c_m \rightarrow$$

$$t(c_0, \ldots, c_{m+n}, d_0, \ldots, d_{s-1}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1}, d_0, \ldots, d_{s-1}).$$  

(IIIb) For any $n + r + s + 1$-ary Skolem term $t$, $T$ contains the sentence:

$$t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{r+s}) < d_r \rightarrow$$

$$t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{r+s}) = t(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{r-1}, d_{r+s+1}, \ldots, d_{r+2s+1}).$$  

Proof: A simple argument by contradiction establishes (IIa) from $\text{sup}(X) = \lambda$ and similarly, (IIb) from $\text{sup}(Y) = \nu$. An argument as for $0^\#$ establishes (IIIa) and (IIIb) from the minimality of the $\omega$th elements of $X$ and $Y$ respectively and the fact that $\lambda$ and $\nu$ are cardinals. \(\square\)

If an EM blueprint satisfies (I), then for any $\alpha, \gamma$ temporarily let

$$\langle \chi^T_{\eta, \alpha, \gamma} \mid \eta < \alpha \rangle, \langle \xi^T_{\alpha, \gamma} \mid \xi < \gamma \rangle$$

and $\chi^T_{\alpha, \gamma}$ denote the double set of indiscernibles and the measurable cardinal of $\mathcal{M}(T, \alpha, \gamma)$ respectively. The $\chi$ notation anticipates the following:

1.5 Lemma: Suppose that $T$ is an EM blueprint satisfying (I)-(III). Then for any $\beta, \delta$:

(a) $\{\chi^T_{\eta, \beta, \delta} \mid \eta < \beta \}$ is a closed set of ordinals, unbounded in $\chi^T_{\beta, \beta, \delta}$ if $\beta$ is a limit ordinal $\geq \omega$.

(b) If $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$ with $\alpha, \gamma$ limit ordinals, then

$$\chi^T_{\eta, \alpha, \gamma} = \chi^T_{\eta, \beta, \delta}$$

for every $\eta \leq \alpha$.

Proof: (a)(IIa) implies that the set is unbounded in $\chi^T_{\beta, \beta, \delta}$ if $\beta$ is a limit ordinal $\geq \omega$. (IIIa) implies that the set is closed (by an argument as for $0^\#$, also used below).
(b) Let $H$ be the Skolem hull of
\[ \{ \chi^T_{\eta,\beta,\delta} \mid \eta < \alpha \} \cup \{ \iota^T_{\xi,\beta,\delta} \mid \xi < \gamma \} \]
in $M(T, \beta, \delta)$. Then its transitive collapse is $M(T, \alpha, \gamma)$ by uniqueness. We shall show that $\chi^T_{\alpha,\beta,\delta} \subseteq$ the domain of $H$. This suffices, since the collapsing isomorphism consequently fixes every member of $\{ \chi^T_{\eta,\beta,\delta} \mid \eta < \alpha \}$ making this set the lower set of the double set of indiscernibles for $M(T, \alpha, \gamma)$, and also $\chi^T_{\alpha,\gamma} = \chi^T_{\alpha,\beta,\delta}$ by (a) for $\alpha$ as well as for $\beta$.

To show that $\chi^T_{\alpha,\beta,\delta} \subseteq$ the domain of $H$, let $\sigma < \chi^T_{\alpha,\beta,\delta}$ be arbitrary. Suppressing the superscript $T_{\beta,\delta}$ from our indiscernibles for convenience,
\[ \sigma = t^{M(T, \beta, \delta)}[\chi_{\eta_0}, \ldots, \chi_{\eta_{m-1}}, \chi_{\xi_0}, \ldots, \chi_{\xi_n}, \iota_{\xi_0}, \ldots, \iota_{\xi_{s-1}}] \]
for some Skolem term $t$ and the indiscernibles listed in ascending order with $\eta_{m-1} < \alpha \leq \xi_0$. Applying (IIb) with $r = 0$ we can replace $\xi_i$ by $i$ for $i < s$, and then applying (IIa) we can replace $\xi_i$ by $\eta_{m-1} + i$ for $i < n + 1$. Since $\alpha$ and $\gamma$ are limit ordinals, the resulting expression shows that $\sigma$ is in the domain of $H$. $\dagger$

1.6 Lemma: Suppose that $T$ is an EM blueprint satisfying (I)-(III), $\omega \leq \gamma \leq \delta$, and $\gamma$ and $\alpha$ are limit ordinals (allowing $\alpha = 0$). Then if $M(T, \alpha, \delta) = \langle L_\xi[D], \epsilon, D \rangle$, say, the Skolem hull of

\[ \{ \chi^T_{\eta,\alpha,\delta} \mid \eta < \alpha \} \cup \{ \iota^T_{\xi,\alpha,\delta} \mid \xi < \gamma \} \]
in $M(T, \alpha, \delta)$ is $\langle L_\xi[D], \epsilon, D \cap L_\xi[D] \rangle$, where $\iota = \iota^T_{\gamma,\alpha,\delta}$. Consequently,

$M(T, \alpha, \gamma) = \langle L_\xi[D], \epsilon, D \cap L_\xi[D] \rangle$ and $\iota^{T,\alpha,\gamma} = \iota^{T,\alpha,\delta}$ for every $\xi < \gamma$.

Proof: Let $H$ be the stated Skolem hull. Then its transitive collapse is $M(T, \alpha, \gamma)$ by uniqueness. By the argument for 1.5(b) with $\alpha = \beta$ (and a simple version for $\alpha = 0$), $\chi^T_{\alpha,\beta,\delta} \subseteq$ the domain of $H$, so that $M(T, \alpha, \gamma)$ must be of form $\langle L_\xi[D], \epsilon, D \cap L_\xi[D] \rangle$ for some $\iota$. We can now complete the proof by showing that $\iota = \iota^T_{\gamma,\alpha,\delta}$ just as for the $0^{th}$ theory. $\dagger$

By 1.5 and 1.6, if $T$ is an EM blueprint satisfying (I)-(III), then for any $\eta, \xi$, and $\alpha$ with $\alpha$ a limit ordinal (possibly 0), we can unambiguously set

\[
\begin{align*}
\chi^T_{\eta,\beta,\gamma} & = \chi^T_{\eta,\beta,\delta} \text{ for any limit ordinals } \beta, \gamma \text{ with } \beta > \eta, \\
\iota^T_{\xi,\alpha,\gamma} & = \iota^T_{\xi,\alpha,\delta} \text{ for any limit ordinal } \gamma > \xi, \\
\gamma^T_{\xi,\alpha} & = \{ \iota^T_{\xi,\alpha} \mid \xi \in On \}.
\end{align*}
\]

Finally, we specify that if $\alpha$ is a limit ordinal,
$D^T_{\alpha}$ is the normal ultrafilter over $\chi^T_{\alpha}$ in the sense of $\mathcal{M}(T, \alpha, \omega)$.

1.7 Lemma: Suppose that $T$ is an EM blueprint satisfying (I)-(III) and $\alpha$ is a limit ordinal. Then:
(a) $(L[D^T_{\alpha}], \in, D^T_{\alpha})$ is the $\chi^T_{\alpha}$-model, and whenever $\xi \leq \zeta$,

\[
(L_{\xi_{\alpha}}[D^T_{\alpha}], \in, D^T_{\alpha}) \prec (L_{\chi_{\alpha}}[\chi^T_{\alpha}], \in, D^T_{\alpha}).
\]

(b) $|\chi^T_{\eta}| = |\eta| + \aleph_0$ for every $\eta$, and $|\nu^T_{\xi, \alpha}| = |\xi| + |\alpha|$ for every $\xi$.
(c) $\{\chi^T_{\eta} \mid \eta \in \text{On}\}$ and $Y^{T, \alpha}$ are closed unbounded classes of ordinals.
(d) For any cardinal $\lambda > \omega$, $\chi^T_{\lambda} = \lambda$ and if $\lambda > \alpha$, $\nu^T_{\lambda, \alpha} = \lambda$ and so $\mathcal{M}(T, \alpha, \lambda) = (L_\lambda[D^T_{\alpha}], \in, D^T_{\alpha})$.
(e) If $\overline{T}$ is any EM blueprint satisfying (I)-(III), then $\overline{T} = T$.

Proof: For (a), note that 1.6 implies that there is a $D$ such that

\[
\mathcal{M}(T, \alpha, \gamma) = (L_{\chi^T_{\alpha}[D]}, \in, D \cap L_{\chi^T_{\alpha}[D]}) \text{ for any limit ordinal } \gamma \geq \omega.
\]

Hence, $(L[D], \in, D)$ is the $\chi^T_{\alpha}$-model. By indiscernibility, $(\chi^T_{\alpha})^{L[D]} < \nu^T_{\xi, \alpha}$ for any $\xi$. In particular, $D \subseteq L_{\nu^T_{\alpha}}[D]$ by the proof of the GCH in $L[D]$, and so $D = D^T_{\alpha}$. The rest of (a) and the lemma is just as for the $0^\#$ theory.

We shall soon derive more information about the $\chi^T_{\alpha}$'s and $\nu^T_{\alpha}$'s, incorporating successor $\alpha$'s into the scheme using iterated ultrapowers. As with $0^\#$, the hypothesis of 1.7 implies through its (a) and (d) that for any limit ordinal $\alpha \geq \omega$,

the satisfaction relation for $(L[D^T_{\alpha}], \in, D^T_{\alpha})$ is definable in ZFC.

We point out without further mention that because of this, various upcoming assertions like the following about inner models are directly formalizable.

1.8 Lemma: For any limit $\alpha$, $\{\chi^T_{\eta} \mid \eta < \alpha\}$, $Y^{T, \alpha}$ is a double class of indiscernibles for the $\chi^T_{\alpha}$-model such that the Skolem hull of $\{\chi^T_{\eta} \mid \eta < \alpha\} \cup Y^{T, \alpha}$ in the model is again the model.

With 1.7(e) in hand, we stipulate that

$0^\dagger$ is the unique EM blueprint satisfying (I)-(III)

if there is one, and use the accepted
with the intended meaning. Through a recursive arithmetization of \( \mathbb{Z}^* \), \( 0^\dagger \) is regarded as a subset of \( \omega \). The following summarizing theorem highlights some of the features:

1.9 Theorem (Solovay):

(a) If there is a \( \kappa \)-model for some ordinal \( \kappa \) and a Ramsey cardinal \( > \kappa \) (e.g. if there are two measurable cardinals), then \( 0^\dagger \) exists.

(b) \( 0^\dagger \) exists iff for every cardinal \( \lambda > \omega \), there is a \( \lambda \)-model and a double class \( \langle X, Y \rangle \) of indiscernibles for it such that: \( X \subseteq \lambda \) is closed unbounded, \( Y \subseteq On \sim \lambda + 1 \) is a closed unbounded class, \( X \cup \{ \lambda \} \cup Y \) contains every cardinal \( > \omega \) and the Skolem hull of \( X \cup Y \) in the \( \lambda \)-model is again the model. \( \dagger \)

In this last situation, since there are regular cardinals and there are strong limit cardinals, every member of \( Y \) is inaccessible in the \( \lambda \)-model by indiscernibility. Hence, as Solovay noted, the existence of \( 0^\dagger \) implies the existence of set, transitive \( \epsilon \)-models of measurability. To be sure, we can say much more about the size of the indiscernibles, along the lines of 9.17.

Solovay also established the direct analogues for \( 0^\dagger \) of the results on the definability of \( 0^\# \).

1.10 Theorem (ZF+DC)(Solovay): The relation \( R \subseteq {}^\omega \omega \) defined by

\[
R(x) \iff 0^\dagger \text{ exists } \land x \in {}^\omega 2 \land \{m \mid x(m) = 1\} = 0^\dagger
\]

is \( \Pi^1_2 \). \( \dagger \)

1.11 Corollary:

(a) \( 0^\dagger \) is absolute for transitive \( \epsilon \)-models of ZF+DC such that \( \omega_1 \subseteq M \) in the following sense: \( M \models \text{ There is an EM blueprint satisfying (I)-(III) iff } 0^\dagger \in M \), in which case \( M \models 0^\dagger \) is the unique EM blueprint satisfying (I)-(III).

(b) \( 0^\dagger \) is a \( \Delta^1_3 \) subset of \( \omega \) which is not a member of the \( \kappa \)-model for any ordinal \( \kappa \).

Proof: (a) uses the Shoenfield Absoluteness Theorem. For (b), if to the contrary \( 0^\dagger \) were a member of the \( \kappa \)-model for some \( \kappa \), by iterating ultrapowers we can assume that \( \kappa \) is a cardinal. But this is a contradiction by (a) and 1.9(b). \( \dagger \)

§2. Relationships among the Classes of Indiscernibles

Having developed the analogue of the \( 0^\# \) theory, we now proceed to derive more information with iterated ultrapowers that sharpens the focus. For the rest of this paper, we stipulate that
\((L[U], \in, U)\) is the \(\kappa\)-model where \(\kappa\) is least possible, and
\((L[U_\alpha], U_\alpha, \kappa_\alpha, i_{\alpha\beta})_{\alpha < \beta < \alpha} \in_0 \) is the iteration of \((L[U], \in, U)\).

By a previous remark, all the \(\kappa\)-models for various \(\kappa\) appear in this iteration.

The results quickly follow from the basic uniqueness property of 1.1 tailored for \(0^\dagger\) and \(\kappa\)-models and stated here for emphasis.

2.1 Lemma: For any \(\alpha\), there is at most one double class \((X, Y)\) of indiscernibles for the \(\kappa_\alpha\)-model with \(X \subseteq \kappa_\alpha\) of ordertype \(\alpha\) and \(Y \subseteq On \sim (\kappa_\alpha + 1)\) such that:

(a) For any formula \(\psi(v_1, \ldots, v_{n+1})\) in \(L\), \(x_1 < \ldots < x_n\) all \(\in X\), and \(y_1 < \ldots < y_s\) all \(\in Y\), the \(\kappa_\alpha\)-model satisfies \(\psi[x_1, \ldots, x_n, y_1, \ldots, y_s]\) iff \(\psi(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in 0^\dagger\).

(b) The Skolem hull of \(X \cup Y\) in the \(\kappa_\alpha\)-model is again the model. 

Proof: In brief, if \((X, Y)\) and \((\overline{X}, \overline{Y})\) were two such classes, then the order-preserving injection \(X \cup Y \rightarrow \overline{X} \cup \overline{Y}\) extends to an isomorphism of the \(\kappa_\alpha\)-model into itself and hence must have been the identity. 

Dropping the superscript \(T\) by uniqueness, in the presence of \(0^\dagger\) there are corresponding classes

\[\{x_\eta \mid \eta \in On\}\] and \(Y^\alpha = \{z_\xi \mid \xi \in On\}\)

for every limit \(\alpha\) as defined before 1.7 and with the properties ascribed by it. By 1.8, for every limit ordinal \(\alpha\), \(\{(x_\eta \mid \eta < \alpha), Y^\alpha\}\) satisfies 2.1(a)(b) for the \(\chi_\alpha\)-model. In particular, the \((X, Y)\) of 2.1 need not determine an EM blueprint: \(X\) can be finite, with \((\emptyset, Y^0)\) for the \(\chi_0\)-model being an example. In fact, \(Y^0\) is the basis of the next lemma, one which is not surprising in view of the canonicity properties of \(0^\dagger\).

2.2 Lemma:

(a) For every \(\alpha\), \((X, Y) = \{(x_\eta \mid \eta < \alpha), i_{o\alpha} Y^0\}\) satisfies 2.1(a)(b) for the \(\kappa_\alpha\)-model.

(b) \(x_\eta = \kappa_\eta\) for every \(\eta\), and \(i_{o\alpha} Y^0 = Y^\alpha\) for every limit ordinal \(\alpha\).

Proof: We first show that \(\chi_0 = \kappa_0\). Clearly, \(\kappa_0 \leq \chi_0\) by definition of \(\kappa_0\). For the converse, we work in some generality for a later inference. Let \(\lambda\) be a regular cardinal such that \(\kappa_\lambda = \lambda = \chi_\lambda\); there are arbitrarily large such \(\lambda\) by the iterated ultrapower theory and 1.7(d). In what follows, we rely on the fact that by 1.8, \(\{(x_\eta \mid \eta < \lambda), Y^\lambda\}\) satisfies 2.1(a)(b) for the \(\lambda\)-model. By Kunen[70]3.3, if \(E\) is the proper class of cardinals \(> \lambda\) fixed by \(i_{0\lambda}\), then \(\{(x_\eta \mid \eta < \lambda), E\}\) is a double class of indiscernibles for the \(\lambda\)-model. Note that this pair satisfies 2.1(a) for this model: There are infinitely many \(x_\eta\)'s in the set \(\{(x_\eta \mid \eta < \lambda), E\}\), and \(E \subseteq Y^\lambda\) as \(Y^\lambda\) contains every cardinal \(> \lambda\). Let \(N\) be the transitive collapse of the Skolem hull of \(\{(x_\eta \mid \eta < \lambda) \cup E\}\) in the \(\lambda\)-model, and \(\pi\) the collapsing isomorphism. Then \(N\) is
again the \( \lambda \)-model since \( \pi(\lambda) = \lambda \). Hence, \( \pi(\kappa_\eta) = \chi_\eta \) for every \( \eta < \lambda \) by 2.1. Consequently, \( \chi_\eta \leq \kappa_\eta \) for every \( \eta < \lambda \) since \( \pi \) is collapsing.

Now that we know \( \kappa_0 = \chi_0 \), we can argue as follows for any \( \alpha \): Remembering that \( \langle L[U_\alpha], \epsilon, U_\alpha \rangle \) is the \( \kappa_\alpha \)-model, by the representation of iterated ultrapowers in Kunen [70][22] any \( x \in L[U_\alpha] \) is of form \( x = i_{0\alpha}(f)(\gamma_1, \ldots, \gamma_n) \) for some function \( f \in L[U_0] \) and \( \gamma_1 < \cdots < \gamma_n < \alpha \). Since \( \kappa_0 = \chi_0 \), \( f \) is definable in the \( \kappa_0 \)-model from elements in \( Y^0 \), and so \( i_{0\alpha}(f) \) is definable in the \( \kappa_\alpha \)-model from elements in \( i_{0\alpha}'' Y^0 \). Hence, \( \{\kappa_\eta \mid \eta < \alpha\}, i_{0\alpha}'' Y^0 \) satisfies 2.1(b) for the \( \kappa_\alpha \)-model.

Next, take any regular cardinal \( \lambda \geq \alpha \) such that \( \kappa_\lambda = \lambda = \chi_\lambda \). It follows from a previous remark that \( \{\kappa_\eta \mid \eta < \lambda\}, i_{0\lambda}'' Y^0 \) satisfies 2.1(a) for the \( \lambda \)-model. Hence, the elementarity of \( i_{0\lambda} \) shows that \( \{\kappa_\eta \mid \eta < \alpha\}, i_{0\alpha}'' Y^0 \) satisfies 2.1(a) for the \( \kappa_\alpha \)-model.

(b) of the lemma is now a direct consequence of 1.8 and 2.1. \( \dashv \)

We can now define \( Y^\alpha \) for successor ordinals \( \alpha \) by:

\[
Y^\alpha = i_{0\alpha}'' Y^0.
\]

These \( Y^\alpha \)'s are also closed unbounded classes by the usual argument from (IIIb) and (IIIb) of 0\(^\dagger\). Save for a small initial shift, the \( Y^\alpha \)'s are the same:

2.3 Lemma: For any \( \alpha \leq \beta \):

(a) \( Y^\beta = i_{\alpha\beta}''(Y^\alpha \cap (\beta + 1) \cup (Y^\alpha \sim (\beta + 1))) \), a disjoint union.

(b) If \( \beta < \min(Y^\alpha) \), then \( Y^\beta = Y^\alpha \).

Proof: Any \( \iota \in Y^\alpha \) is inaccessible in the \( \kappa_\alpha \)-model. Since the iteration of inner models of measurability beyond the \( \kappa_\alpha \)-model can be defined in the model using \( U_\alpha, i_{\alpha\beta}(\iota) = \iota \) for any \( \beta < \iota \). \( \dashv \)

The following overall theorem describes the global coherence of \( \kappa_\alpha \)-models and their indiscernibles.

2.4 Theorem: Assume \( 0^\dagger \) exists. Then for every \( \alpha \) there is a class \( Y^\alpha \) of ordinals characterized by:

(a) \( Y^\alpha \) is closed unbounded and \( \{\kappa_\eta \mid \eta < \alpha\}, Y^\alpha \) is a double class of indiscernibles for the \( \kappa_\alpha \)-model.

(b) The Skolem hull of \( \{\kappa_\eta \mid \eta < \alpha\} \cup Y^\alpha \) in the \( \kappa_\alpha \)-model is again the model. Moreover, for any \( \alpha \leq \beta, i_{\alpha\beta}'' Y^\alpha = Y^\beta \), and if \( \pi^\beta_\alpha \) is a collapsing isomorphism from the Skolem hull of \( \{\kappa_\eta \mid \eta < \alpha\} \cup Y^\alpha \) in the \( \kappa_\beta \)-model into its transitive collapse, then: \( \pi^\beta_\alpha(\kappa_\eta) = \kappa_\eta \) for \( \eta < \alpha \), \( \pi^\beta_\alpha(\kappa_\beta) = \kappa_\alpha \), and \( \pi^\beta_\alpha'' Y^\beta = Y^\alpha \).

Proof: The characterization of \( Y^\alpha \) follows from 2.1 and the fact that two closed unbounded classes have many common members.
For the assertion about $\pi_\alpha^\beta$, the transitive collapse must be the $\kappa$-model for some $\kappa$ with 
$\langle \{ \pi_\alpha^\beta(\kappa_\eta) \mid \eta < \alpha \}, \pi_\alpha^\beta \gamma^\beta \rangle$ satisfying 2.1(a)(b) for that $\kappa$-model. Because of the ordertype $\alpha$ of the lower set of this pair, the only possibility by uniqueness is $\kappa = \kappa_\alpha$, and the conclusions follow, also by uniqueness. ⊡

Thus, the existence of 0↑ leads to remarkable conclusions about the simple generation of inner models of measurability and their relation to each other. Taking into account the absoluteness result 1.11(a), for every $\alpha$ the $\kappa_\alpha$-model is a subclass of $L[0↑]$, and is moreover definable in $L[0↑]$ as $\bigcup \mathcal{M}(0↑, \alpha, \gamma)$, i.e. by taking the lower set of indiscernibles of ordertype $\alpha$ and “stretching” the upper set. The resulting systems of indiscernibles are closely interrelated by iterated ultrapowers and Skolem hulls as described in 2.4.

§3. When 0↑ Exists

We finally review some characterizations of the existence of 0↑. Kunen’s well-known result that 0# exists iff there is a (non-trivial) elementary embedding: $L \prec L$ has the following analogue:

3.1 Theorem: The following are equivalent:
(a) $0↑$ exists.
(b) The $\kappa$-model exists for some ordinal $\kappa$ and there is an elementary embedding of the model into itself with critical point $> \kappa$.

Proof: In the forward direction, for any $\alpha$ let $Y^\alpha$ be the closed unbounded class for the $\kappa_\alpha$-model as given by 0↑ and $h$ any order-preserving injection of $\{ \kappa_\eta \mid \eta < \alpha \} \cup Y^\alpha$ into itself such that $h(\kappa_\eta) = \kappa_\eta$ for $\eta < \alpha$ and $h(\iota) > \iota$ for some $\iota \in Y^\alpha$. Then the usual argument shows that $h$ induces an elementary embedding of the $\kappa_\alpha$-model into itself as desired.

The converse can be established by following any of the 0# arguments of Jech[1], Kanamori-Magidor[KM], or Dodd[Do], using the following basic observation: If $M$ and $N$ are transitive, $j : M \prec N$, and $j|_{\kappa_\alpha + 1}$ is the identity on $\kappa_\alpha + 1$, then $M = L[U_\alpha]$ iff $N = L[U_\alpha]$. The forward direction is for ultrapower arguments, and the latter, for Skolem hull arguments. ⊡

One can go on to show that every embedding as in 3.1(b) is induced by an $h$ as described in the proof.

The existence of 0↑ does not imply the existence of measurable cardinals, only the $\kappa_\alpha$-models ($L[U_\alpha], \in, U_\alpha$). The following is a slight reformulation of an observation of Kunen in the presence of a measurable cardinal.

3.2 Proposition (Kunen[70]): Suppose that $\kappa$ is a measurable cardinal and $\langle L[U], \in, U \rangle$ the $\kappa$-model. Then the following are equivalent:
(a) $0↑$ exists.
(b) $\kappa^{+ + L[U]} \leq 2^\kappa$. 

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Proof: The forward direction is clear; every (real) cardinal > \kappa is in the class \(Y^\kappa\) of indiscernibles for the \(\kappa\)-model given by \(0^\dagger\), and hence large in the model by simple indiscernibility arguments.

For the converse, first note that in our continuing terminology \(\kappa = \kappa_\alpha\) and \(U = U_\alpha\) for some \(\alpha\). Since the iteration of \(<L[U], \in, U\) can be defined in \(L[U]\) using \(U_\alpha\), \(\kappa^{\omega+L[U]} < \kappa^{++L[U]}\). Also, if \(F\) is the filter over \(\kappa^{\omega+L[U]}\) generated \(\{\kappa^{\alpha+n} : n \in \omega\}\), i.e.

\[
X \in F \iff \exists m \{\kappa^{\alpha+n} : m \leq n < \omega\} \subseteq X,
\]

then \(L[F]\) is the \(\kappa^{\omega+L[U]}\)-model.

Suppose now that \(W\) is any \(\kappa\)-complete ultrafilter over \(\kappa\) and \(j_W : V \prec M_W \cong V^\kappa/W\). Then by assumption

\[
\kappa^{\omega+L[U]} < \kappa^{++L[U]} \leq 2^\kappa < j_W(\kappa).
\]

Also \(\{\kappa^{\alpha+n} : n \in \omega\} \in M_W\) since \(\omega M_W \subseteq M_W\), and hence \(F \cap M_W \in M_W\) so that \(L[F] = L[F \cap M_W]\) is definable in \(M_W\). Consequently,

\[
M_W \models \text{There is a } \rho\text{-model for some } \rho < j_W(\kappa),
\]

so that by elementarity there is a \(\rho\)-model for some \(\rho < \kappa\). By 1.3 this entails the existence of \(0^\dagger\). \(\dashv\)

As Kunen observed, it follows that if there is a measurable cardinal \(\kappa\) such that \(\kappa^+ < 2^\kappa\), then \(0^\dagger\) exists. This was the first inkling of a genuine impediment to forcing at measurable cardinals: the measurability of \(\kappa\) imposes sufficient constraints on \(P(\kappa)\) so that achieving \(\kappa^+ < 2^\kappa\) requires strong hypotheses and presumably a new forcing approach. Such an approach was to be discovered by Silver (see Jech\cite{J} or Kanamori-Magidor\cite{KM}\S25).

The further results on hypotheses sufficient to imply the existence of \(0^\dagger\) depend on the theory of the Core Model \(K\); see Donder-Koepke\cite{DK}.
References


