THE COMPLEAT $0^\dagger$

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Soon after Solovay [So] and Silver [Si] established the basic transcendence properties over L of the existence of $0^\#$, Solovay formulated an analogous set of integers $0^\dagger$ ("zero dagger") for inner models of measurability. Its definition became known through a timely survey of set theory by Mathias which appeared in typescript in 1968 and later as Mathias [M]. Results about $0^\dagger$ appeared in Kunen[K], and it was further described in Dodd[Do]. Here, we finally provide a detailed presentation of the theory of $0^\dagger$, establishing intimate connections between the various generating classes of indiscernibles for inner models of measurability.

As for the ambient context, we assume familiarity with the theory of $0^\#$. Kanamori-Magidor[KM], Jech[J], and Devlin[De] all provide the necessary details; we shall follow the development of the first. We also assume familiarity with the basic inner model structure theory from Kunen[K]. In order to establish some terminology we review the major results: An *inner model of measurability* is an inner model of ZFC of form $L[U]$, where for some ordinal κ, $L[U] \models U$ is a normal ultrafilter over κ. Incorporating U as a predicate, we call $\langle L[U], \in, U\rangle$ *the κ-model*, since it is known that the only dependence is on κ: If $L[U_0]$ and $L[U_1]$ are both inner models of measurability for the same κ, then $U_0 = U_1$. For convenience, we say that $\langle L[U_\alpha], U_\alpha, \kappa_\alpha, i_{\alpha\beta}\rangle_{\alpha\leq\beta\in On}$ is the *iteration of* $\langle L[U], \in, U\rangle$ meaning that the αth iterated ultrapower of $\langle L[U], \in, U\rangle$ is the κ_α-model $\langle L[U_\alpha], \in, U_\alpha\rangle$ with $i_{\alpha\beta}$ the corresponding elementary embedding of the κ_α-model into the κ_β-model. The κ_α's comprise a closed unbounded class of ordinals, and every ρ-model for $\rho > \kappa$ appears in the iteration.

As for the organization of this paper, in §1 we formulate the necessary Ehrenfeucht-Mostowski theory, show how a sufficently strong hypothesis generates inner models of measurability with indiscernibles, and formulate $0^\dagger$. In §2 we establish connections between the classes of indiscernibles for the κ-models for various κ. Finally, in §3 we review various characterizations of the existence of $0^\dagger$.

§1 Indiscernibles for the κ-models

If $\langle L[U], \in, U\rangle$ is the κ-model for some ordinal κ, there exists under sufficient assumptions a set $U^\# \subseteq \kappa$ analogous to $0^\#$ that generates a closed unbounded class of indiscernibles for the structure $\langle L[U], \in, U, \xi\rangle_{\xi<\kappa}$. However, as the κ-models for various κ are merely iterated ultrapowers of each other, one might expect a *unifying* transcendence principle. This is successfully realized by the existence of the set of integers $0^\dagger$. The basic idea behind $0^\dagger$ is to develop a canonical theory for structures of form $\langle L_\zeta[U], \in, U\rangle \models$ "U is a normal ultrafilter over κ" with *two* sets of indiscernibles, one below κ and one above, that together generate the structure. Proceeding to the development, we follow the main steps of Kanamori-Magidor[KM]§7.

If $\mathcal{A}$ is a structure and X and Y are subsets of the domain of $\mathcal{A}$ so that $X \cup Y$ is linearly ordered by a relation $<$, then $\langle X, Y, <\rangle$ (or in context, just $\langle X, Y\rangle$) is a *double set of* *indiscernibles for* $\mathcal{A}$ *iff* for every formula $\psi(v_1, \ldots, v_{n+s})$ in the language of $\mathcal{A}$; $x_1 < \ldots < x_n$ and $\overline{x}_1 < \ldots < \overline{x}_n$ all $\in X$; and $y_1 < \ldots < y_s$ and $\overline{y}_1 < \ldots < \overline{y}_s$ all $\in Y$,

$$\mathcal{A} \models \psi[x_1, \ldots, x_n, y_1, \ldots, y_s] \text{ iff } \mathcal{A} \models \psi[\overline{x}_1, \ldots, \overline{x}_n, \overline{y}_1, \ldots, \overline{y}_s].$$

We next recall that there is a formula $\phi(v_0, v_1)$ that for any set A defines in $L[A]$ a well-ordering $<_{L[A]}$ of $L[A]$ such that: for any limit $\delta > \omega$ and $x, y \in L_\delta[A]$, $x <_{L[A]} y$ *iff* $\langle L_\delta[A], \in, A \cap L_\delta[A] \rangle \models \phi[x, y]$. In such $\langle L_\delta[A], \in, A \cap L_\delta[A] \rangle$ we can define Skolem functions for every formula ψ by taking $<_{L[A]}$-least witnesses; it is crucial to observe that using ϕ, the definition of the Skolem function for ψ can be taken to be the same for all such structures. Consequently, if a structure $\langle M, E, R \rangle$ is elementarily equivalent to one of form $\langle L_\delta[A], \in, A \rangle$ for some limit ordinal $\delta > \omega$, then for any $X \subseteq M$ we can consider *the* Skolem hull of X in $\langle M, E, R \rangle$ to be well-defined, and given by Skolem terms closed under composition.

Let $\overline{\mathcal{L}}$ be the language of set theory augmented by one unary predicate symbol $\dot{U}$, and let $\overline{\mathcal{L}}^*$ be $\overline{\mathcal{L}}$ further augmented by new constants $\{c_k \mid k \in \omega\} \cup \{d_k \mid k \in \omega\}$. By an *EM blueprint* in this context we mean the theory in $\overline{\mathcal{L}}^*$ of some structure

$$\langle L_\zeta[U], \in, U, x_k, y_k \rangle_{k \in \omega}$$

where ζ is a limit ordinal $> \omega$; for some ordinal κ, $\langle L_\zeta[U], \in, U \rangle \models U$ is a normal ultrafilter over κ; and $\langle x_k \mid k \in \omega \rangle$ and $\langle y_k \mid k \in \omega \rangle$ are ascending sequences of ordinals such that $\langle \langle x_k \mid k \in \omega \rangle, \langle y_k \mid k \in \omega \rangle \rangle$ is a double set of indiscernibles for $\langle L_\zeta[U], \in, U \rangle$ satisfying

$$x_k < \kappa < y_k \quad \text{for every} \quad k.$$

A basic observation to keep in mind is that any structure $\langle L_\zeta[U], \in, U \rangle$ where ζ is a limit ordinal $> \omega$ and $\langle L_\zeta[U], \in, U \rangle \models U$ is a normal ultrafilter over κ, a double set $\langle X, Y \rangle$ of ordinal indiscernibles satisfying $X \subseteq \kappa$ and $Y \cap (\kappa + 1) = \emptyset$ uniquely determines an EM blueprint, *so long as* X *and* Y *are both infinite.*

The next two lemmata are as for $0^\#$. For any theory T in $\overline{\mathcal{L}}^*$, let T^- denote its restriction to $\overline{\mathcal{L}}$. Note for here and later that if $\dot{U}$ is interpreted by a normal ultrafilter in a structure, then $\bigcup \dot{U}$ is a way of denoting the corresponding measurable cardinal in the structure.

1.1 Lemma: *Suppose that* T *is an EM blueprint. Then for any* α *and* γ, *there is a model* $\mathcal{M} = \mathcal{M}(T, \alpha, \gamma)$ *of* T^- *unique up to isomorphism such that:*

(a) There is a double set $\langle X, Y \rangle$ *of indiscernibles for* $\mathcal{M}$ *with* $X \cup Y \subseteq On^{\mathcal{M}}$, X *of ordertype* α *and* Y *of ordertype* γ *under* $<^{\mathcal{M}}$, *and* $x <^{\mathcal{M}} \bigcup \dot{U}^{\mathcal{M}} <^{\mathcal{M}} y$ *for every* $x \in X$ *and* $y \in Y$. *Moreover, for any formula* $\psi(v_1, \ldots, v_{n+s})$ *in* $\overline{\mathcal{L}}$, $x_1 <^{\mathcal{M}} \ldots <^{\mathcal{M}} x_n$ *all* $\in X$, *and* $y_1 <^{\mathcal{M}} \ldots <^{\mathcal{M}} y_s$ *all* $\in Y$,

$$\mathcal{M} \models \psi[x_1, \ldots, x_n, y_1, \ldots, y_s] \text{ iff } \psi(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in T.$$

(b) The Skolem hull of $X \cup Y$ in $\mathcal{M}$ is again $\mathcal{M}$. ⊣

If $\mathcal{M}(T,\alpha,\gamma)$ is well-founded, then its transitive collapse is of form $\langle L_\delta[U], \in, U\rangle$ for some limit ordinal $\delta > \omega$. In this case,

we identify $\mathcal{M}(T,\alpha,\gamma)$ with $\langle L_\delta[U], \in, U\rangle$.

1.2 Lemma: *Suppose that T is an EM blueprint. Then $\mathcal{M}(T,\alpha,\gamma)$ is well-founded for every α, γ iff*

$$(I) \qquad \mathcal{M}(T,\alpha,\gamma) \text{ is well-founded for every } \alpha,\gamma < \omega_1.$$

⊣

We next describe a sufficient hypothesis that leads quickly to EM blueprints with potent properties.

1.3 Lemma: *Suppose that there is a κ-model for some ordinal κ and a Ramsey cardinal $> \kappa$. Then there is an EM blueprint satisfying (I) of 1.2.*

Proof: Let $\langle L[U], \in, U\rangle$ be the κ-model with iteration $\langle L[U_\alpha], U_\alpha, \kappa_\alpha, i_{\alpha\beta}\rangle_{\alpha\leq\beta\in On}$, and ν a Ramsey cardinal $> \kappa$. Let λ be a cardinal such that $\kappa_\lambda = \lambda < \nu$. (This is for later extractions from this proof; $\lambda = \omega_1$ suffices for present purposes.) In what follows, we rely on Kunen[70]3.9 concerning how the $i_{\alpha\beta}$'s move ordinals. The set

$$Z = \{\theta < \nu \mid \theta > \lambda \ \wedge \ i_{0\lambda}(\theta) = \theta\}$$

has cardinality ν, so let $Y \in [Z]^\nu$ be a set of indiscernibles for the structure

$$\langle L_\nu[U_\lambda], \in, U_\lambda, \kappa_n\rangle_{n\in\omega}.$$

Set $X = \{\kappa_\alpha \mid \alpha < \lambda\}$. $i_{0\lambda}$ fixes every member of $Z \cup \{\nu\}$, so Kunen[70]3.3 implies that X is a set of indiscernibles for $\langle L_\nu[U_\lambda], \in, U_\lambda\rangle$ allowing parameters from Z. Consequently, a simple argument shows that $\langle X, Y\rangle$ is a double set of indiscernibles for $\langle L_\nu[U_\lambda], \in, U_\lambda\rangle$. Hence, $\langle X, Y\rangle$ determines an EM blueprint, and since X and Y are uncountable, this EM blueprint satisfies (I) by an argument as for $0^\#$. ⊣

Assuming the hypothesis of 1.3, we can deduce the existence of an EM blueprint fully analogous to $0^\#$. (Actually, a weaker partition property than Ramsey will do, but this involves distracting technical details.) On the basis of the proof of 1.3, specify that

(i) $\lambda < \nu$ are uncountable cardinals (in V) and $\langle L_\nu[U], \in, U\rangle \models U$ is a normal ultrafilter over λ,

(ii) $\langle X, Y\rangle$ is a double set of indiscernibles with $X \in [\lambda]^\lambda$ and $Y \in [\nu \sim (\lambda+1)]^\nu$, and

(iii) X and Y have the least possible ωth elements. Finally,

(iv) T_0 is the corresponding EM blueprint.

1.4 Lemma: *The following conditions hold for $T = T_0$:*

(IIa) For any $n+s$-ary Skolem term t, T contains the sentence:

$$t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{s-1}) \in \bigcup \dot{U} \rightarrow t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{s-1}) < c_n.$$

(IIb) For any $n+s$-ary Skolem term t, T contains the sentence:

$$t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{s-1}) \in On \rightarrow t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{s-1}) < d_s.$$

(IIIa) For any $m+n+s+1$-ary Skolem term t, T contains the sentence:

$$t(c_0,\ldots,c_{m+n},d_0,\ldots,d_{s-1}) < c_m \rightarrow$$

$$t(c_0,\ldots,c_{m+n},d_0,\ldots,d_{s-1}) = t(c_0,\ldots,c_{m-1},c_{m+n+1},\ldots,c_{m+2n+1},d_0,\ldots,d_{s-1}).$$

(IIIb) For any $n+r+s+1$-ary Skolem term t, T contains the sentence:

$$t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{r+s}) < d_r \rightarrow$$

$$t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{r+s}) = t(c_0,\ldots,c_{n-1},d_0,\ldots,d_{r-1},d_{r+s+1},\ldots,d_{r+2s+1}).$$

Proof: A simple argument by contradiction establishes (IIa) from $sup(X) = \lambda$ and similarly, (IIb) from $sup(Y) = \nu$. An argument as for $0^{\#}$ establishes (IIIa) and (IIIb) from the minimality of the ωth elements of X and Y respectively and the fact that λ and ν are cardinals. $\dashv$

If an EM blueprint satisfies (I), then for any α,γ temporarily let

$$\langle\langle \chi_\eta^{T,\alpha,\gamma} \mid \eta < \alpha\rangle, \langle \iota_\xi^{T,\alpha,\gamma} \mid \xi < \gamma\rangle\rangle \text{ and } \chi_\alpha^{T,\alpha,\gamma}$$

denote the double set of indiscernibles and the measurable cardinal of $\mathcal{M}(T,\alpha,\gamma)$ respectively. The χ notation anticipates the following:

1.5 Lemma: *Suppose that T is an EM blueprint satisfying (I)-(III). Then for any β,δ:*

(a) $\{\chi_\eta^{T,\beta,\delta} \mid \eta < \beta\}$ is a closed set of ordinals, unbounded in $\chi_\beta^{T,\beta,\delta}$ if β is a limit ordinal $\geq \omega$.

(b) If $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$ with α,γ limit ordinals, then

$$\chi_\eta^{T,\alpha,\gamma} = \chi_\eta^{T,\beta,\delta} \text{ for every } \eta \leq \alpha.$$

Proof: (a)(IIa) implies that the set is unbounded in $\chi_\beta^{T,\beta,\delta}$ if β is a limit ordinal $\geq \omega$. (IIIa) implies that the set is closed (by an argument as for $0^{\#}$, also used below).

(b) Let $\mathcal{H}$ be the Skolem hull of

$$\{\chi_\eta^{T,\beta,\delta} \mid \eta < \alpha\} \bigcup \{\iota_\xi^{T,\beta,\delta} \mid \xi < \gamma\}$$

in $\mathcal{M}(T,\beta,\delta)$. Then its transitive collapse is $\mathcal{M}(T,\alpha,\gamma)$ by uniqueness. We shall show that $\chi_\alpha^{T,\beta,\delta} \subseteq$ the domain of $\mathcal{H}$. This suffices, since the collapsing isomorphism consequently fixes every member of $\{\chi_\eta^{T,\beta,\delta} \mid \eta < \alpha\}$ making this set *the* lower set of the double set of indiscernibles for $\mathcal{M}(T,\alpha,\gamma)$, and also $\chi_\alpha^{T,\alpha,\gamma} = \chi_\alpha^{T,\beta,\delta}$ by (a) for α as well as for β.

To show that $\chi_\alpha^{T,\beta,\delta} \subseteq$ the domain of $\mathcal{H}$, let $\sigma < \chi_\alpha^{T,\beta,\delta}$ be arbitrary. Suppressing the superscript ${}^{T,\beta,\delta}$ from our indiscernibles for convenience,

$$\sigma = t^{\mathcal{M}(T,\beta,\delta)}[\chi_{\eta_0},\ldots,\chi_{\eta_{m-1}},\chi_{\zeta_0},\ldots,\chi_{\zeta_n},\iota_{\xi_0},\ldots,\iota_{\xi_{s-1}}]$$

for some Skolem term t and the indiscernibles listed in ascending order with $\eta_{m-1} < \alpha \leq \zeta_0$. Applying (IIIb) with $r = 0$ we can replace ξ_i by i for $i < s$, and then applying (IIIa) we can replace ζ_i by $\eta_{m-1} + i$ for $i < n+1$. Since α and γ are limit ordinals, the resulting expression shows that σ is in the domain of $\mathcal{H}$. ⊣

1.6 Lemma: *Suppose that T is an EM blueprint satisfying (I)-(III), $\omega \leq \gamma \leq \delta$, and γ and α are limit ordinals (allowing $\alpha = 0$). Then if $\mathcal{M}(T,\alpha,\delta) = \langle L_\zeta[D], \in, D\rangle$, say, the Skolem hull of*

$$\{\chi_\eta^{T,\alpha,\delta} \mid \eta < \alpha\} \bigcup \{\iota_\xi^{T,\alpha,\delta} \mid \xi < \gamma\}$$

in $\mathcal{M}(T,\alpha,\delta)$ is $\langle L_\iota[D], \in, D \cap L_\iota[D]\rangle$, where $\iota = \iota_\gamma^{T,\alpha,\delta}$. Consequently,

$$\mathcal{M}(T,\alpha,\gamma) = \langle L_\iota[D], \in, D \cap L_\iota[D]\rangle \text{ and } \iota_\xi^{T,\alpha,\gamma} = \iota_\xi^{T,\alpha,\delta} \text{ for every } \xi < \gamma.$$

Proof: Let $\mathcal{H}$ be the stated Skolem hull. Then its transitive collapse is $\mathcal{M}(T,\alpha,\gamma)$ by uniqueness. By the argument for 1.5(b) with $\alpha = \beta$ (and a simple version for $\alpha = 0$), $\chi_\alpha^{T,\alpha,\delta} \subseteq$ the domain of $\mathcal{H}$, so that $\mathcal{M}(T,\alpha,\gamma)$ must be of form $\langle L_\iota[D], \in, D \cap L_\iota[D]\rangle$ for some ι. We can now complete the proof by showing that $\iota = \iota_\gamma^{T,\alpha,\delta}$ just as for the $0^\#$ theory. ⊣

By 1.5 and 1.6, if T is an EM blueprint satisfying (I)-(III), then for any η, ξ, and α with α a limit ordinal (possibly 0), we can unambiguously set

$$\begin{aligned}
\chi_\eta^T &= \chi_\eta^{T,\beta,\gamma} \text{ for any limit ordinals } \beta,\gamma \text{ with } \beta > \eta, \text{ and} \\
\iota_\xi^{T,\alpha} &= \iota_\xi^{T,\alpha,\gamma} \text{ for any limit ordinal } \gamma > \xi, \text{ and} \\
Y^{T,\alpha} &= \{\iota_\xi^{T,\alpha} \mid \xi \in On\}.
\end{aligned}$$

Finally, we specify that if α is a limit ordinal,

D^T_α is the normal ultrafilter over χ^T_α in the sense of $\mathcal{M}(T, \alpha, \omega)$.

1.7 Lemma: *Suppose that T is an EM blueprint satisfying (I)-(III) and α is a limit ordinal. Then:*

(a) $\langle L[D^T_\alpha], \in, D^T_\alpha\rangle$ is the χ^T_α-model, and whenever $\xi \leq \zeta$,

$$\langle L_{\iota^{T,\alpha}_\xi}[D^T_\alpha], \in, D^T_\alpha\rangle \prec \langle L_{\iota^{T,\alpha}_\zeta}[D^T_\alpha], \in, D^T_\alpha\rangle.$$

(b) $|\chi^T_\eta| = |\eta| + \aleph_0$ for every η, and $|\iota^{T,\alpha}_\xi| = |\xi| + |\alpha|$ for every ξ.

(c) $\{\chi^T_\eta \mid \eta \in On\}$ and $Y^{T,\alpha}$ are closed unbounded classes of ordinals.

(d) For any cardinal $\lambda > \omega$, $\chi^T_\lambda = \lambda$ and if $\lambda > \alpha$, $\iota^{T,\alpha}_\lambda = \lambda$ and so $\mathcal{M}(T, \alpha, \lambda) = \langle L_\lambda[D^T_\alpha], \in, D^T_\alpha\rangle$.

(e) If $\overline{T}$ is any EM blueprint satisfying (I)-(III), then $\overline{T} = T$.

Proof: For (a), note that 1.6 implies that there is a D such that

$$\mathcal{M}(T, \alpha, \gamma) = \langle L_{\iota^{T,\alpha}_\gamma}[D], \in, D \cap L_{\iota^{T,\alpha}_\gamma}[D]\rangle \text{ for any limit ordinal } \gamma \geq \omega.$$

Hence, $\langle L[D], \in, D\rangle$ is the χ^T_α-model. By indiscernibility, $(\chi^T_\alpha)^{+L[D]} < \iota^{T,\alpha}_\xi$ for any ξ. In particular, $D \subseteq L_{\iota^{T,\alpha}_\omega}[D]$ by the proof of the GCH in $L[D]$, and so $D = D^T_\alpha$. The rest of (a) and the lemma is just as for the $0^\#$ theory. ⊣

We shall soon derive more information about the χ^T_η's and $\iota^{T,\alpha}_\xi$'s, incorporating successor α's into the scheme using iterated ultrapowers. As with $0^\#$, the hypothesis of 1.7 implies through its (a) and (d) that for any limit ordinal $\alpha \geq \omega$,

the satisfaction relation for $\langle L[D^T_\alpha], \in, D^T_\alpha\rangle$ is definable in ZFC.

We point out without further mention that because of this, various upcoming assertions like the following about inner models are directly formalizable.

1.8 Lemma: *For any limit α, $\langle\{\chi^T_\eta \mid \eta < \alpha\},\ Y^{T,\alpha}\rangle$ is a double class of indiscernibles for the χ^T_α-model such that the Skolem hull of $\{\chi^T_\eta \mid \eta < \alpha\} \ \cup\ Y^{T,\alpha}$ in the model is again the model.* ⊣

With 1.7(e) in hand, we stipulate that

$0^\dagger$ is the unique EM blueprint satisfying (I)-(III)

if there is one, and use the accepted

$$0^{\dagger} \text{ exists}$$

with the intended meaning. Through a recursive arithmetization of $\overline{\mathcal{L}}^*$, $0^{\dagger}$ is regarded as a subset of ω. The following summarizing theorem highlights some of the features:

1.9 Theorem (Solovay):

(a) If there is a κ-model for some ordinal κ and a Ramsey cardinal $> \kappa$ (e.g. if there are two measurable cardinals), then $0^{\dagger}$ exists.

(b) $0^{\dagger}$ exists iff for every cardinal $\lambda > \omega$, there is a λ-model and a double class $\langle X, Y \rangle$ of indiscernibles for it such that: $X \subseteq \lambda$ is closed unbounded, $Y \subseteq On \sim \lambda + 1$ is a closed unbounded class, $X \cup \{\lambda\} \cup Y$ contains every cardinal $> \omega$ and the Skolem hull of $X \cup Y$ in the λ-model is again the model. $\dashv$

In this last situation, since there are regular cardinals and there are strong limit cardinals, every member of Y is inaccessible in the λ-model by indiscernibility. Hence, as Solovay noted, the existence of $0^{\dagger}$ implies the existence of set, transitive $\in$-models of measurability. To be sure, we can say much more about the size of the indiscernibles, along the lines of 9.17.

Solovay also established the direct analogues for $0^{\dagger}$ of the results on the definability of $0^{\#}$.

1.10 Theorem (ZF+DC)(Solovay): *The relation $R \subseteq {}^{\omega}\omega$ defined by*

$$R(x) \leftrightarrow 0^{\dagger} \text{ exists} \wedge x \in {}^{\omega}2 \wedge \{m \mid x(m) = 1\} = 0^{\dagger}$$

is Π^1_2. $\dashv$

1.11 Corollary:

(a) $0^{\dagger}$ is absolute for transitive $\in$-models of ZF+DC such that $\omega_1 \subseteq M$ in the following sense: $M \models$ There is an EM blueprint satisfying (I)-(III) iff $0^{\dagger} \in M$, in which case $M \models 0^{\dagger}$ is the unique EM blueprint satisfying (I)-(III).

(b) $0^{\dagger}$ is a Δ^1_3 subset of ω which is not a member of the κ-model for any ordinal κ.

Proof: (a) uses the Shoenfield Absoluteness Theorem. For (b), if to the contrary $0^{\dagger}$ were a member of the κ-model for some κ, by iterating ultrapowers we can assume that κ is a cardinal. But this is a contradiction by (a) and 1.9(b). $\dashv$

§2. Relationships among the Classes of Indiscernibles

Having developed the analogue of the $0^{\#}$ theory, we now proceed to derive more information with iterated ultrapowers that sharpens the focus. For the rest of this paper, we stipulate that

$\langle L[U], \in, U\rangle$ is the $\overline{\kappa}$-model where $\overline{\kappa}$ is least possible, and
$\langle L[U_\alpha], U_\alpha, \kappa_\alpha, i_{\alpha\beta}\rangle_{\alpha\leq\beta\in On}$ is the iteration of $\langle L[U], \in, U\rangle$.

By a previous remark, all the κ-models for various κ appear in this iteration.

The results quickly follow from the basic uniqueness property of 1.1 tailored for $0^\dagger$ and κ-models and stated here for emphasis.

2.1 Lemma: *For any α, there is at most one double class $\langle X, Y\rangle$ of indiscernibles for the κ_α-model with $X \subseteq \kappa_\alpha$ of ordertype α and $Y \subseteq On \sim (\kappa_\alpha + 1)$ such that:*

(a) For any formula $\psi(v_1, \ldots, v_{n+s})$ in $\overline{\mathcal{L}}$, $x_1 < \ldots, < x_n$ all $\in X$, and $y_1 < \ldots < y_s$ all $\in Y$

the κ_α-model satisfies $\psi[x_1, \ldots, x_n, y_1, \ldots, y_s]$ iff $\psi(c_0, \ldots, c_{n-1}, d_0, \ldots, d_{s-1}) \in 0^\dagger$.

(b) The Skolem hull of $X \cup Y$ in the κ_α-model is again the model. $\dashv$

Proof: In brief, if $\langle X, Y\rangle$ and $\langle\overline{X}, \overline{Y}\rangle$ were two such classes, then the order-preserving injection $X \cup Y \to \overline{X} \cup \overline{Y}$ extends to an isomorphism of the κ_α-model into itself and hence must have been the identity. $\dashv$

Dropping the superscript ^{T} by uniqueness, in the presence of $0^\dagger$ there are corresponding classes

$$\{\chi_\eta \mid \eta \in On\} \text{ and } Y^\alpha = \{\iota^\alpha_\xi \mid \xi \in On\}$$

for every limit α as defined before 1.7 and with the properties ascribed by it. By 1.8, for every limit ordinal α, $\langle\{\chi_\eta \mid \eta < \alpha\}, Y^\alpha\rangle$ satisfies 2.1(a)(b) for the χ_α- model. In particular, the $\langle X, Y\rangle$ of 2.1 need not determine an EM blueprint: X can be finite, with $\langle\emptyset, Y^0\rangle$ for the χ_0-model being an example. In fact, Y^0 is the basis of the next lemma, one which is not surprising in view of the canonicity properties of $0^\dagger$.

2.2 Lemma:

(a) For every α, $\langle X, Y\rangle = \langle\{\kappa_\eta \mid \eta < \alpha\}, i_{0\alpha}``Y^0\rangle$ satisfies 2.1(a)(b) for the κ_α-model.

(b) $\chi_\eta = \kappa_\eta$ for every η, and $i_{0\alpha}``Y^0 = Y^\alpha$ for every limit ordinal α.

Proof: We first show that $\chi_0 = \kappa_0$. Clearly, $\kappa_0 \leq \chi_0$ by definition of κ_0. For the converse, we work in some generality for a later inference. Let λ be a regular cardinal such that $\kappa_\lambda = \lambda = \chi_\lambda$; there are arbitrarily large such λ by the iterated ultrapower theory and 1.7(d). In what follows, we rely on the fact that by 1.8, $\langle\{\chi_\eta \mid \eta < \lambda\}, Y^\lambda\rangle$ satisfies 2.1(a)(b) for the λ-model. By Kunen[70]3.3, if E is the proper class of cardinals $> \lambda$ fixed by $i_{0\lambda}$, then $\langle\{\kappa_\eta \mid \eta < \lambda\}, E\rangle$ is a double class of indiscernibles for the λ-model. Note that this pair satisfies 2.1(a) for this model: There are infinitely many χ_η's in the set $\{\kappa_\eta \mid \eta < \lambda\}$, and $E \subseteq Y^\lambda$ as Y^λ contains every cardinal $> \lambda$. Let $\mathcal{N}$ be the transitive collapse of the Skolem hull of $\{\kappa_\eta \mid \eta < \lambda\} \cup E$ in the λ-model, and π the collapsing isomorphism. Then $\mathcal{N}$ is again the λ-model since $\pi(\lambda) = \lambda$. Hence, $\pi(\kappa_\eta) = \chi_\eta$ for every $\eta < \lambda$ by 2.1. Consequently, $\chi_\eta \leq \kappa_\eta$ for every $\eta < \lambda$ since π is collapsing.

Now that we know $\kappa_0 = \chi_0$, we can argue as follows for any α: Remembering that $\langle L[U_\alpha], \in, U_\alpha \rangle$ is the κ_α-model, by the representation of iterated ultrapowers in Kunen [70]§2 any $x \in L[U_\alpha]$ is of form $x = i_{0\alpha}(f)(\kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n})$ for some function $f \in L[U_0]$ and $\gamma_1 < \ldots < \gamma_n < \alpha$. Since $\kappa_0 = \chi_0$, f is definable in the κ_0-model from elements in Y^0, and so $i_{0\alpha}(f)$ is definable in the κ_α-model from elements in $i_{0\alpha}$"Y^0. Hence, $\langle \{\kappa_\eta \mid \eta < \alpha\},\ i_{0\alpha}"Y^0 \rangle$ satisfies 2.1(b) for the κ_α-model.

Next, take any regular cardinal $\lambda \geq \alpha$ such that $\kappa_\lambda = \lambda = \chi_\lambda$. It follows from a previous remark that $\langle \{\kappa_\eta \mid \eta < \lambda\}, i_{0\lambda}"Y^0 \rangle$ satisfies 2.1(a) for the λ-model. Hence, the elementarity of $i_{\alpha\lambda}$ shows that $\langle \{\kappa_\eta \mid \eta < \alpha\}, i_{0\alpha}"Y^0 \rangle$ satisfies 2.1(a) for the κ_α-model.

(b) of the lemma is now a direct consequence of 1.8 and 2.1. $\dashv$

We can now define Y^α for *successor* ordinals α by:

$$Y^\alpha = i_{0\alpha}"Y^0.$$

These Y^α's are also closed unbounded classes by the usual argument from (IIb) and (IIIb) of $0^\dagger$. Save for a small initial shift, the Y^α's are the same:

2.3 Lemma: *For any $\alpha \leq \beta$:*
(a) $Y^\beta = i_{\alpha\beta}"(Y^\alpha \cap (\beta+1) \cup (Y^\alpha \sim (\beta+1)))$, a disjoint union.
(b) If $\beta < min(Y^\alpha)$, then $Y^\beta = Y^\alpha$.

Proof: Any $\iota \in Y^\alpha$ is inaccessible in the κ_α-model. Since the iteration of inner models of measurability beyond the κ_α-model can be defined in the model using U_α, $i_{\alpha\beta}(\iota) = \iota$ for any $\beta < \iota$.) $\dashv$

The following overall theorem describes the global coherence of κ_α-models and their indiscernibles.

2.4 Theorem: *Assume $0^\dagger$ exists. Then for every α there is a class Y^α of ordinals characterized by:*

(a) Y^α is closed unbounded and $\langle \{\kappa_\eta \mid \eta < \alpha\}, Y^\alpha \rangle$ is a double class of indiscernibles for the κ_α-model.

(b) The Skolem hull of $\{\kappa_\eta \mid \eta < \alpha\} \cup Y^\alpha$ in the κ_α-model is again the model.

Moreover, for any $\alpha \leq \beta$, $i_{\alpha\beta}"Y^\alpha = Y^\beta$, and if π^β_α is a collapsing isomorphism from the Skolem hull of $\{\kappa_\eta \mid \eta < \alpha\} \cup Y^\beta$ in the κ_β-model into its transitive collapse, then: $\pi^\beta_\alpha(\kappa_\eta) = \kappa_\eta$ for $\eta < \alpha$, $\pi^\beta_\alpha(\kappa_\beta) = \kappa_\alpha$, and $\pi^\beta_\alpha"Y^\beta = Y^\alpha$.

Proof: The characterization of Y^α follows from 2.1 and the fact that two closed unbounded classes have many common members.

For the assertion about π^β_α, the transitive collapse must be the κ-model for some κ with $\langle\{\pi^\beta_\alpha(\kappa_\eta) \mid \eta < \alpha\}, \pi^\beta_\alpha``Y^\beta\rangle$ satisfying 2.1(a)(b) for that κ-model. Because of the ordertype α of the lower set of this pair, the only possibility by uniqueness is $\kappa = \kappa_\alpha$, and the conclusions follow, also by uniqueness. ⊣

Thus, the existence of $0^\dagger$ leads to remarkable conclusions about the simple generation of inner models of measurability and their relation to each other. Taking into account the absoluteness result 1.11(a), for every α the κ_α-model is a subclass of $L[0^\dagger]$, and is moreover definable in $L[0^\dagger]$ as $\bigcup_\gamma \mathcal{M}(0^\dagger, \alpha, \gamma)$, i.e. by taking the lower set of indiscernibles of ordertype α and "stretching" the upper set. The resulting systems of indiscernibles are closely interrelated by iterated ultrapowers and Skolem hulls as described in 2.4.

§3. When $0^\dagger$ Exists

We finally review some characterizations of the existence of $0^\dagger$. Kunen's well-known result that $0^\#$ exists *iff* there is a (non-trivial) elementary embedding: $L \prec L$ has the following analogue:

3.1 Theorem: *The following are equivalent:*

(a) $0^\dagger$ *exists.*

(b) The κ-model exists for some ordinal κ and there is an elementary embedding of the model into itself with critical point $> \kappa$.

Proof: In the forward direction, for any α let Y^α be the closed unbounded class for the κ_α-model as given by $0^\dagger$ and h any order-preserving injection of $\{\kappa_\eta \mid \eta < \alpha\} \cup Y^\alpha$ into itself such that $h(\kappa_\eta) = \kappa_\eta$ for $\eta < \alpha$ and $h(\iota) > \iota$ for some $\iota \in Y^\alpha$. Then the usual argument shows that h induces an elementary embedding of the κ_α-model into itself as desired.

The converse can be established by following any of the $0^\#$ arguments of Jech[J], Kanamori-Magidor[KM], or Dodd[Do], using the following basic observation: If M and N are transitive, $j : M \prec N$, and $j|\kappa_\alpha + 1$ is the identity on $\kappa_\alpha + 1$, then $M = L[U_\alpha]$ *iff* $N = L[U_\alpha]$. The forward direction is for ultrapower arguments, and the latter, for Skolem hull arguments. ⊣

One can go on to show that *every* embedding as in 3.1(b) is induced by an h as described in the proof.

The existence of $0^\dagger$ does not imply the existence of measurable cardinals, only the κ_α-models $\langle L[U_\alpha], \in, U_\alpha\rangle$. The following is a slight reformulation of an observation of Kunen in the presence of a measurable cardinal.

3.2 Proposition (Kunen[70]): *Suppose that κ is a measurable cardinal and $\langle L[U], \in, U\rangle$ the κ-model. Then the following are equivalent:*

(a) $0^\dagger$ *exists.*

(b) $\kappa^{++L[U]} \le 2^\kappa$.

Proof: The forward direction is clear; every (real) cardinal $> \kappa$ is in the class Y^κ of indiscernibles for the κ-model given by $0^\dagger$, and hence large in the model by simple indiscernibility arguments.

For the converse, first note that in our continuing terminology $\kappa = \kappa_\alpha$ and $U = U_\alpha$ for some α. Since the iteration of $\langle L[U], \in, U\rangle$ can be defined in $L[U]$ using U, $\kappa_{\alpha+\omega} < \kappa^{++L[U]}$. Also, if F is the filter over $\kappa_{\alpha+\omega}$ generated $\{\kappa_{\alpha+n} \mid n \in \omega\}$, i.e.

$$X \in F \ \textit{ iff } \ \exists m\{\kappa_{\alpha+n} \mid m \leq n < \omega\} \subseteq X,$$

then $L[F]$ is the $\kappa_{\alpha+\omega}$-model.

Suppose now that W is any κ-complete ultrafilter over κ and $j_W : V \prec M_W \approx V^\kappa/W$. Then by assumption

$$\kappa_{\alpha+\omega} < \kappa^{++L[U]} \leq 2^\kappa < j_W(\kappa).$$

Also $\{\kappa_{\alpha+n} \mid n \in \omega\} \in M_W$ since ${}^\omega M_W \subseteq M_W$, and hence $F \cap M_W \in M_W$ so that $L[F] = L[F \cap M_W]$ is definable in M_W. Consequently,

$$M_W \models \text{There is a } \rho\text{-model for some } \rho < j_W(\kappa),$$

so that by elementarity there is a ρ-model for some $\rho < \kappa$. By 1.3 this entails the existence of $0^\dagger$. $\dashv$

As Kunen observed, it follows that *if there is a measurable cardinal* κ *such that* $\kappa^+ < 2^\kappa$, *then* $0^\dagger$ *exists.* This was the first inkling of a genuine impediment to forcing at measurable cardinals: the measurability of κ imposes sufficient constraints on $\mathcal{P}(\kappa)$ so that achieving $\kappa^+ < 2^\kappa$ requires strong hypotheses and presumably a new forcing approach. Such an approach was to be discovered by Silver (see Jech[J] or Kanamori-Magidor[KM]§25).

The further results on hypotheses sufficient to imply the existence of $0^\dagger$ depend on the theory of the Core Model K; see Donder-Koepke[DK].

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